(Work in progress: beware signs and powers of 2)

A level 1 genus 2 Jacobi's derivative formula and applications to the analytic theory of genus 2 curves

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Modular forms

- Central in number theory: few but ubiquitous

- $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, χ character on Γ, $k \in \mathbb{Z}$, $\tau \in \mathfrak{h} = \{x + iy | y > 0\}$

- f modular of weight k and character χ if for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f(\frac{a\tau + b}{c\tau + d}) = \chi(\gamma)(c\tau + d)^k f(\tau)$$

- f analytic on $\mathfrak{h}$, cusps of $\mathfrak{h}/\Gamma$. If vanishes at cusps called a cusp form.

Genus 1 case

- $L \subset \mathbb{C}$ lattice, $E = \mathbb{C}/L$, $z \in \mathbb{C}$,

$$\wp(z, L) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in L} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{1}{z^2} + \sum_{n \geq 2} e_n z^{2n-2}.$$

$$\wp(\alpha z, \alpha L) = \alpha^{-2} \wp(z, L),$$

- $\tau \in \mathfrak{h}$, $L_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$. If $\wp(z, \tau) = \wp(z, L_\tau)$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$\wp(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2 \wp(z, \mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)) = (c\tau + d)^2 \wp(z, \tau)$$

$$e_n(\tau) = e_n(L_\tau) \implies e_n(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{2n} e_n(\tau)$$

- e_n is forced to be a modular form, because of modularity of $\wp$ AND that of z

- Forces coefficients of defining equation for E,

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

to be modular ($g_2 = 60e_2$, $g_3 = 140e_3$)

Application

- Rubin and Silverberg (following Gross, Stark) used modular coefficients of elliptic curve to count points on elliptic curves over finite fields.

- Apply to “CM" method.

- Goal is to build modular models of genus 2 curves (do history later)

- Nick Alexander (Silverberg student) is using to generalize Rubin-Silverberg to genus 2.

Siegel modular forms of genus 2

- $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ Consists of:

- Integral 2×2 matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$^t\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

- $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ act on $\mathfrak{h}_2$ via $\gamma \circ \tau = (A\tau + B)(C\tau + D)^{-1}$.

- $k \in \mathbb{Z}$, χ a character of Γ.

- Siegel modular form of degree 2 on Γ of weight k and character χ, is holomorphic functions f on $\mathfrak{h}_2$ satisfying

$$f(\gamma \circ \tau) = \chi(\gamma) j_\gamma(\tau)^k f(\tau),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, where $j_\gamma(\tau) = \mathrm{det}(C\tau + D)$.

- Build with theta functions.

Theta Functions

- $\tau \in \mathfrak{h}_g,\ a, b \in \frac{1}{2}\mathbb{Z}^g,\ z \in \mathbb{C}^g$.

- Theta function with characteristic $[{a \atop b}]$ is

$$\theta[{a \atop b}](z,\tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i^t(n+a)\tau(n+a)+2\pi i^t(n+a)(z+b)}.$$

- $[{a \atop b}]$ is a *theta characteristic*. It is *even* or *odd* depending on whether $\theta[{a \atop b}](z,\tau)$ is an even or odd function, i.e., where $e^{4\pi i ab} = \pm 1$.

Transformation formula

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{Sp}_{2g}(\mathbb{Z}),\ a, b \in \mathbb{R}^g,\ z \in \mathbb{C}^g,\ \tau \in \mathfrak{h}_g,$$

$$\theta \begin{bmatrix} a \\ b \end{bmatrix}^{\gamma} ({}^t(C\tau+D)^{-1}z, \gamma\circ\tau) = \zeta(\gamma, a, b) j_\gamma(\tau)^{1/2} e^{\pi i {}^t z (C\tau+D)^{-1} C z} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau),$$

$$\begin{aligned} \zeta(\gamma, a, b) &= \rho(\gamma)\kappa(\gamma, a, b), \\ \kappa(\gamma, a, b) &= e^{\pi i ({}^t(Da-Cb)(-Ba+Ab+(A^tB)_0) - {}^t ab)} \\ \rho(\gamma) &= \text{an eighth root of 1}, \\ \begin{bmatrix} a \\ b \end{bmatrix}^{\gamma} &= \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (C^tD)_0 \\ (A^tB)_0 \end{bmatrix}, \end{aligned}$$

For matrix M, $(M)_0$ is column vector of diagonal entries of M
$j_\gamma(\tau)^{1/2}$ is a choice of branch of square root of $j_\gamma(\tau)$.

g=1, Jacobi's Derivative Formula

- $\tau \in \mathfrak{h}$, $\theta[{}^{1/2}_{1/2}](0,\tau)' = -\pi\theta[{}^{0}_{0}](0,\tau)\theta[{}^{1/2}_{0}](0,\tau)\theta[{}^{0}_{1/2}](0,\tau)$.

- $[{}^{1/2}_{1/2}]$ is the lone odd theta characteristic mod 1, and $[{}^{0}_{0}]$, $[{}^{0}_{1/2}]$, $[{}^{1/2}_{0}]$ represent the 3 even theta characteristic mod 1.

- For $\gamma \in \Gamma$, the map $[{}^{a}_{b}] \to [{}^{a}_{b}]^{\gamma}$ mod 1 gives an action on theta characteristic mod 1 that preserves the parity of theta characteristics.

Quick proof

- Transformation formula shows both sides of formula to the eighth power are modular forms of weight 12 for Γ. Their Fourier expansions show they are cusp forms. There is a unique such up to constants.The Fourier expansions give the constant.

- Formula was generalized by Rosenhain to $\tau \in \mathfrak{h}_2$, by Thomae to τ the period matrix of hyperelliptic curves, and by Igusa to all $\tau \in \mathfrak{h}_g$. Still active area (Farkas & Kra, Grushevsky & Manni.)

g=2: Rosenhain's Theorem

THEOREM: If δ_i, $i = 1, 2$, are distinct odd theta characteristics, then there are even theta characteristics ϵ_k, $1 \leq k \leq 4$, depending on the δ_i, such that

$$\mathrm{Det}_{1 \leq i,j \leq 2}\left[\frac{\partial \theta[\delta_i](0,\tau)}{\partial z_j}\right] = \pm \pi^2 \prod_{k=1}^{4} \theta[\epsilon_k](0,\tau).$$

- Let $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$, $\Gamma_\delta = \{\gamma \in \Gamma | [\delta]^\gamma = [\delta] \mod 1\}$

- Unlike genus 1, both sides of formula only modular on $\Gamma_{\delta_1} \cap \Gamma_{\delta_2}$ (so not on all of Γ, which would be "level 1")

First goal will be to describe a version of Rosenhain's formula that is modular for all of Γ.

Set up

- There are 10 even theta characteristics mod 1 for degree 2 theta functions. Choose representatives for these mod 1 and define
$$D(\tau) = \prod_{\epsilon \text{ even}} \theta[\epsilon](0, \tau).$$
- Let Z be orbit of $\tau_{12} = 0$ in $\mathfrak{h}_2$ under action of $\mathrm{Sp}_4(\mathbb{Z})$. Then $D(\tau)$ has a zero of order 1 on Z and no other zeroes. (So $D(\tau) \neq 0$ precisely when τ is the period matrix of a curve of genus 2.)
- D is up to a constant the Siegel modular form (with character ψ) on Γ and weight 5.

Definition of $X[\delta]$

- For an odd theta characteristic δ, set

$$X[\delta](z,\tau) = \theta[\delta](z,\tau)^3 \text{Det}_{1\leq i,j\leq 2}[\frac{\partial^2 \log\theta[\delta](z,\tau)}{\partial z_i \partial z_j}],$$

  which a computation with partial derivatives shows is entire.

- $X[\delta]$ was not chosen out of thin air. The function $\frac{X[\delta](z,\tau)}{\theta[\delta](z,\tau)^3}$ plays a pivotal role in the function theory of the abelian variety $A_\tau = \mathbb{C}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$ when $D(\tau) \neq 0$.

Degree 2 generalization of Jacobi's formula

THEOREM: For any odd theta characteristic δ, $\mathrm{Det}_\delta =$

$$\mathrm{Det}\begin{pmatrix} \frac{\partial\theta[\delta](0,\tau)}{\partial z_1} & \frac{\partial\theta[\delta](0,\tau)}{\partial z_2} \\ \frac{\partial X[\delta](0,\tau)}{\partial z_1} & \frac{\partial X[\delta](0,\tau)}{\partial z_2} \end{pmatrix} = \pm 2\pi^6 D(\tau).$$

Sketch of Proof

- Transformation formula show Det_δ is Siegel modular form (with character) on Γ_δ that vanishes on Z, and Γ permutes the Det_δ.

- Since holds for ALL odd theta characteristics δ, $\mathrm{Det}_\delta/D(\tau)$ is holomorphic, so Siegel modular form of weight 0, i.e., a constant.

- Constant from the lead term of Taylor expansion in τ_{12}.

- Must employ Jacobi's derivative formula to find constant! (Similar argument gives quick proof of Rosenhain's theorem.)

Analytic jacobian

- Start with genus 2 curve

$$C : y^2 = x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5$$

  ($\infty =$ point at infinity.)

- Differentials of the first kind $\mu_1 = dx/y, \mu_2 = xdx/y$.

- Symplectic basis for $H_1(C, \mathbb{Z})$, A and B-loops generators.

- Form period matrices

$$\omega = [\omega_{ij}],\ \omega' = [\omega'_{ij}],\ \tau = \omega^{-1}\omega'$$

Analytic jacobian, continued

- Get $\tau \in \mathfrak{h}_2$. Set $L = \omega\mathbb{Z}^2 \oplus \omega'\mathbb{Z}^2$

- $J = \mathbb{C}^2/L$

- Embed $C \to J$ via

$$P \to \int_{\infty}^{P} (\mu_1, \mu_2) \mod L$$

- Image is a divisor Θ, which is zeros of a theta function with odd characteristic.

Torelli

- Can recover C from (J, Θ).

- Rosenhain form (λ_i = ratio of Thetanullwerte)

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

(coefficients modular on $\cap_{i=1}^{6} \Gamma_{\delta_i}$, index 720 in Γ)

- Guardia form

$$y^2 = x(x^4 + ax^3 + bx^2 + c)$$

(roots involve derivatives of theta functions)

(coefficients modular on $\Gamma_{\delta_1} \cap \Gamma_{\delta_2}$, index 30 in Γ)

- Our result has coefficients modular on Γ_δ

(index 6 in Γ)

- Done by viewing C as zeroes of a theta function, and by using “modular parameters"

Set up

- Start with τ such that $D(\tau) \neq 0$. Set $L_\tau = \mathbb{Z}^2\tau \oplus \mathbb{Z}^2$.

- Pick odd theta characteristic δ, $\tilde{\Theta}$ zeroes of $\theta[\delta](z,\tau)$ in $\mathbb{C}^2$. Descends to divisor Θ on $A_\tau = \mathbb{C}^2/L_\tau$.

- Get precisely those (A_τ, Θ) not product of elliptic curves

- Means A_τ is a jacobian of a curve of genus two

- Means that Θ is smooth

Modular parameters

- For $\gamma \in \Gamma_\delta$, (*)

  $\theta[\delta](^t(C\tau+D)^{-1}z, \gamma\circ\tau) = \zeta_8(\gamma)j_\gamma(\tau)^{1/2}e^{\pi i^tz(C\tau+D)^{-1}Cz}\theta[\delta](z,\tau)$

- Set $u_1 = \theta_{z,1}[\delta](0,\tau)z_1 + \theta_{z,2}[\delta](0,\tau)z_2$, the linear term, so

  $$u_1^\gamma = \zeta_8(\gamma)j_\gamma(\tau)^{1/2}u_1$$

  (We let subscript indices $z, ijk...$ denote partial derivatives with respect to the correspondingly indexed variables in z.)

- To avoid sign ambiguities, set $\ell_\gamma = u_1^\gamma/u_1$

- Let $\chi(\gamma) = (u_1^\gamma)^2 / \det(C\tau + D) u_1^2$

- character of Γ_δ of order 4: $\chi^2 = \psi$.

Modular parameters (continued)

- $H =$ Hessian, $h = \det H$. So $X[\delta](z,\tau) = \theta[\delta](z,\tau)^3 h(\log\theta[\delta](z,\tau))$.

- Taking logs and hessians of (*) gives

  $h(\log\theta[\delta]({}^t(C\tau{+}D)^{-1}z, \gamma{\circ}\tau)) = j_\gamma(\tau)^2 \det(\mu{+}H(\log\theta[\delta](z,\tau)))$,

  where $\mu = 2\pi i(C\tau + D)^{-1}C$

- Hence $X[\delta]({}^t(C\tau + D)^{-1}z, \gamma \circ \tau) =$

  $\ell_\gamma(\tau)^3(\gamma) j_\gamma(\tau)^2 e^{3\pi i\, {}^t z(C\tau+D)^{-1}Cz}\theta[\delta](z,\tau)^3 \det(\mu{+}H(\log\theta[\delta](z,\tau)))$.

- Now

  $$\theta[\delta](z,\tau)^3 \det(\mu + H(\log\theta[\delta](z,\tau))) = X[\delta](z,\tau) + \theta[\delta](z,\tau) g_\gamma(z,\tau),$$

  where $g_\gamma(z,\tau)$ is analytic.

- So if $u_2 = X_{z,1}[\delta](0,\tau)z_1 + X_{z,2}[\delta](0,\tau)z_2$,

  the linear term of $X[\delta](z,\tau)$, then

  $$u_2^\gamma = \psi(\gamma)\ell_\gamma(\tau)^7 (u_2 + g_\gamma(0,\tau)u_1).$$

Recap

- Have Γ acting on $\mathbb{C}^2 \times \mathfrak{h}_2$ via $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ sending (z, τ) to $({}^t(C\tau + D)^{-1} z, \gamma(\tau))$.

- Let (z_1, z_2) be the complex coordinates on $\mathbb{C}^2$. We introduced new coordinates:

$$ {}^t(u_1, u_2) = \begin{pmatrix} \frac{\partial \theta[\delta](0,\tau)}{\partial z_1} & \frac{\partial \theta[\delta](0,\tau)}{\partial z_2} \\ \frac{\partial X[\delta](0,\tau)}{\partial z_1} & \frac{\partial X[\delta](0,\tau)}{\partial z_2} \end{pmatrix} {}^t(z_1, z_2) = M {}^t(z_1, z_2). $$

- The theorem tells us that these are parameters for $\mathbb{C}^2$.

$$u_1^\gamma = \ell_\gamma u_1, u_2^\gamma = \psi(\gamma){\ell_\gamma}^7(u_2 + \beta_\gamma(\tau)u_1).$$

- The point is that although the pair (z_1, z_2) transform like a vector valued modular function, u_1 transforms like a modular function, and u_2 almost does.

- First order of business is to modify the definition of u_2 to a parameter which actually does transform like a modular function.

Taylor expansion of theta function in u

- Taking Hessian in definition of X with respect to u gives: $\frac{1}{(2\pi^6 D(\tau))^2} X[\delta](u,\tau) =$

$$\theta[\delta](u,\tau)(\theta[\delta]_{u,11}(u,\tau)\theta[\delta]_{u,22}(u,\tau) - \theta[\delta]_{u,12}(u,\tau)^2)$$

$$-\theta[\delta]_{u,11}(u,\tau)\theta[\delta]_{u,2}(u,\tau)^2 - \theta[\delta]_{u,22}(u,\tau)\theta[\delta]_{u,1}(u,\tau)^2 +$$

$$2\theta[\delta]_{u,12}(u,\tau)\theta[\delta]_{u,1}(u,\tau)\theta[\delta]_{u,2}(u,\tau).$$

- Comparing linear terms gives the linear term of $\theta[\delta]_{u,22}(u,\tau)$ as $-u_2/(4\pi^{12}D(\tau)^2)$

- Hence $\theta[\delta](u,\tau) = u_1 + ?u_1u_2^2 - u_2^3/(12\pi^{12}D(\tau)^2) + ...$

Handwaving over messy details

- Have $\theta_{u,222}(0,\tau) = \frac{-2}{4\pi^{12}D(\tau)^2}$

- Studying the cubic and quintic terms in the expansion of (*) gives $w_1 = u_1$, and $w_2 = u_2 - \frac{1}{10}\frac{\theta_{u,22222}[\delta](0,\tau)}{\theta_{u,222}[\delta](0,\tau)^2}u_1$ are modular.

- For $\gamma \in \Gamma_\delta$,

$$w_1^\gamma = \ell_\gamma w_1, w_2^\gamma = \psi(\gamma){\ell_\gamma}^7 w_2,$$

so w_1 and w_2 are our desired “modular" parameters.

Modular model of the curve

- Since we took $D(\tau) \neq 0$, we have Θ is a smooth curve C of genus 2, A_τ is the Jacobian of C, and what we seek is to use the function theory on A_τ to define a model for C entirely in terms of τ.

- Θ goes through origin, and can expand there via the implicit function theorem to solve identically for

$$w_1 = \rho(w_2) = w_2^3/(12\pi^{12}D(\tau)^2) + ... = \sum_{i \geq 3} a_i(\tau) w_2^i$$

where ρ is a power series containing only terms of odd degree (i.e., $a_i = 0$ for even i) since $\theta[\delta](w,\tau)$ is an odd function.

- Worth noting that $a_5(\tau) = 0$. In fact, we formed w_2 by modifying u_2 by the unique multiple of u_1 that makes $a_5(\tau)$ vanish

- Since w_1 and w_2 are modular, the a_i are modular, too.

Defining the x-coordinate

- Since $\theta[\delta](w, \tau)$ vanishes on Θ, first derivatives have the same factor of automorphy:

- For $w \in \mathbb{C}^2$, $\lambda \in L_\tau$, have a linear function $r_\lambda(w)$:

$$\theta[\delta](w + \lambda, \tau) = e^{r_\lambda(w)}\theta[\delta](w, \tau)$$

- $i = 1, 2$, $w \in \tilde{\Theta}$, $\theta[\delta]_{w,i}(w + \lambda, \tau) = e^{r_\lambda(w)}\theta[\delta]_{w,i}(w, \tau)$.

- Hence $w \in \tilde{\Theta}$,

$$x(w) = x(w, \tau) = -\theta[\delta]_{w,1}(w, \tau)/2\theta[\delta]_{w,2}(w, \tau)$$

gives a function on Θ.

Properties of x

- A is jacobian of C, so Riemann's vanishing theorem $\Longrightarrow$

  For a generic point $v \in \tilde{\Theta}$, and w a variable point,

  $\theta[\delta](v + w, \tau) = 0$ for precisely 2 choices of w mod L_τ

- Since $\theta_{w,1}[\delta](w, \tau)$ and $\theta_{w,2}[\delta](w, \tau)$, have same factor of automorphy as $\theta[\delta](w, \tau)$, also have 2 zeros on $\tilde{\Theta}$ mod L_τ.

- For $\theta_{w,2}(w) = -w_2^2/4\pi^{12}D(\tau)^2 + ...$, both at origin,

- so x is a function on C with a double pole at ∞ (as we will call the origin as a point of C) and nowhere else.

Expansion of x coordinate at origin.

- Since lead term about the origin of $\theta_{w,1}[\delta](w,\tau)$ is 1, expansion of x is
$$\frac{(4\pi^6 D(\tau))^2}{w_2^2} + \sum_{n\geq 0} c_{2n} w_2^{2n}.$$

- c_{2n} determined by a_{2m+1} and are modular.

  In particular $a_5 = 0$ means $c_0 = 0$

- Take derivative of $\theta[\delta]((\rho(w_2), w_2), \tau) = 0$.

- Get $dw_1/dw_2 = \rho'(w_2) = 1/2x(w)$ [see deJong].

- Get for all $\gamma \in \Gamma_\delta$,

$$x(w^\gamma, \gamma(\tau)) = \chi(\gamma) j_\gamma(\tau)^3 x(w, \tau),$$

- x transforms like a Jacobi-Siegel form for $w \in \tilde{\Theta}$ and $\gamma \in \Gamma_\delta$

Defining the y-coordinate

- dx/dw_1, is function on C, poles only where x has poles or w_1 is not a local parameter.

- Since Θ is smooth only happens where $\theta_{w,2}[\delta](w,\tau) = 0$, which is just the origin.

- So $dx/dw_1 = (dx/dw_2)/(dw_1/dw_2)$ has a pole of order 5 at infinity and no other poles on C.

- Compute

$$\frac{dx}{dw_1} = -\frac{d}{dw_1}\frac{\theta[\delta]_{w,1}(\rho(w_2), w_2, \tau)}{2\theta[\delta]_{w,2}(\rho(w_2), w_2, \tau)} =$$

$$\frac{1}{2\theta_{w,2}(\rho(w_2),w_2)^2}\cdot$$

$$[-\theta_{w,2}(\rho(w_2),w_2)(\theta_{w,11}(\rho(w_2),w_2)+\theta_{w,12}(\rho(w_2),w_2)\frac{dw_2}{dw_1})$$

$$-\theta_{w,1}(\rho(w_2),w_2)(\theta_{w,12}(\rho(w_2),w_2)+\theta_{w,22}(\rho(w_2),w_2)\frac{dw_2}{dw_1})]$$

$$=\frac{1}{2\theta_{w,2}(\rho(w_2),w_2)^3}\cdot$$

$$[-\theta_{w,2}(\rho(w_2),w_2)^2\theta_{w,11}(\rho(w_2),w_2)+2\theta_{w,12}(\rho(w_2),w_2)\theta_{w,1}(\rho(w_2),w_2)$$

$$-\theta_{w,1}(\rho(w_2),w_2)^2\theta_{w,22}(\rho(w_2),w_2)]$$

(here we suppress the $[\delta]$ and τ from the notation to improve readability)

Expansion of y-coordinate

- Numerator is just $X[\delta](w, \tau)$ restricted to $\tilde{\Theta}$,

- We denote this quotient by $y(w)/16\pi^6 D(\tau)$, $w \in \tilde{\Theta}$

- So $y(w) = y(w, \tau)$ is a function on C, and tranforms for $\gamma \in \Gamma_\delta$ as

$$y(w^\gamma, \gamma(\tau)) = \ell_\gamma^{15} y(w, \tau),$$

and the expansion at ∞ of y is

$$\frac{(\pi^6 D(\tau))^5}{w_2^5} + \dots$$

Defining equation

- Note that x is an even function on C and y is an odd function.

- From expansions, there are $b_i \in \mathbb{C}$, such that
$$y^2 = f(x) = x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5.$$

- Since y is odd it vanishes at the 5 points W_i of order 2 on J which lie on Θ

- $f(x)$ has distinct roots a_i, $1 \le i \le 5$, and the equation gives an affine model for C. (Equation gives a recursion to find all c_{2n} as a polynomial in c_2, c_4, c_6 and c_8.)

Finding Weierstrass points

- These $a_i = x(W_i)$ are determined by the criterion that $\Theta_i = T^*_{W_i}\Theta$ is zeroes of an odd theta function $\theta[\delta_i](w,\tau)$

$$a_i = -\frac{\theta_{w,1}[\delta](W_i,\tau)}{2\theta_{w,2}[\delta](W_i,\tau)} = -\frac{\frac{\partial}{\partial w_1}\theta[\delta_i](0,\tau)}{2\frac{\partial}{\partial w_2}\theta[\delta_i](0,\tau)}$$

Writing ${}^t(u_1,u_2) = M^t(z_1,z_2)$, ${}^t(w_1,w_2) = N^t(z_1,z_2)$, we have ${}^t(\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2}) = {}^tM^{-1t}(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2})$, and ${}^t(\frac{\partial}{\partial w_1},\frac{\partial}{\partial w_2}) = {}^tN^{-1t}(\frac{\partial}{\partial u_1},\frac{\partial}{\partial u_2})$, so $a_i =$

$$-\frac{\frac{\partial}{\partial u_1}\theta[\delta_i](0,\tau) + \frac{1}{10}\frac{\theta_{u,22222}[\delta](0,\tau)}{\theta_{u,222}[\delta](0,\tau)^2}\frac{\partial}{\partial u_2}\theta[\delta_i](0,\tau)}{2\frac{\partial}{\partial u_2}\theta[\delta_i](0,\tau)} =$$

$$-\frac{1}{2}\frac{\frac{\partial X[\delta](0,\tau)}{\partial z_2}\frac{\partial\theta[\delta_i](0,\tau)}{\partial z_1}-\frac{\partial X[\delta](0,\tau)}{\partial z_1}\frac{\partial\theta[\delta_i](0,\tau)}{\partial z_2}}{-\frac{\partial\theta[\delta](0,\tau)}{\partial z_2}\frac{\partial\theta[\delta_i](0,\tau)}{\partial z_1}+\frac{\partial\theta[\delta](0,\tau)}{\partial z_1}\frac{\partial\theta[\delta_i](0,\tau)}{\partial z_2}}-\frac{1}{20}\frac{\theta_{u,22222}[\delta](0,\tau)}{\theta_{u,222}[\delta](0,\tau)^2}$$

$$=\frac{J(X[\delta](0,\tau),\theta[\delta_i](0,\tau))}{2J(\theta[\delta](0,\tau),\theta[\delta_i](0,\tau))}-\frac{1}{20}\frac{\theta_{u,22222}[\delta](0,\tau)}{\theta_{u,222}[\delta](0,\tau)^2},$$

which is a modular function of weight 3 (automorphy factor $\psi(\gamma)\ell_\gamma(\tau)^6$) on $\Gamma_\delta\cap\Gamma_{\delta_i}$.

- This follows from the transformational properties of x
- Here J is jacobian matrix with respect to z_1, z_2.
- Will find an alternative expression for a_i.

Weierstrass points from Thetanullwerte

- $J(\theta[\delta](0,\tau),\theta[\delta_i](0,\tau))$ is given by Rosenhain's generalization of Jacobi's derivative formula. Let $\eta_i = \delta_i - \delta$. Then

$$J(\theta[\delta](0,\tau),\theta[\delta_i](0,\tau)) = \pm\pi^2 \prod_{j\neq i} \theta[\delta+\eta_i+\eta_j](0),$$

which vanishes only if $D(\tau)=0$. Likewise

$$J(\theta[\delta_i](0,\tau),\theta[\delta_j](0,\tau)) = \pm\pi^2\theta[\delta+\eta_i+\eta_j](0,\tau) \prod_{k,\ell\neq i,j} \theta[\delta+\eta_k+\eta_l](0).$$

One calculates for $i \neq j$, $\{1,2,3,4,5\} = \{i,j,k,l,m\}$, that $a_i - a_j =$

$$\frac{J(X[\delta](0,\tau),\theta[\delta_i](0,\tau))}{2J(\theta[\delta](0,\tau),\theta[\delta_i](0,\tau))} - \frac{J(X[\delta](0,\tau),\theta[\delta_j](0,\tau))}{2J(\theta[\delta](0,\tau),\theta[\delta_j](0,\tau))}$$

$$= \frac{J(X[\delta](0,\tau), \theta[\delta](0,\tau))J(\theta[\delta_i](0,\tau), \theta[\delta_j](0,\tau))}{2J(\theta[\delta](0,\tau), \theta[\delta_i](0,\tau))J(\theta[\delta](0,\tau), \theta[\delta_j](0,\tau))}$$

$$= \frac{\pm\pi^2\theta[\delta+\eta_i+\eta_j](0,\tau)\prod_{k,\ell\neq i,j}\theta[\delta+\eta_k+\eta_l](0)(2\pi^6 D(\tau))}{(\pm\pi^2\prod_{k\neq i}\theta[\delta+\eta_i+\eta_k](0))(\pm\pi^2\prod_{k\neq j}\theta[\delta+\eta_j+\eta_k](0))}$$

$$= \pm\pi^4\theta[\delta+\eta_k+\eta_\ell](0,\tau)^2\theta[\delta+\eta_\ell+\eta_m](0,\tau)^2\theta[\delta+\eta_k+\eta_m](0,\tau)^2.$$

- $c_0(\tau) = 0$ implies that $b_1 = 0$, (i.e., that $\sum_{i=1}^{5} a_i = 0$)

  Hence $a_i = \frac{1}{5} \sum_{j \neq i} a_i - a_j =$

$$= \frac{1}{10} \frac{J(X[\delta](0,\tau), \theta[\delta](0,\tau))}{J(\theta[\delta](0,\tau), \theta[\delta_i](0,\tau)))} \sum_{j \neq i} \frac{J(\theta[\delta_i](0,\tau), \theta[\delta_j](0,\tau))}{J(\theta[\delta](0,\tau), \theta[\delta_j](0,\tau))}$$

$$= \pi^4 \sum_{j \neq i} \pm \prod_{k,\ell \notin \{i,j\}} \theta[\delta + \eta_k + \eta_\ell](0,\tau)^2.$$

- This gives another way to use analytic functions to solve quintic equations!

Applications

- Easy proof of Thomae's Theorem in genus 2.

  $$(a_i-a_j)(a_k-a_\ell)(a_\ell-a_m)(a_m-a_k) = \frac{\pm 1}{\pi^8} 16 D(\tau)^4 \theta[\delta+\eta_i+\eta_j](0,\tau)^4,$$

  (for our model, $\det(\omega) = D(\tau)$.)

- Quick derivation of cross-ratios of branch points:

  $$\frac{a_i - a_k}{a_j - a_k} = \pm \frac{\theta[\delta + \eta_j + \eta_\ell](0,\tau)^2 \theta[\delta + \eta_j + \eta_m](0,\tau)^2}{\theta[\delta + \eta_i + \eta_\ell](0,\tau)^2 \theta[\delta + \eta_i + \eta_m](0,\tau)^2}.$$

Relationship to function theory on J

- Haven't needed sigma function yet!

- For $\gamma \in \Gamma_\delta$

$$\theta[\delta]({}^t(C\tau + D)^{-1}w, \gamma \circ \tau) = k(\gamma, \delta) j_\gamma(\tau)^{1/2} e^{\pi i q_\tau(w)} \theta[\delta](w, \tau),$$

$q_\tau(w)$ is a quadratic form whose coefficients depend on τ.

- Modify $\theta[\delta](w, \tau)$ by a trivial theta function so that the quadratic form appearing in the transformation formula vanishes.

$$\theta[\delta](w, \tau) = w_1 - w_2^3/12\pi^{12} D(\tau)^2 + w_1(a_{11}w_1^2 + 2a_{12}w_1w_2 + a_{22}w_2^2) + \ldots$$

- Let $q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$.

- DEFINE $\sigma[\delta](w,\tau) = e^{-{}^t wqw}\theta[\delta](w,\tau)$

- Expansion at the origin is just $w_1 - w_2^3/12\pi^{12}D(\tau)^2 + ...$

- Resulting transformation

$$\sigma[\delta]^\gamma({}^t(C\tau + D)^{-1}w, \gamma \circ \tau) = k(\gamma,\delta)j_\gamma(\tau)^{1/2}\sigma[\delta](w,\tau)$$

- Every coefficient in the expansion of σ in w_1 and w_2 is a modular function of half-integral weight on Γ_δ.

- Other advantage of σ over θ: if we define

$$X[\delta](w,\tau) = \mathrm{Det}_{1\leq i,j\leq 2}[\frac{\partial^2 \log \sigma[\delta](w,\tau)}{\partial w_i \partial w_j}],$$

then $\sigma[\delta](w,\tau)^3 X[\delta](w,\tau) = w_2 + ...$

as before, but now transforms like a Siegel-Jacobi form of weight 2, i.e., for any $\gamma \in \Gamma_\delta$,

$$X[\delta]({}^t(C\tau + D)^{-1}w, \gamma(\tau)) = \det(C\tau + D)^2 X[\delta](w,\tau).$$

Hyperelliptic $\wp$-functions

First let us multiply w_1 and w_2 by $2\pi^6 D(\tau)$ and divide $\sigma(w,\tau)$ by $2\pi^6 D(\tau)$ so that the expansion at the origin is just:

$$w_1 - w_2^3/3 + ...$$

For $i, j = 1, 2$, let $\wp_{ij} = -\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j} \log \sigma[\delta](w,\tau)$.

$$\sigma(w,\tau) = w_1 + ...$$

$$\sigma_1(w,\tau) = 1 + ...$$

$$\sigma_2(w,\tau) = -w_2^2 + ...$$

$$\sigma_{11}(w,\tau) = 0 + ...,\ \sigma_{12}(w,\tau) = 0 + ...$$

$$\sigma_{22}(w,\tau) = -2w_2 + ...$$

$$\sigma^2(w,\tau)\wp_{11}(w,\tau) = 1 + ...$$

$$\sigma^2(w,\tau)\wp_{12}(w,\tau) = -w_2^2 + ...$$

$$\sigma^2(w,\tau)\wp_{22}(w,\tau) = 2w_1w_2 + ...$$

- Hence $1, \wp_{11}, \wp_{12}, \wp_{22}$ are a basis for the 4-dimensional space $\mathcal{L}(2\Theta)$.

- Definition in terms of partial derivatives then shows that $\wp_{22}$ is the unique function $f \in L(2\Theta)$ up to affine transformation such that there exist $g, h \in L(2\Theta)$ such that $g/f|_{\Theta} = x^2$, $h/f|_{\Theta} = -x$, and up to affine transformation, the unique such g and h are $\wp_{11}$ and $\wp_{12}$.

Algebraic jacobian

- A is birational to the symmetric product $C^{(2)}$ so functions on A are symmetric functions in two independent generic points (x_1, y_1) , (x_2, y_2) on C.

- Basis for $L(2\Theta)$ is 1, $X_{22} = x_1 + x_2$, $X_{12} = -x_1 x_2$, $X_{11} =$
$$\frac{X_{22}X_{12}^2 + 2b_1 X_{12}^2 - b_2 X_{22} X_{12} - 2b_3 X_{12} + b_4 X_{22} + 2b_5 - 2y_1 y_2}{(x_1 - x_2)^2}.$$

- One can check that $X_{11}/X_{22}|_\Theta = x^2,\ X_{12}/X_{22}|_\Theta = -x$.

  So there exist constants α_{ij}, β_{ij} such that $\wp_{ij} = \alpha_{ij} X_{ij} + \beta_{ij}$ for $i, j = 1, 2$.

Finding the α_{ij}

The α_{ij} can be found by taking independent complex variables z, z' and looking at the lead terms in the expansions of both X_{ij} and $\wp_{ij}$ in terms of $s = z + z'$ and $p = zz'$ gotten by setting

$$(x_1, y_1) = (\rho(z), z),\ (x_2, y_2) = (\rho(z'), z'),\ w = (\rho(z)+\rho(z'), z+z').$$

For example, $\sigma(w) = 0$ if $z = 0, z' = 0$, or $z' = -z$. So the expansion of σ is divisible by p and s. On the other hand its lead term is the lead term of $\rho(z) + \rho(z') - (z + z')^3/3$ which is ps. So $\sigma(w)/ps$ is an invertible power series. Note that $\sigma_2(w)$ and $\sigma_{22}(s)$ are divisible each by s, and their lead terms are $-s^2$ and $-2s$, so the expansion of $\wp_{22}(w) = \frac{1}{p^2}(s^2 - 2p + ...)$. Likewise $X_{22} = \frac{1}{\rho(z)} + \frac{1}{\rho(z')} = \frac{1}{p^2}(s^2 - 2p + ...)$. Hence $\alpha_{22} = 1$. Similar calculations show that $\alpha_{11} = \alpha_{12} = 1$. Determining β_{ij} takes a little more work.

Finding the β_{ij}

Given the expansions we have, one can give Baker's proof of Baker's formula:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{11}(v) - \wp_{11}(u) + \wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u).$$

On the other hand, general theory gives that a function on $A \times A$ with the same divisor as either side of Baker's formula is

$$X_{11}(v) - X_{11}(u) + X_{12}(u)X_{22}(v) - X_{12}(v)X_{22}(u),$$

which shows that $\beta_{12} = \beta_{22} = 0$. It turns out that $\wp_{11}$ and X_{11} differ by a multiple of b_3. One can *redefine* σ, so that they coincide.

Zeta functions

- $\zeta_i(w) = \frac{\sigma_i(w)}{\sigma(w)}$, for $i = 1, 2$, $w \in \mathbb{C}^2$ are quasiperiodic functions, but do not restrict to functions on Θ.

- Rather, for $w \in \tilde{\Theta}$, $\xi_i(w) = \frac{\sigma_{ii}(w)}{\sigma_i(w)}$ are quasiperiodic

  (with twice the quasiperiods of ζ_i.)

- Hence their derivatives are functions on C.

- A currently messy calculation shows that $\frac{d}{dw_2}\xi_2(w) = -2x$.

- Like in genus 1, x is a derivative of a quasi-periodic function.

- Gives another way to invert the abelian integral in genus 2!