ADDITION FORMULAE OVER THE JACOBIAN PRE-IMAGE OF HYPERELLIPTIC WIRTINGER VARIETIES

J.C. EILBECK, V.Z. ENOLSKII, S. MATSUTANI, Y. ÔNISHI AND E. PREVIATO

Abstract. Using results on Frobenius-Stickelberger-type relations for hyperelliptic curves (Y. Ônishi, Proc. Edinb. Math. Soc. (2), **48** (2005) p.705-742), we provide certain addition formulae for any symmetric power of the curves, which hold on the strata W_k, the pre-images in the Jacobian of the classical Wirtinger varieties. As an appendix, we give similar relations for a trigonal curve $y^3 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)$.

1. Introduction

In this paper we investigate certain types of addition laws on the Jacobian variety of hyperelliptic curves.

For an elliptic curve C_1 given by $y^2 = 4x^3 - g_2x - g_3$, the addition law is determined by the addition formula

$$\frac{\sigma(u_1 + u_2)\sigma(u_1 - u_2)}{\sigma(u_1)^2\sigma(u_2)^2} = -\wp(u_1) + \wp(u_2), \tag{1.1}$$

where σ and $\wp$ are the usual functions of Weierstrass. If we apply $\frac{d}{du_1}(\frac{d}{du_1} + \frac{d}{du_2}) \log$ to both sides, we obtain addition formulae for the $\wp$-function, $\wp(u) = -\frac{d^2}{du^2} \log \sigma(u)$.

Since we can regard $(\wp(u), \frac{d\wp(u)}{du})$ as a point of the curve C_1, we write the relation (1.1) as

$$\frac{\sigma(u_1 + u_2)\sigma(u_1 - u_2)}{\sigma(u_1)^2\sigma(u_2)^2} = -x_1 + x_2, \quad \text{for } u_a = \int_\infty^{(x_a,y_a)} \frac{dx}{y}. \tag{1.2}$$

We shall consider (1.1) and (1.2) as coming from two different kinds of general formulae. Formula (1.1) was generalized to the case of higher-genus hyperelliptic curves by using Klein's sigma function, which is a natural generalization of the Weierstrass elliptic σ-function to higher genera, for the case $g = 2,\ 3$ by Baker [Ba2], and for general genus by Buchstaber *et al.* in [BEL].

In this article, we study addition formulae of type (1.2) over subvarieties in the Jacobian $\mathcal{J}_g$ of a hyperelliptic curve where the sigma function vanishes; derivatives of sigma will occur instead. The case $n = m = g = 1$ of Theorem 5.1 corresponds to (1.2).

For a hyperelliptic curve C_g given by the equation $y^2 = \prod_{i=1}^{2g+1}(x - b_i)$, where the b_is are distinct complex numbers, and a smooth point ∞ at infinity, the Jacobian $\mathcal{J}_g$ of C_g is a complex torus $\mathbb{C}^g/\Lambda$ with $\Lambda \subset \mathbb{C}^g$ a complete lattice. The Abel-Jacobi theorem says that we have a birational map ϕ_g, depending on the choice of a basepoint in C_g, from the symmetric product $\mathrm{S}^g(C_g)$ to the Jacobian $\mathcal{J}_g$. Henceforth we fix the basepoint to be ∞ and we transform the natural stratification of $\mathrm{S}^g(C_g)$ to $\mathcal{J}_g$. We consider the k-th symmetric product $\mathrm{S}^k(C_g)$ for $k \geqq 1$ and the analogous maps ϕ_k from $\mathrm{S}^k(C_g)$ to $\mathcal{J}_g$. We introduce subvarieties in $\mathcal{J}_g$, whose 2:1 images under the level-two theta map are classically known as the Wirtinger varieties [G],

$$W_k = \phi_k[\mathrm{S}^k(C_g)]$$

Then we see $W_k \subset W_{k+1}$ in general, and $W_n = \mathcal{J}_g$ if $n \geq g$. The natural projection is denoted by

$$\kappa : \mathbb{C}^g \to \mathcal{J}_g = \mathbb{C}^g/\Lambda.$$

For hyperelliptic curves, Riemann's singularity theorem (cf. [ACGH] VI.1) characterizes each W_k as the zero-locus of multi-derivatives of the Riemann theta function in the affine coordinate $u \in \mathbb{C}^g = \kappa^{-1}(\mathcal{J}_g)$. Indeed, our choice of a base point which is a Weierstrass point makes it possible to relate the dimension of a linear series with its degree. In general, it is still a difficult and important question to relate the Wirtinger varieties to vanishing properties of the theta function (cf. [G] for a survey).

By investigating the algebraic and analytic structure of the subvarieties W_k, some among the present authors [EMO] extended the formula (1.2) to the subvarieties W_k, based on the results in [O1], as follows.

Theorem 1.1. *Let (m, n) be a pair of positive integers (m, n) such that $m + n \leq g + 1$. Let $(x_1, y_1) \cdots , (x_m, y_m)$, $(x'_1, y'_1) \cdots , (x'_n, y'_n) \in C_g$, $u \in \kappa^{-1}(W_m)$, and $v \in \kappa^{-1}(W_n)$ be points satisfying $\kappa(u) = \phi_m((x_1, y_1) \cdots , (x_m, y_m))$ and $\kappa(v) = \phi_n((x'_1, y'_1), \cdots , (x'_n, y'_n))$. Then the following relation holds:*

$$\frac{\sigma_{\natural^{m+n}}(u - v)\sigma_{\natural^{m+n}}(u + v)}{\sigma_{\natural^m}(u)^2\sigma_{\natural^n}(v)^2} = \delta(g, n) \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i - x'_j),$$

Here $\sigma_{\natural^m}$ is a certain (higher) derivative of σ given in Table 1, *and* $\delta(g, n) = (-1)^{gn+\frac{1}{2}n(n-1)}$.

Note that we changed the notation of $\delta(g, n)$ from [EMO]. In this article, we generalize this theorem *for all pairs* of positive integers m and n by a more direct application of the results in [O1] than [EMO]. In an Appendix, we give similar relations for a trigonal curve $y^3 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)$ as an application of the generalized Frobenius-Stickelberger formulae for this curve([O2]). In essence, the strategy in [O1] and [O2] consists of comparing zero and pole-divisors of two meromorphic functions on the Jacobian, and finally conclude they are equal by determining the leading term(s) of their power-series expansions in suitable abelian coordinates. The present contribution consists in defining the appropriate derivatives of the sigma function, and exploiting previously found indentities to obtain a cancellation.

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2. Geometrical setting of hyperelliptic curves

Let us consider a hyperelliptic curve defined by

$$y^2 = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_0$$

together with a smooth point ∞ at infinity.

We fix a basis of holomorphic one-forms

$$du_1 = \frac{dx}{2y}, \quad du_2 = \frac{xdx}{2y}, \quad \cdots, \quad du_g = \frac{x^{g-1}dx}{2y}.$$

We also fix a homology basis for the curve X so that

$$\mathrm{H}_1(X, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j,$$

where the intersections are given by $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = -[\beta_i, \alpha_j]\delta_{ij}$. We take the half-period matrices of X with respect to the given bases,

$$\omega' = \frac{1}{2}\left[\int_{\alpha_j} du_i\right], \quad \omega'' = \frac{1}{2}\left[\int_{\beta_j} du_i\right], \quad \omega = \begin{bmatrix}\omega' \\ \omega''\end{bmatrix}.$$

Let Λ be the lattice in $\mathbb{C}^g$ generated by the column vectors in $2\omega'$ and $2\omega''$. The Jacobian variety of X is denoted by $\mathcal{J}_g$ and is identified with $\mathbb{C}^g/\Lambda$. We denote by κ the map given by modulo Λ:

$$\kappa : \mathbb{C}^g \to \mathbb{C}^g/\Lambda.$$

For a non-negative integer k, we define the Abel map from k-th symmetric product $\mathrm{S}^k(X)$ of the curve X to $\mathcal{J}_g$ by,

$$\phi_k : \mathrm{S}^k(X) \to \mathcal{J}_g, \quad \phi_k((x_1, y_1), \cdots, (x_k, y_k)) = \sum_{i=1}^{k} \int_{\infty}^{(x_i,y_i)} \begin{pmatrix} du_1 \\ \vdots \\ du_g \end{pmatrix} \bmod \Lambda.$$

The image of ϕ_k is denoted by $W_k = \phi_k(\mathrm{S}^k(X_g))$. The mapping ϕ_g is surjective by Abel's theorem, and is injective if we restrict the map to the pre-image of the complement of a specific connected Zariski closed subset of dimension at most $g - 2$ in $\mathcal{J}_g$, by Jacobi's theorem (see Theorem 5.13 in [I]).

3. Sigma function and its derivatives

In this section, we will introduce the hyperelliptic θ-functions, and the σ-function, which is a natural generalization of the Weierstrass σ-function.

We define differentials of the second kind,

$$dr_j = \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k dx, \quad (j=1,\cdots,g)$$

and complete hyperelliptic integrals of the second kind

$$\eta' = \frac{1}{2}\left[\int_{\alpha_j} dr_i\right], \quad \eta'' = \frac{1}{2}\left[\int_{\beta_j} dr_i\right].$$

For this basis of the $2g$-dimensional space of meromorphic differentials, the half-periods $\omega', \omega'', \eta', \eta''$ satisfy the generalized Legendre relation

$$\mathfrak{M}\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}\mathfrak{M}^T = \frac{\imath\pi}{2}\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}. \tag{3.1}$$

where $\mathfrak{M} = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix}$. Let $\mathbb{T} = \omega'^{-1}\omega''$. The theta function on $\mathbb{C}^g$ with modulus $\mathbb{T}$ and characteristics $\mathbb{T}a + b$ is given by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\mathbb{T}) = \sum_{n\in\mathbb{Z}^g}\exp\left[2\pi i\left\{\frac{1}{2}\,{}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b)\right\}\right],$$

for g-dimensional complex vectors a and b. The σ-function ([Ba1], p.336,[BEL]), an analytic function on the space $\mathbb{C}^g$ and a theta series having modular invariance of a given weight with respect to $\mathfrak{M}$, is given by the formula

$$\sigma(u) = \gamma_0\exp\left\{-\frac{1}{2}\,{}^t u\eta'\omega'^{-1}u\right\}\vartheta\begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}(\frac{1}{2}\omega'^{-1}u;\mathbb{T}),$$

where δ and δ' are half-integer characteristics giving the vector of Riemann constants with basepoint at ∞ and γ_0 is a certain non-zero constant. The σ-function vanishes only on $\kappa^{-1}(W_{g-1})$ (see for example [Ba1], p. 252).

Let $\{\varphi_i\}$ be an ordered set of $\mathbb{C}\cup\{\infty\}$-valued functions over X defined as follows

$$\varphi_i = \begin{cases} x^i & \text{for } i \le g, \\ x^{\lfloor (i-g)/2\rfloor + g} & \text{for } i > g, i-g \text{ even}, \\ x^{\lfloor (i-g)/2\rfloor}y & \text{for } i > g, i-g \text{ odd}. \end{cases} \tag{3.2}$$

Following [O1], we introduce a multi-index $\natural^n$. For n with $1 \le n < g$, we let $\natural^n$ be the set of positive integers i such that $n+1 \le i \le g$ with $i \equiv n+1 \bmod 2$. Namely,

$$\natural^n = \begin{cases} \{n+1, n+3, \cdots, g-1\} & \text{if } g-n \equiv 0 \mod 2, \\ \{n, n+2, \cdots, g\} & \text{if } g-n \equiv 1 \mod 2; \end{cases}$$

and partial derivative over the multi-index $\natural^n$

$$\sigma_{\natural^n} = \left(\prod_{i\in\natural^n} \frac{\partial}{\partial u_i}\right)\sigma(u).$$

For $n \geq g$, we define $\natural^n$ as empty and $\sigma_{\natural^n}$ as σ itself. The first few examples are given in the following table, where we let $\sharp$ denote $\natural^1$ and $\flat$ denote $\natural^2$.

For $u \in \mathbb{C}^g$, we denote by u' and u'' the unique vectors in $\mathbb{R}^g$ such that

$$u = u'2\omega' + u''2\omega''.$$

We define

$$\begin{aligned} L(u, v) &= {}^t u(2\eta' v' + 2\eta'' v''), \\ \chi(\ell) &= \exp\left\{2\pi\imath\left({}^t\ell'\delta'' - {}^t\ell''\delta' + \tfrac{1}{2}{}^t\ell'\ell''\right)\right\} \ (\in \{1, -1\}) \end{aligned}$$

for u, $v \in \mathbb{C}^g$ and for ℓ $(= \ell'2\omega' + \ell''2\omega'') \in \Lambda$. Then $\sigma_{\natural^n}(u)$ for $u \in \kappa^{-1}(W_1)$ satisfies the translational relation ([O1], Lemma 6.3):

$$\sigma_{\natural^n}(u + \ell) = \chi(\ell)\sigma_{\natural^n}(u) \exp L(u + \tfrac{1}{2}\ell, \ell) \quad \text{for } u \in \kappa^{-1}(W_1). \tag{3.3}$$

Further for $n \leqq g$, we note that

$$\begin{aligned} \sigma_{\natural^n}(-u) &= (-1)^{ng+\frac{1}{2}n(n-1)}\sigma_{\natural^n}(u) \quad \text{for } u \in \kappa^{-1}(W_n), \text{ especially,} \\ \sigma_\flat(-u) &= -\sigma_\flat(u) \quad \text{for } u \in \kappa^{-1}(W_2) \\ \sigma_\sharp(-u) &= (-1)^g\sigma_\sharp(u) \quad \text{for } u \in \kappa^{-1}(W_1) \end{aligned} \tag{3.4}$$

by Proposition 6.5 in [O1].

Table 1

genus	$\sigma_\sharp \equiv \sigma_{\natural^1}$	$\sigma_\flat \equiv \sigma_{\natural^2}$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$	$\cdots$
1	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
2	σ_2	σ	σ	σ	σ	σ	σ	σ	$\cdots$
3	σ_2	σ_3	σ	σ	σ	σ	σ	σ	$\cdots$
4	σ_{24}	σ_3	σ_4	σ	σ	σ	σ	σ	$\cdots$
5	σ_{24}	σ_{35}	σ_4	σ_5	σ	σ	σ	σ	$\cdots$
6	σ_{246}	σ_{35}	σ_{46}	σ_5	σ_6	σ	σ	σ	$\cdots$
7	σ_{246}	σ_{357}	σ_{46}	σ_{57}	σ_6	σ_7	σ	σ	$\cdots$
8	σ_{2468}	σ_{357}	σ_{468}	σ_{57}	σ_{68}	σ_7	σ_8	σ	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

4. Generalized Frobenius-Stickelberger formula

We recall the generalized Frobenius-Stickelberger formula, which was given by one of the authors ([O1], Theorem 7.2).

Definition 4.1. For a positive integer $n \geq 1$ and a point $(x_1, y_1) \cdots, (x_n, y_n)$ in X, we define

$$\Delta_n((x_1, y_1), \cdots, (x_n, y_n))$$
$$= \begin{vmatrix} 1 & \varphi_1(x_1, y_1) & \varphi_2(x_1, y_1) & \cdots & \varphi_{n-2}(x_1, y_1) & \varphi_{n-1}(x_1, y_1) \\ 1 & \varphi_1(x_2, y_2) & \varphi_2(x_2, y_2) & \cdots & \varphi_{n-2}(x_2, y_2) & \varphi_{n-1}(x_2, y_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \varphi_1(x_{n-1}, y_{n-1}) & \varphi_2(x_{n-1}, y_{n-1}) & \cdots & \varphi_{n-2}(x_{n-1}, y_{n-1}) & \varphi_{n-1}(x_{n-1}, y_{n-1}) \\ 1 & \varphi_1(x_n, y_n) & \varphi_2(x_n, y_n) & \cdots & \varphi_{n-2}(x_n, y_n) & \varphi_{n-1}(x_n, y_n) \end{vmatrix}$$

Proposition 4.2. *For a positive integer $n > 1$, let $(x_1, y_1), \cdots, (x_n, y_n)$ in X, and $u^{(1)}, \cdots, u^{(n)}$ in $\kappa^{-1}(W_1)$ be points such that $\kappa(u^{(i)}) = \phi_1((x_i, y_i))$. Then the following relation holds*:

$$\frac{\sigma_{\natural^n}(\sum_{i=1}^n u^{(i)}) \prod_{i<j} \sigma_\flat(u^{(i)} - u^{(j)})}{\prod_{i=1}^n \sigma_\sharp(u^{(i)})^n} = \epsilon_n \Delta_n((x_1, y_1), \cdots, (x_n, y_n)), \tag{4.1}$$

where $\epsilon_n = (-1)^{g+n(n+1)/2}$ for $n \leq g$ and $\epsilon_n = (-1)^{(2n-g)(g-1)/2}$ for $n \geq g+1$.

5. The generalized addition formula

Now we are ready to describe the main result. Our motivation is found in [EEP] and [BES]. A special case of our main result appeared in [EEP] (see formula (34) and the analogous formula above (32)) in the process of deriving Baker's addition formula by the technique given in [O1] for genus 2. Another special case is the formula (3.21) in [BES].

Theorem 5.1. *Assume that (m, n) is a pair of positive integers. Let (x_i, y_i) $(i = 1, \cdots, m)$, (x'_j, y'_j) $(j = 1, \cdots, n)$ in X and $u \in \kappa^{-1}(W_m)$, $v \in \kappa^{-1}(W_n)$ be points such that $\kappa(u) = \phi_m((x_1, y_1), \cdots, (x_m, y_m))$ and $\kappa(v) = \phi_n((x'_1, y'_1), \cdots, (x'_n, y'_n))$. Then the following relation holds*:

$$\begin{aligned} &\frac{\sigma_{\natural^{m+n}}(u+v)\sigma_{\natural^{m+n}}(u-v)}{\sigma_{\natural^m}(u)^2\sigma_{\natural^n}(v)^2} \\ &= \delta(g,m,n)\frac{\prod_{i=0}^1 \Delta_{m+n}((x_1, y_1), \cdots, (x_m, y_m), (x'_1, (-1)^i y'_1), \cdots, (x'_n, (-1)^i y'_n))}{(\Delta_m((x_1, y_1) \cdots, (x_m, y_m))\Delta_n((x'_1, y'_1) \cdots, (x'_n, y'_n)))^2} \\ &\times \prod_{i=1}^m \prod_{j=1}^n \frac{1}{\Delta_2((x_i, y_i), (x'_j, y'_j))} \end{aligned} \tag{5.1}$$

where $\delta(g, m, n) = (-1)^{gn + \frac{1}{2}n(n-1) + mn}$.

To show this, we introduce the following symbol:

Definition 5.2. Let k be a positive integer. In general, for any k points $u^{(j)}$ $(j = 1, \cdots, k)$ in $\kappa^{-1}(W_1)$, we define

$$q(u^{(1)}, \cdots, u^{(k)}) = \frac{\sigma_{\natural^k}(u^{(1)} + \cdots + u^{(k)})}{\prod_{j=1}^{k} \sigma_\sharp(u^{(j)})}.$$

Remark 5.3. By the translational relation (3.3), we see that $q(u^{(1)}, \cdots, u^{(k)})^2$ is a function on $\mathrm{S}^k(X)$. However, $q(u^{(1)}, \cdots, u^{(k)})$ is in general only a function on the fibre product taken over $\mathcal{J}_g$ with respect to the Abel map $\psi_k : \mathrm{S}^k(X) \to \mathcal{J}_g$ and the duplication map $[2] : \widetilde{\mathcal{J}}_g \to \mathcal{J}_g$, where $\widetilde{\mathcal{J}}_g$ is an Abelian variety which has $\mathcal{J}_g$ as a 2:1 image.

Proof of Theorem 5.1. In the course of this proof, we let $u^{(i)}$ and $v^{(j)}$ be points such that $\kappa(u^{(i)}) = \phi_1((x_i, y_i))$ and $\kappa(u^{(j)}) = \phi_1((x'_j, y'_j))$ in W_1, respectively. Then we have $\kappa(-u^{(i)}) = \phi_1((x_i, -y_i))$ and $\kappa(-v^{(j)}) = \phi_1((x'_j, -y'_j))$. By (4.1) and (3.4), we have

$$\begin{aligned}
&q(u^{(1)}, \cdots, u^{(m)}, \pm v^{(1)}, \cdots, \pm v^{(n)}) \\
&\qquad \cdot \prod_{i_1<i_2} q(u^{(i_1)}, \pm u^{(i_2)}) \prod_{i,j} q(u^{(i)}, \pm v^{(j)}) \prod_{j_1<j_2} q(\pm v^{(j_1)}, v^{(j_2)}) \\
&= \frac{\sigma_{\natural^{m+n}}(u^{(1)} + \cdots + u^{(m)} \pm v^{(1)} \pm \cdots \pm v^{(n)})}{\prod_{i=1}^{m} \sigma_\sharp(u^{(i)}) \prod_{j=1}^{n} \sigma_\sharp(\pm v^{(j)})} \\
&\qquad \cdot \prod_{i_1<i_2} \frac{\sigma_\flat(u^{(i_1)} - u^{(i_2)})}{\sigma_\sharp(u^{(i_1)})\sigma_\sharp(-u^{(i_2)})} \cdot \prod_{i,j} \frac{\sigma_\flat(u^{(i)} \mp v^{(j)})}{\sigma_\sharp(u^{(i)})\sigma_\sharp(\mp v^{(j)})} \cdot \prod_{j_1<j_2} \frac{\sigma_\flat(\pm v^{(j_1)} \mp v^{(j_2)})}{\sigma_\sharp(u^{(j_1)})\sigma_\sharp(\mp v^{(j_2)})} \\
&= (-1)^{g(\frac{1}{2}m(m-1)+mn+\frac{1}{2}n(n-1))}\epsilon_{m+n} \\
&\qquad \cdot \Delta_{m+n}((x_1, y_1), \cdots, (x_m, y_m), (x'_1, \pm y'_1), \cdots, (x'_n, \pm y'_n)),
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
q(u^{(1)}, \cdots, u^{(m)}) \prod_{i_1<i_2} q(u^{(i_1)}, -u^{(i_2)}) &= (-1)^{\frac{1}{2}gm(m-1)}\epsilon_m \Delta_m((x_1, y_1), \cdots, (x_m, y_m)), \\
q(v^{(1)}, \cdots, v^{(n)}) \prod_{j_1<j_2} q(v^{(j_1)}, -v^{(j_2)}) &= (-1)^{\frac{1}{2}gn(n-1)}\epsilon_n \Delta_n((x'_1, y'_1), \cdots, (x'_n, y'_n)).
\end{aligned} \tag{5.3}$$

We regard both sides of (5.1) as functions on $\mathrm{S}^m(X) \times \mathrm{S}^n(X)$. By (5.2) and (5.3), the left hand side of (5.1) is equal to

$$(-1)^{gn} \frac{q(u^{(1)}, \cdots, u^{(m)}, v^{(1)}, \cdots, v^{(n)})\, q(u^{(1)}, \cdots, u^{(m)}, -v^{(1)}, \cdots, -v^{(n)})}{q(u^{(1)}, \cdots, u^{(m)})^2\, q(v^{(1)}, \cdots, v^{(n)})^2}. \tag{5.4}$$

Using (5.2) and (5.3), we see that (5.4) is equal to

$$(-1)^{gn} \frac{\prod_\pm \Delta_{m+n}((x_1, y_1), \cdots, (x_m, y_m), (x'_1, \pm y'_1), \cdots, (x'_n, \pm y'_n))}{\Delta_n((x_1, y_1), \cdots, (x_m, y_m))^2 \Delta_m((x'_1, y'_1), \cdots, (x'_n, y'_n))^2} \times Q,$$

where

$$Q = \frac{\prod_{i<j}^{m} q(u^{(i)}, -u^{(j)})^2 \prod_{i<j}^{n} q(v^{(i)}, -v^{(j)})^2}{\prod_{\pm} \left[\prod_{i<j}^{m} q(u^{(i)}, -u^{(j)}) \prod_{i<j}^{n} q(\pm v^{(i)}, \mp v^{(j)}) \prod_{i=1}^{m} \prod_{j=1}^{n} q(u^{(i)}, \mp v^{(j)}) \right]}.$$

Since

$$q(-u, v) = \frac{\sigma_\flat(-u+v)}{\sigma_\sharp(-u)\sigma_\sharp(v)} = \frac{-\sigma_\flat(u-v)}{\sigma_\sharp(u)\sigma_\sharp(-v)} = -q(u, -v) \tag{5.5}$$

for u, v in $\kappa^{-1}(W_1)$, we have

$$\begin{aligned} Q &= (-1)^{n(n-1)/2} \frac{1}{\prod_{\pm} \left[\prod_{i=1}^{m} \prod_{j=1}^{n} q(u^{(i)}, \mp v^{(j)}) \right]} \\ &= (-1)^{n(n-1)/2} \frac{1}{\prod_{i=1}^{m} \prod_{j=1}^{n} q(u^{(i)}, v^{(j)})\, q(u^{(i)}, -v^{(j)})}. \end{aligned}$$

Using (3.4) and (4.1) again, we see

$$\begin{aligned} q(u, v) q(u, -v) &= \frac{\sigma_\flat(u+v)\sigma_\flat(u-v)}{\sigma_\sharp(u)^2 \sigma_\sharp(v) \sigma_\sharp(-v)} \\ &= (-1)^g \frac{\sigma_\flat(u+v)\sigma_\flat(u-v)}{\sigma_\sharp(u)^2 \sigma_\sharp(v)^2} \\ &= -\Delta_2((x, y), (x', y')) \end{aligned} \tag{5.6}$$

for u, v in $\kappa^{-1}(W_1)$ and (x, y), (x', y') in X such that $\kappa(u) = \phi_1((x, y))$ and $\kappa(v) = \phi_1((x', y'))$. Hence we obtain

$$Q = (-1)^{n(n-1)/2} (-1)^{mn} \frac{1}{\prod_{i=1}^{m} \prod_{j=1}^{n} \Delta_2((x_i, y_i), (x'_j, y'_j))}$$

and have proved the assertion. □

Note that Theorem 1.1 follows as a special case, since $\Delta_2((x, y), (x', y')) = -(x - x')$.

Remark 5.4. We note that definition 5.2 for the q's is behind the Frobenius-Stickelberger relations. This is a relation that governs the automorphy factor of the σ function. In turn, such automorphy factor defines the line bundle corresponding to the point of the Jacobian, cf. [F], (5). The Jacobian appears in the long exact sequence of cohomology for

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1,$$

so the behavior of the automorphy factor derives from the exact sequence,

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1.$$

Indeed, if $f : \mathbb{C} \to \mathbb{C}^*$ is a group homomorphism,

$$\frac{f(u^{(1)} + u^{(2)} + \cdots + u^{(n)})}{\prod_{i=1}^{n} f(u^{(i)})} = 1.$$

Appendix : The case of purely trigonal genus 3 curves

Our technique can be generalized for the recent result on the Frobenius-Stickelberger relation for the curve whose affine part is given by [O2],

$$y^3 = x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

where λ_js are complex numbers. We denote the curve defined by this equation with a unique point ∞ at infinity as X. In this appendix, we will give an analog for X of Theorem 5.1.

Geometrical setting. We take a basis of the holomorphic one-forms:

$$du_1 = \frac{dx}{3y^2}, \quad du_2 = \frac{xdx}{3y^2}, \quad du_3 = \frac{dx}{3y}.$$

We use the same symbol as in the case of hyperelliptic curves, hoping that this causes no confusion. Then, the rest of the notation for the period matrices, the coordinate space $\mathbb{C}^3$ associated to the forms above, the lattice Λ in the coordinate space, the Jacobian variety $\mathcal{J}_3 = \mathbb{C}^3/\Lambda$, the Abel maps ϕ_k and the varieties W_k, is the same as in section 2 above. The relevant modulo-Λ map is $\kappa : \mathbb{C}^3 \to \mathcal{J}_3$. The Abel-Jacobi theorem ensures that $\kappa^{-1}(W_k)$ fills $\mathbb{C}^3$ for $k \geq 3$ and ϕ_3 is a birational map from $\mathrm{S}^3(X)$ to $\mathcal{J}_3$. We have $W_1 \subset W_2 \subset W_3 = W_4 = \cdots = \mathcal{J}_3$. We should be careful to note that W_k for $k = 1$ and 2 is not stable under the operation of taking the additive inverse, $[-1]{:}(u_1, u_2, u_3) \mapsto (-u_1, -u_2, -u_3)$ in $\mathcal{J}_3$.

Let $\{\varphi_i\}$ be the sequence of $\mathbb{C} \cup \{\infty\}$-valued functions over X defined by

$$\varphi_i = \begin{cases} x & \text{for } i = 3 \\ y & \text{for } i = 4 \\ x^{(i-3)/3+2} & \text{for } i > 3, i \equiv 0 \text{ modulo } 3, \\ x^{(i-3)/3+1}y & \text{for } i > 3, i \equiv 1 \text{ modulo } 3, \\ x^{(i-3)/3}y^2 & \text{for } i > 3, i \equiv 2 \text{ modulo } 3. \end{cases} \tag{A.1}$$

We need again a sigma function $\sigma(u) = \sigma(u_1, u_2, u_3)$ for X; this is a theta function on the space $\mathbb{C}^3$, only the quadratic form entering its definition is not the same as Riemann's. We omit this definition and the details, referring the reader to [O2]. Following [O2], we define (partial) derivatives over the multi-index $\natural^n$:

$$\sigma_{\natural^n}(u) = \begin{cases} \frac{\partial^2}{\partial u_3^2}\sigma(u) & \text{for } n = 1, \\ \frac{\partial}{\partial u_3}\sigma(u) & \text{for } n = 2, \\ \sigma(u) & \text{for } n > 2. \end{cases}$$

Let ζ be a primitive cube root of 1. The automorphism of the curve, $(x, y) \mapsto (x, \zeta y)$, induces an action $[\zeta]$ on $\kappa^{-1}(W_n)$, namely, for

$$u = (u_1, u_2, u_3) = \left(\int_\infty^{(x_1,y_1)} + \int_\infty^{(x_2,y_2)} + \int_\infty^{(x_3,y_3)} \right) \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix},$$

we let

$$[\zeta]u = (\zeta u_1, \zeta u_2, \zeta^2 u_3) = \left(\int_\infty^{(x_1,\zeta y_1)} + \int_\infty^{(x_2,\zeta y_2)} + \int_\infty^{(x_3,\zeta y_3)} \right) \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}.$$

Then $[\zeta]^2$ is induced by multiplication of the coordinates by ζ^2 and $[\zeta]^3$ is the identity. Moreover, we have

$$\sigma([\zeta]u) = \zeta\sigma(u), \quad \sigma_{\natural^2}([\zeta]u) = \sigma_{\natural^2}(u), \quad \sigma_{\natural^1}([\zeta]u) = \zeta^2\sigma_{\natural^1}(u). \tag{A.2}$$

Generalized Frobenius-Stickelberger formula. The following generalized Frobenius-Stickelberger formula was given by one of the authors([O2], Theorem 4.3).

Definition A.1. *For a positive integer $n \geq 1$ and a point $(x_1, y_1) \cdots, (x_n, y_n)$ in X, we define*

$$\begin{aligned}
&\Delta_n((x_1, y_1), \cdots, (x_n, y_n)) \\
&= \begin{vmatrix}
1 & \varphi_1(x_1, y_1) & \varphi_2(x_1, y_1) & \cdots & \varphi_{n-2}(x_1, y_1) & \varphi_{n-1}(x_1, y_1) \\
1 & \varphi_1(x_2, y_2) & \varphi_2(x_2, y_2) & \cdots & \varphi_{n-2}(x_2, y_2) & \varphi_{n-1}(x_2, y_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \varphi_1(x_{n-1}, y_{n-1}) & \varphi_2(x_{n-1}, y_{n-1}) & \cdots & \varphi_{n-2}(x_{n-1}, y_{n-1}) & \varphi_{n-1}(x_{n-1}, y_{n-1}) \\
1 & \varphi_1(x_n, y_n) & \varphi_2(x_n, y_n) & \cdots & \varphi_{n-2}(x_n, y_n) & \varphi_{n-1}(x_n, y_n)
\end{vmatrix} \\
&\cdot \begin{vmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-2} & x_{n-1}^{n-1} \\
1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1}
\end{vmatrix}.
\end{aligned}$$

Proposition A.2. *For a positive integer $n > 1$, let $(x_1, y_1), \cdots, (x_n, y_n)$ in X and $u^{(1)}$, $\cdots$, $u^{(n)}$ in $\kappa^{-1}(W_1)$ be points such that $\kappa(u^{(i)}) = \phi_1((x_i, y_i))$. Then the following relation holds:*

$$\frac{\sigma_{\natural^n}(\sum_{i=1}^n u^{(i)}) \prod_{i<j} \sigma_{\natural^2}(u^{(i)} + [\zeta]u^{(j)})\sigma_{\natural^2}(u^{(i)} + [\zeta^2]u^{(j)})}{\prod_{i=1}^n \sigma_{\natural^1}(u^{(i)})^{2n-1}} = \Delta_n((x_1, y_1), \cdots, (x_n, y_n)).$$

The generalized addition formulae. For our trigonal curve X, the following generalized addition formulae hold over W_k.

Theorem A.3. *Assume that (m, n) is a pair of positive integers $(n, m \geq 1)$. Let (x_i, y_i) $(i = 1, \cdots, m)$, (x'_j, y'_j) $(j = 1, \cdots, n)$ be in X and $u \in \kappa^{-1}(W_m)$, $v \in \kappa^{-1}(W_n)$ be points such that $\kappa(u) = \phi_m((x_1, y_1), \cdots, (x_m, y_m))$, $\kappa(v) = \phi_n((x'_1, y'_1), \cdots, (x'_n, y'_n))$, Then the* *following relation holds*:

$$
\begin{aligned}
&\frac{\sigma_{\natural^{m+n}}(u+v)\,\sigma_{\natural^{m+n}}(u+[\zeta]v)\,\sigma_{\natural^{m+n}}(u+[\zeta^2]v)}{\sigma_{\natural^m}(u)^3\,\sigma_{\natural^n}(v)^3}\\
&= \frac{\prod_{i=0}^{2}\Delta_{m+n}((x_1,y_1),\cdots,(x_m,y_m),(x_1',\zeta^i y_1'),\cdots,(x_n',\zeta^i y_n'))}{\big(\Delta_m((x_1,y_1)\cdots,(x_m,y_m))\Delta_n((x_1',y_1')\cdots,(x_n',y_n'))\big)^3}\\
&\times\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{\Delta_2((x_i,y_i),(x_j',y_j'))^2}.
\end{aligned}
\tag{A.3}
$$

Proof. This is proved by the same way as Theorem 5.1. □

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