Finite Element Approximation for electro-rheological fluids

Dominic Breit
Luigi C. Berselli & Lars Diening

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Stationary Navier-Stokes equations

Find a velocity field $\mathbf{v} : \Omega \to \mathbb{R}^d$ and a pressure function $\pi : \Omega \to \mathbb{R}$ satisfying the following

system of partial differential equations

$$\left\{ \begin{aligned} -(\nabla \mathbf{v})\mathbf{v} + \operatorname{div} \mathbf{S} &= \nabla \pi - \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega, \\ \mathbf{v} &= 0 && \text{on } \partial\Omega. \end{aligned} \right. \tag{0.1}$$

- Ω denotes a domain in $\mathbb{R}^d$, $d \in \{2, 3\}$;
- $\mathbf{f} : \Omega \to \mathbb{R}^d$ is a system of volume forces;
- $\mathbf{S} : \Omega \to \mathbb{R}^{d \times d}$ is the stress deviator.

Electro-rheological fluids (1)

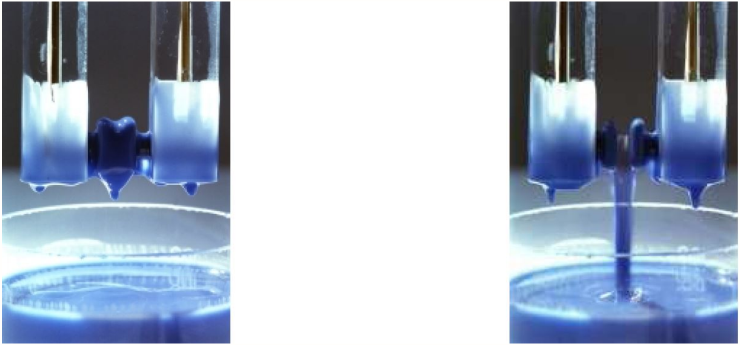

Abbildung: ERF with and without electric tension

Electro-rheological fluids (2)

Smart materials with change of material properties as reaction to electric field. Increase of viscosity times 1000 in 1ms.

Constitutive law (Rajagopal–Růžička, '96)

$$\mathbf{S} = \mathbf{S}(x, \varepsilon(\mathbf{v})) \approx |\varepsilon(\mathbf{v})|^{p(x)-2}\varepsilon(\mathbf{v}). \tag{0.2}$$

- Symmetric gradient $\varepsilon(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$;
- Applications: construction of clutches and shock absorbers;
- Equation of motion for slow flows:

$$\operatorname{div}\left(|\varepsilon(\mathbf{v})|^{p(\cdot)-2}\varepsilon(\mathbf{v})\right) = \nabla\pi - \mathbf{f}.$$

Finite Element spaces

Let $p : \Omega \to (1, \infty)$ be continuous

$$L^{p(\cdot)}(\Omega) := \Big\{ u \in L^1(\Omega) : \int_\Omega |u(x)|^{p(x)} \,\mathrm{d}x < \infty \Big\},$$
$$W^{1,p(\cdot)}(\Omega) := \Big\{ u \in W^{1,1}(\Omega) : \; u \in L^{p(\cdot)}(\Omega), \; \nabla u \in L^{p(\cdot)}(\Omega) \Big\}.$$

- Unique weak solution $(\mathbf{v}, \pi) \in W^{1,p(\cdot)}_{0,\mathrm{div}}(\Omega) \times L^{p'(\cdot)}_0(\Omega)$ to (0.1);
- Discrete spaces for FEM:

$$V_h := \mathcal{P}_2 \cap W^{1,p(\cdot)}_0(\Omega), \quad Q_h := \mathcal{P}_0 \cap L^{p'(\cdot)}_0(\Omega),$$
$$V_{h,\mathrm{div}} := \{ \mathbf{w}_h \in V_h \; : \; \langle \mathrm{div}\, \mathbf{w}_h, \eta_h \rangle = 0 \; \forall \, \eta_h \in Q_h \}.$$

Finite Element Methods

Theorem (Berselli, Breit, Diening), $p \in C^{0,\alpha}(\overline{\Omega})$, $\mathbf{F}(x,\xi) = |\xi|^{\frac{p(x)-2}{2}}$

$$\|\mathbf{F}(\cdot,\nabla \mathbf{v}) - \mathbf{F}(\cdot,\nabla \mathbf{v}_h)\|_2 \le c\Big(h^{\min\{1,\frac{(p^+)'}{2}\}} + h^\alpha\Big),$$

$$\|\pi - \pi_h\|_{p'(\cdot)} \le c\Big(h^{\frac{\min\{((p^+)')^2,4\}}{2(p^-)'}} + h^\alpha\Big).$$

- FEM solution $\mathbf{v}_h \in V_{h,\text{div}}$ such that for all $\varphi_h \in V_{h,\text{div}}$

$$\int_\Omega |\varepsilon(\mathbf{v}_h)|^{p(x)-2}\varepsilon(\mathbf{v}_h) : \varepsilon(\varphi_h)\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\varphi_h\,\mathrm{d}x\,.$$

- Discrete pressure $\pi_h \in Q_h$ such that for all $\varphi_h \in V_h$

$$\int_\Omega |\varepsilon(\mathbf{v}_h)|^{p(x)-2}\varepsilon(\mathbf{v}_h) : \varepsilon(\varphi_h)\,\mathrm{d}x - \int_\Omega \operatorname{div}\varphi_h\,\pi_h\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\varphi_h\,\mathrm{d}x\,.$$