Existence theory for generalized Newtonian fluids

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Outline

Navier-Stokes equations

Find a velocity field $\mathbf{v} : Q \to \mathbb{R}^d$ and a pressure function $\pi : Q \to \mathbb{R}$ satisfying the following

system of partial differential equations

$$\begin{cases} -\partial_t \mathbf{v} + \operatorname{div} \mathbf{S} = (\nabla \mathbf{v})\mathbf{v} + \nabla \pi - \mathbf{f} & \text{in} \quad Q, \\ \operatorname{div} \mathbf{v} = 0 & \text{in} \quad Q, \\ \mathbf{v} = 0 & \text{on} \quad \partial G, \\ \mathbf{v}(0, \cdot) = \mathbf{v}_0 & \text{in} \quad G, \end{cases}$$

- $Q := (0, T) \times G$ with $G \subset \mathbb{R}^d$, $d \in \{2, 3\}$ and $T > 0$;
- $\mathbf{f} : Q \to \mathbb{R}^d$ is a system of volume forces;
- $\mathbf{S} : Q \to \mathbb{R}^{d \times d}$ is the stress deviator.

Constitutive law

In order to characterize the specific fluid under consideration we need a constitutive law, which relates $\boldsymbol{\sigma}$ and the symmetric gradient

$$\varepsilon(\mathbf{v}) := \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right).$$

- Newtonian fluid: $\mathbf{S} = \nu\varepsilon(\mathbf{v})$ (water, air and the most oils);
- Generalized Newtonian fluid: $\mathbf{S} = \nu(|\varepsilon(\mathbf{v})|)\varepsilon(\mathbf{v})$;
- the viscosity ν is a function of the shear rate $|\varepsilon(\mathbf{v})|$;
- ν increasing $\Rightarrow$ shear thickening[1] (batter);
- ν decreasing $\Rightarrow$ shear thinning[2] (blood, ketchup).

[1] https://www.youtube.com/watch?v=uQrqGVAqalY
[2] https://www.youtube.com/watch?v=uNlJqNaN8Sc

Power law model

Most popular model among rheologists

for $1 < p < \infty$ and $\nu_0 > 0$

$$\nu(|\varepsilon(\mathbf{v})|) = \nu_0 |\varepsilon(\mathbf{v})|^{p-2},$$
$$\nu(|\varepsilon(\mathbf{v})|) = \nu_0 (1 + |\varepsilon(\mathbf{v})|)^{p-2}.$$

- $p > 2 \Rightarrow$ shear thickening fluid (batter);
- $p < 2 \Rightarrow$ shear thinning fluid (blood, ketchup);
- $p = 2 \Rightarrow$ Newtonian fluid.

Mathematical questions

- In which function spaces do we have existence of solutions?
- Under which assumptions is the solution unique?
- How are the regularity properties of solutions?

The stationary p-Navier-Stokes problem (1)

Find $\mathbf{v} : G \to \mathbb{R}^d$ and $\pi : G \to \mathbb{R}$ satisfying

$$\left\{ \begin{aligned} \operatorname{div}\left(|\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v})\right) &= \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla\pi - \mathbf{f} && \text{in} \quad G, \\ \operatorname{div}\mathbf{v} &= 0 && \text{in} \quad G, \\ \mathbf{v} &= 0 && \text{on} \quad \partial G. \end{aligned} \right.$$

- Function space for $\mathbf{v}$:

$$W^{1,p}_{0,\mathrm{div}}(G) := \left\{\mathbf{u} \in W^{1,p}(G) : \ \mathbf{u}|_{\partial\Omega} = 0, \ \operatorname{div}\mathbf{u} = 0\right\};$$

- Function space for π:

$$L^{p'}_0(G) := \left\{ u \in L^{p'}(\Omega) : \ \int_G u\,dx = 0 \right\}.$$

The stationary p-Navier-Stokes problem (2)

Existence theory for stationary generalized Newtonian fluids.

Find $\mathbf{v} \in W^{1,p}_{0,\mathrm{div}}(G)$ such that for all $\varphi \in C^{\infty}_{0,\mathrm{div}}(G)$

$$\int_G |\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v}) : \varepsilon(\varphi)\, dx = \int_G \mathbf{v} \otimes \mathbf{v} : \varepsilon(\varphi)\, dx + \int_G \mathbf{f} \cdot \varphi \, dx.$$

- No variational approach available;
- Consider an approximated system whose solution $\mathbf{v}^n$ is known to exist together with

$$\mathbf{v}^n \rightharpoonup: \mathbf{v} \quad \text{in} \quad W^{1,p}(G).$$

The stationary p-Navier-Stokes problem (3)

Convergence of the convective term

$$\int_G \mathbf{v}^n \otimes \mathbf{v}^n : \nabla\varphi\, dx \longrightarrow \int_G \mathbf{v} \otimes \mathbf{v} : \nabla\varphi\, dx$$

- Compact embedding

$$W^{1,p}(G) \hookrightarrow L^2(G), \quad p > \frac{2d}{d+2}.$$

The stationary p-Navier-Stokes problem (4)

Energy convergence

$$\int_G |\varepsilon(\mathbf{v}^n)|^{p-2}\varepsilon(\mathbf{v}^n) : \varepsilon(\varphi)\, dx \longrightarrow \int_G |\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v}) : \varepsilon(\varphi)\, dx$$

- Almost everywhere-convergence $\varepsilon(\mathbf{v}^n) \to \varepsilon(\mathbf{v})$.
- Monotone-operator theory for $\mathbf{S}(\xi) = |\xi|^{p-2}\xi$:

$$\int_G \big(\mathbf{S}(\varepsilon(\mathbf{v}^n)) - \mathbf{S}(\varepsilon(\mathbf{v}))\big) : \varepsilon\left(\mathbf{v}^n - \mathbf{v}\right)\, dx \longrightarrow 0,$$

$$\big(\mathbf{S}(\zeta) - \mathbf{S}(\xi)\big) : (\zeta - \xi) > 0 \quad \text{if} \quad \zeta \neq \xi.$$

The stationary p-Navier-Stokes problem (5)

Test the equation by

$$\mathbf{u}^n = \mathbf{v}^n - \mathbf{v}.$$

- Standard if $p > \frac{9}{5}$; many interesting fluids are between $[\frac{3}{2}, 2]$;
- L^∞-truncation if $p \geq \frac{3}{2}$ by Frehse, Málek, Steinhauer in '97:

$$\mathbf{u}_\lambda = \mathbf{u} \quad \text{on} \quad \{x : \ |\mathbf{u}(x)| \leq \lambda\}, \quad \|\mathbf{u}_\lambda\|_\infty \leq \lambda.$$

- For blood we have $p \approx 1.21$;
- Lipschitz truncation if $p > \frac{6}{5}$ by Frehse, Málek, Steinhauer '03:

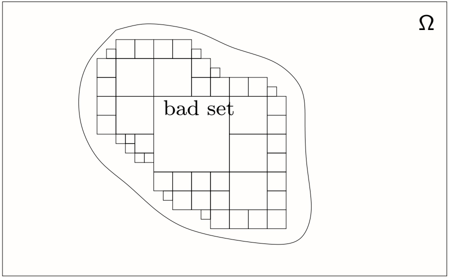
$$\mathbf{u}_\lambda = \mathbf{u} \quad \text{on} \quad \{x : \ M(|\nabla \mathbf{u}|)(x) \leq \lambda\}, \quad \|\nabla \mathbf{u}_\lambda\|_\infty \leq \ c\,\lambda.$$

Lipschitz truncation (1)

For $\mathbf{u} \in W^{1,1}(G)$ define

$$\mathbf{u}_\lambda := \begin{cases} \mathbf{u} & \text{on good set} \\ \sum_j \varphi_j \mathbf{u}_j & \text{on bad set} \end{cases}$$

- $\mathbf{u}_j = \fint_{Q_j} \mathbf{u}\, dx$;
- $(\varphi_j)_j$ partition of unity;
- $|\nabla \mathbf{u}_\lambda| \leq \fint_{4Q_j} \left| \frac{\mathbf{u}-\mathbf{u}_{4j}}{r_j} \right|$ on Q_j.



Lipschitz truncation (2)

- If $\mathbf{u} \in W_0^{1,p}(G)$ then $\mathbf{u}_\lambda \in W_0^{1,\infty}(G)$;
- Lipschitz truncation of Sobolev-functions goes back to Acerbi and Fusco (1988);
- Firstly used in fluid mechanics by Frehse, Málek, Steinhauer in 2003;
- Advanced by Diening, Málek, Steinhauer in 2006;
- Existence theory for the stationary p-Navier-Stokes problem provided

$$p > \frac{6}{5}.$$

Prandtl-Eyring fluids (1)

Eyring suggested in 1936 the constitutive law

$$\mathbf{S}(\varepsilon(\mathbf{v})) = \frac{\log(1+|\varepsilon(\mathbf{v})|)}{|\varepsilon(\mathbf{v})|}\varepsilon(\mathbf{v}).$$

- $\varepsilon(\mathbf{v}) \in L^h(G)$ where $h(t) := t\log(1+t)$;
- Lipschitz truncation nonlinear $\Rightarrow \operatorname{div}\mathbf{u}_\lambda \neq 0$;
- no continuity of Bog in $L^h(G)$;
- no pressure reconstruction in $L^h(G)$;
  $\Rightarrow$ solenoidal Lipschitz truncation.

Solenoidal Lipschitz truncation stationary

For $\mathbf{u} \in W^{1,p}_{0,\mathrm{div}}(G)$ define

$$\mathbf{u}_\lambda := \begin{cases} \mathbf{u} & \text{on good set} \\ \operatorname{curl}\left(\sum_j \varphi_j (\operatorname{curl}^{-1}\mathbf{u})_j\right) & \text{on bad set} \end{cases}$$

- $\operatorname{curl}^{-1} : W^{1,p}_{0,\mathrm{div}}(G) \to W^{2,p}_{\mathrm{div}}(G)$,
- $\mathbf{u}_\lambda$ has the same properties and $\operatorname{div}\mathbf{u}_\lambda = 0$;
- firstly invented by Breit, Diening, Fuchs (JDE, 2012).

Prandtl-Eyring fluids (2)

Theorem (Breit, Diening, Fuchs, 2012)

Let $G \subset \mathbb{R}^2$ be an open bounded set. Then there exists at least one weak solution $v \in V^{1,h}_{0,\mathrm{div}}(G)$ to the Prandtl-Eyring fluid model.

- Function space $(h(t) = t\log(1+t))$

$$V^{1,h}_{0,\mathrm{div}}(G) := \left\{\mathbf{u} \in W^{1,1}_{0,\mathrm{div}}(G) : \int_G h(|\varepsilon(\mathbf{u})|)\, dx < \infty\right\}.$$

- Compactness of $V^{1,h}_0(G) \hookrightarrow L^2(G)$ if $d = 2$;
- Korn's inequality does not hold in $L^h(G)$
  $\Rightarrow$ truncate such that $\varepsilon(\mathbf{u}_\lambda) \in L^\infty(G)$.

Korn's inequality in Orlicz spaces

Let $\varphi : [0, \infty) \to [0, \infty)$ be a Young function.

Korn's inequality holds iff $\varphi \in \Delta_2 \cap \nabla_2$ (Breit, Diening, 2012)

$$\int_G \varphi(|\nabla \mathbf{u}|)\,dx \leq \int_G \varphi(|\varepsilon(\mathbf{u})|)\,dx \quad \forall \mathbf{u} \in C_0^\infty(G).$$

- A Young function is said to have the Δ_2-property iff

$$\varphi(2t) \leq K\varphi(t) \quad \forall t \geq 0.$$

- A Young function is said to have the ∇_2-property iff $\varphi^* \in \Delta_2$

$$\varphi^*(t) = \sup_{s \geq 0}[st - \varphi(s)];$$

- Exlcudes $\varphi(t) = t\log(1+t)$ and $\varphi(t) = t(\exp(t) - 1)$.

The pressure in the Prandtl-Eyring model

Let A, B be two Young functions satisfying some "balance" cond.

Theorem by Breit, Cianchi (2013): Let $\mathbf{H} \in L^A(G)$ such that

$$\int_G \mathbf{H} : \nabla\varphi \, dx = 0 \quad \forall\varphi \in C^\infty_{0,\mathrm{div}}(G)$$

$$\Rightarrow \quad \int_G \mathbf{H} : \nabla\varphi \, dx = \int_G \pi \, \mathrm{div}\, \varphi \, dx \quad \forall\varphi \in C^\infty_0(G)$$

for a unique pressure function $\pi \in L^B_0(G)$.

- If $A \in \Delta_2 \cap \nabla_2$ then $A = B$ (e.g. $A(t) = t^q$ with $q > 1$);
- Prandtl-Eyring, d=2: $\mathbf{H} = \mathbf{S}(\varepsilon(\mathbf{v})) + \mathbf{v} \otimes \mathbf{v} \in L^{t\log^2(1+t)}(G)$
  $\Rightarrow \pi \in L^{t\log(1+t)}(G)$.

The non-stationary p-Navier-Stokes problem

Existence theory for non-stationary generalized Newtonian fluids.

Equation of motion

$$-\partial_t \mathbf{v} + \operatorname{div}\left(|\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v})\right) = \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla\pi - \mathbf{f}$$

Weak formulation: for all $\varphi \in C^\infty_{0,\operatorname{div}}([0,T)\times G)$

$$\int_Q |\varepsilon(\mathbf{v})|^{p-2}\varepsilon(\mathbf{v}) : \varepsilon(\varphi)\,dx\,dt = \int_Q \mathbf{v}\otimes\mathbf{v} : \nabla\varphi\,dx\,dt + \int_Q \mathbf{f}\cdot\varphi\,dx\,dt + \int_Q \mathbf{v}\cdot\partial_t\varphi\,dx\,dt + \int_G \mathbf{v}_0\cdot\varphi(0)\,dx.$$

Existence results

Function space

$$\mathbf{v} \in L^p(0,T;W^{1,p}_{0,\mathrm{div}}(G)) \cap L^\infty(0,T;L^2(G)),$$
$$\partial_t \mathbf{v} \in L^\sigma(0,T;W^{-1,\sigma}_{\mathrm{div}}(G)), \quad \sigma > 1.$$

- Monotone-operator theory provided $p > \frac{11}{5}$ by Ladyshenskaya and Lions late '60;
- L^∞-truncation provided $p > \frac{8}{5}$ by Wolf in 2007;
- Lipschitz truncation+pressure decomposition provided $p > \frac{6}{5}$ by Diening, Růžička, Wolf in 2010;
- Solenoidal Lipschitz truncation provided $p > \frac{6}{5}$ by Breit, Diening, Schwarzacher (M3AS, 2013).

Parabolic problems (1)

Parabolic Poincaré-inequality

$$\fint_{Q_r} \left| \frac{\mathbf{u} - \mathbf{u}_{Q_r}}{r} \right| dx\,dt \le c \fint_{Q_r} |\nabla \mathbf{u}|\,dx\,dt + c\alpha \fint_{Q_r} |\mathbf{H}|\,dx\,dt,$$

$$Q_r := (-\alpha r^2, \alpha r^2) \times B_r.$$

- $\partial_t \mathbf{u} = \operatorname{div} \mathbf{H}$ in $\mathcal{D}'$, where $\mathbf{H} = \mathbf{S}(\varepsilon(\mathbf{v})) + \mathbf{v} \otimes \mathbf{v}$+pressure;
- $\partial_t \mathbf{u} \in \mathcal{D}'_{\mathrm{div}}$ does not suffice $\Rightarrow$ introduction of the pressure;
- $\mathcal{O}_\lambda$ depends on $M(\nabla \mathbf{u})$ and $M(\mathbf{H})$, $\alpha = \lambda^{p-2}$.

Solenoidal Lipschitz truncation still needs the pressure!

Parabolic problems (2)

Letting $\mathbf{w} := \mathrm{curl}^{-1}\,\mathbf{u}$ and $\varphi \in C_0^\infty(Q)$

$$\int_Q \mathbf{w} \cdot \partial_t \Delta\varphi \, dx\, dt = \int_Q \mathbf{H} : \nabla^2 \varphi \, dx\, dt$$

- Control of full distributional time derivative but only for $\Delta\mathbf{w}$
  $\Rightarrow$ Get rid of the harmonic part of $\mathbf{w}$!
- $\mathbf{w} = \mathbf{z} + \mathbf{h}$, where $\mathbf{z} \in \Delta W_0^{2,p}$ and $\Delta\mathbf{h} = 0$

$$\int_Q \mathbf{z} \cdot \partial_t \Delta\varphi \, dx\, dt = \int_Q \mathbf{H} : \nabla^2 \varphi \, dx\, dt$$

$\Rightarrow$ truncate $\mathbf{z}$ using $\|\partial_t \mathbf{z}\| \le c\|\mathbf{H}\|$.

Solenoidal truncation non-stationary

For $\mathbf{u} \in L^p(W^{1,p}_{0,\mathrm{div}}(G)) \cap L^\infty(L^2(G))$ define $\mathbf{w} := \mathrm{curl}^{-1}\,\mathbf{u}$

$$\mathbf{z}_\lambda := \begin{cases} \mathbf{z} & \text{on good set} \\ \sum_j \varphi_j \mathbf{z}_j & \text{on bad set} \end{cases}$$

$$\mathbf{u}_\lambda := \mathrm{curl}\,\mathbf{z}_\lambda + \mathrm{curl}(\mathbf{w} - \mathbf{z})$$

- $\mathbf{u} \in L^p(W^{1,p}) \Rightarrow \mathbf{w} \in L^p(W^{2,p}) \Rightarrow \mathbf{z} \in L^p(W^{2,p})$;
- $\mathcal{O}_\lambda = \mathcal{O}_\lambda(\nabla^2\mathbf{z}; \partial_t\mathbf{z}) \Rightarrow \mathbf{z}_\lambda \in L^\infty(W^{2,\infty})$ with $\|\nabla^2\mathbf{z}_\lambda\|_\infty \le c\lambda$;
- $\mathbf{u}_\lambda \in L^\infty(W^{1,\infty})$ with $\|\nabla\mathbf{u}_\lambda\|_\infty \le c\lambda$ and $\mathrm{div}\,\mathbf{u}_\lambda = 0$;
- by Breit, Diening, Schwarzacher in 2013 (M3AS).

Stochastic Navier-Stokes equations

system of stochastic partial differential equations

$$\begin{cases} d\mathbf{v} = \operatorname{div}\mathbf{S}\,dt - (\nabla\mathbf{v})\mathbf{v}\,dt + \nabla\pi\,dt + \mathbf{f}\,dt + \Phi(\mathbf{v})\,d\mathbf{W}_t & \text{in } Q, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } Q, \\ \mathbf{v} = 0 & \text{on } \partial G, \\ \mathbf{v}(0) = \mathbf{v}_0 & \text{in } G. \end{cases}$$

- $\mathbf{W}$ = Brownian motion, values in Hilbert space with base $(\mathbf{e}_k)$

$$\mathbf{W}(t) = \sum_{k=0}^{\infty} \beta_k(t)\mathbf{e}_k, \quad \beta_k \overset{i.i.d.}{\cong} SBM.$$

- We suppose that Φ grows linearly - roughly speaking $|\Phi(\mathbf{v})| \leq c(1 + |\mathbf{v}|)$ and $|\Phi'(\mathbf{v})| \leq c$

Weak formulation

Weak formulation: $\mathbf{v} : \Omega \times Q \to \mathbb{R}^d$ such that $\mathbb{P} \otimes \mathcal{L}^1$-a.e.

$$\int_G \mathbf{v}(t) \cdot \varphi \, dx = \int_G \mathbf{v}_0 \cdot \varphi \, dx + \int_0^t \int_G \mathbf{H} : \nabla \varphi \, dx \, d\sigma + \int_G \int_0^t \varphi \cdot \Phi(\mathbf{v}) \, d\mathbf{W}_\sigma \, dx, \quad \varphi \in C^\infty_{0,\mathrm{div}}([0,T) \times G).$$

- $(\Omega, \mathcal{F}, \mathbb{P})$ = probability space;
- For $\Phi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T; L_2(U, L^2(G))))$ we have

$$\int_0^t \Phi(\sigma) \, d\mathbf{W}_\sigma = \sum_{k=1}^\infty \int_0^t \Phi(\sigma)(\mathbf{e}_k) \, d\beta_k(\sigma)$$

and the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C[0,T])$.

Weak solutions

Natural function space

$$L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; L^2(G))) \cap L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; W^{1,p}_{0,\mathrm{div}}(G)))$$

- If we have no uniqueness for the deterministic problem we expect martingale solutions, i.e. there is a probability space $(\underline{\Omega}, \underline{\mathcal{F}}, \underline{\mathbb{P}})$ and a Brownian motion $\underline{\mathbf{W}}$ on it such that a weak solutions $\underline{\mathbf{v}}$...
- Many literature for stochastic Navier Stokes equations starting with Bensoussan-Temam in 1973;
- Knowledge about Non-Newtonian problem very limited (periodic boundary...Terasawa–Yoshida, 2011)

Main result

Theorem (Breit, JMFM forth.)

Under appropriate assumotions on $\mathbf{f}$ and $\mathbf{v}_0$ there is a weak martingale solution to the stochastic p-Navier-Stokes equations provided $p > \frac{2d+2}{d+2}$.

- Extension of the results by Wolf to the stochastic case;
- The bound $p > \frac{8}{5}$ (if $d = 3$) includes a wide range of Non-Newtonian fluids;
- L^∞-truncation and harmonic pressure decomposition adapted to the stochastic setting.

Electro-rheological fluids (1)

The fluid reacts on an electric field[3], model by Rajagopal-Růžička

$p : (0, T) \times \Omega \to (1, \infty)$, $p = p(|\mathbf{E}|^2)$,

$$\mathbf{S} = \mathbf{S}(t, x, \varepsilon(\mathbf{v})) \approx |\varepsilon(\mathbf{v})|^{p(t,x)-2}\varepsilon(\mathbf{v}).$$

- ν increases in 1ms for the factor 1000;
- Controlling of fluid properties without mechanical interaction;
- Many technological applications: actuators, clutches, shock absorbers, rehabilitation equipment;
- Firstly oberved by Winslow in 1949.

[3] https://www.youtube.com/watch?v=44430l_VkGs

Electro-rheological fluids (2)

Generalized Lebesgue- and Sobolev-spaces

$$L^{p(\cdot)}(G) := \left\{ u : G \to \mathbb{R} : \int_G |u(x)|^{p(x)}\, dx < \infty \right\},$$
$$W^{1,p(\cdot)}(G) := \left\{ u : G \to \mathbb{R} : u \in L^{p(\cdot)}(G),\ \nabla u \in L^{p(\cdot)}(G) \right\}.$$

- $L^{p(\cdot)}(G)$ is a Banach-space via

$$\|u\|_{p(\cdot)} := \inf \left\{ k : \int_G \left| \frac{u(x)}{k} \right|^{p(x)} dx \le 1 \right\}.$$

Electro-rheological fluids (3)

State of the art and open problems:

- Generalized Lebesgue spaces well-understood;
- Stationary problem solved (Diening-Málek-Steinhauer, 2008): existence of weak solutions if $\inf_{x \in G} p(x) > \frac{6}{5}$;
- No results for the parabolic problem, missing Bochner-space structure: for constant p

$$L^p(0, T; L^p(G)) \cong L^p(Q);$$

- No theory et all for stochastic problems: $p = p(\omega, t, x) \Rightarrow$ random function spaces!