Compressible fluids interacting with a linear-elastic shell

Dominic Breit & Sebastian Schwarzacher

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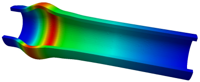
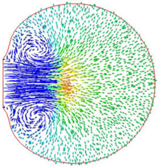
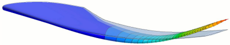
Fluid structure interaction

In this talk we will consider a compressible fluid which is floating in a flexible body.

- The fluid forces are interacting with a membrane that is assumed to be a part of the boundary.
- The geometry changes in time.

Examples:

Blood vessels:  Gas balloon: 



Airplane wing:

The Setting

- $\Omega \in \mathbb{R}^3$ is the **initial geometry** and the **reference geometry**
- $\partial\Omega = \Gamma \cup M$, Γ is the fixed part of the boundary.
- M is the flexible part of the boundary–hence the **domain of definition** for the time-changing coordinates.
- The displacement of the boundary is prescribed via a two dimensional surface representing a **Kirchhoff-Love** plate.
- It is a model reduction assuming small strains and plane stresses parallel to the middle surface.
- $\boldsymbol{\eta} : I \times M \to \mathbb{R}^3$ defines the change of the domain.
- $\Omega_{\boldsymbol{\eta}(t)}$ defines the changed domain: $\partial\Omega_{\boldsymbol{\eta}(t)} = \Gamma \cup \boldsymbol{\eta}(t, M)$.
- Inside the domain we assume a **compressible** fluid. Its motion is characterized by its **velocity**: $\mathbf{u} : I \times \Omega_{\boldsymbol{\eta}(t)} \to \mathbb{R}^3$ and **density**: $\varrho : I \times \Omega_{\boldsymbol{\eta}(t)} \to \mathbb{R}^+$.

The PDE in the interior

The fluid is compressible and viscous

$$\begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) &= 0, && \text{in } I \times \Omega_\eta, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) &= \mu \Delta \mathbf{u} - a \nabla \varrho^\gamma + \varrho \mathbf{f} && \text{in } I \times \Omega_\eta, \\ \mathbf{u}(t, x + \boldsymbol{\eta}(x)) &= \partial_t \boldsymbol{\eta}(t, x) && \text{in } I \times M. \end{aligned}$$

The shell is driven by the **Koiter-energy**

$$K(\boldsymbol{\eta}) = \frac{1}{2}\varepsilon_0 \int_M \mathbf{C} : \boldsymbol{\sigma}(\boldsymbol{\eta}) \otimes \boldsymbol{\sigma}(\boldsymbol{\eta}) \, \mathrm{d}\mathcal{H}^2 + \frac{1}{6}\varepsilon_0^3 \int_M \mathbf{C} : \boldsymbol{\theta}(\boldsymbol{\eta}) \otimes \boldsymbol{\theta}(\boldsymbol{\eta}) \, \mathrm{d}\mathcal{H}^2.$$

If $\boldsymbol{\eta}(t,x) \equiv \eta(t,x)\nu(x)$ the corresponding equation for the shell is

$$\begin{aligned} \partial_t^2 \eta + \Delta^2 \eta &= g + \nu \cdot \big(-\boldsymbol{\tau} \nu_\eta \big) \circ \boldsymbol{\Psi}_{\eta(t)} \big| \det D\boldsymbol{\Psi}_{\eta(t)} \big| \quad \text{in } I \times M, \\ \boldsymbol{\tau} &= \mathbf{S}(\nabla \mathbf{u}) - a \varrho^\gamma \mathcal{I}. \end{aligned}$$

Weak formulation (1)

Let $\eta : I \times M \to \mathbb{R}$ and $\partial\Omega_{\boldsymbol{\eta}(t)} = \Gamma \cup \{x + \eta(t,x)\nu(x) : x \in M\}$.
consider a coordinate map $\boldsymbol{\Psi}_\eta : \Omega \to \Omega_\eta$.
Reynolds transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\eta(t)}} g \,\mathrm{d}x = \int_{\Omega_{\eta(t)}} \partial_t g \,\mathrm{d}x + \int_{\partial\Omega_{\eta(t)}} \partial_t \eta \circ \boldsymbol{\Psi}_\eta^{-1} \nu \cdot \nu_\eta g \,\mathrm{d}\mathcal{H}^2,$$

The weak continuity equation: integration by parts implies for $\psi \in C^\infty(I \times \overline{\Omega})$

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} \varrho\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta} \Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

if $\mathbf{u} \circ \boldsymbol{\Psi}_\eta = \partial_t \eta \nu$ on $\partial\Omega_{\eta(t)}$.

Weak formulation (2)

The weak momentum equation:
For $(b,\varphi) \in C_0^\infty(M) \times C^\infty(\bar{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta \varphi = b\nu$

$$\int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \varphi \,\mathrm{d}x - \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \,\mathrm{d}x \right) \mathrm{d}t$$

$$+ \int_I \int_{\Omega_\eta} \left(\mu \nabla \mathbf{u} : \nabla \varphi + (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \varphi - \varrho^\gamma \operatorname{div} \varphi \right) \mathrm{d}x \,\mathrm{d}t$$

$$+ \int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_M \partial_t \eta b \,\mathrm{d}\mathcal{H}^2 - \int_M \partial_t \eta \,\partial_t b \,\mathrm{d}\mathcal{H}^2 + \int_M K'(\eta)\, b \,\mathrm{d}\mathcal{H}^2 \right) \mathrm{d}t$$

$$= \int_I \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_M g\, b \,\mathrm{d}\mathcal{H}^2 \,\mathrm{d}t.$$

Renormalized continuity equation

The renormalized continuity equation:
For $\psi \in C^\infty(I \times \overline{\Omega})$ and $\theta \in C^1(\mathbb{R}^+)$ positive

$$0 = \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \theta(\varrho)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\theta(\varrho)\partial_t\psi + \theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_I \int_{\Omega_\eta} (\varrho\theta'(\varrho) - \theta(\varrho))\,\mathrm{div}\,\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t.$$

- Major ingredient for compressible Navier–Stokes equations.
- Introduced by Di Perna-Lions '89.

Main theorem

Theorem (Breit, Schwarzacher, ARMA '18)

Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). There is a weak solution $(\eta, \mathbf{u}, \varrho)$. The interval of existence is restricted only in case Ω_η approaches a self intersection. The solution satisfies the energy estimate

$$\sup_{t\in I}\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\,\mathrm{d}x+\sup_{t\in I}\int_{\Omega_\eta}\varrho^\gamma\,\mathrm{d}x+\int_I\int_{\Omega_\eta}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sup_{t\in I}\int_M|\partial_t\eta|^2\,d\mathcal{H}^2+\sup_{t\in I}\int_M|\nabla^2\eta|^2\le c(\mathbf{q}_0,\varrho_0,\mathbf{f},g,\eta_0,\eta_1).$$

The incompressible analogue: Lengeler & Růžička, (ARMA, '14).

Remarks

- The function space for a weak solutions is determined by the left-hand side of energy estimate taking into account the variable domain.
- The bound $\gamma > \frac{12}{7}$ is less restrictive then the bound $\gamma \geq \frac{9}{5}$ appearing in the pioneering work of Lions, but more restrictive than the bound $\gamma > \frac{3}{2}$ arising in the theory by Feireisl et al.
- A major mathematical difficulty is the parabolic-hyperbolic nature of the system $\Rightarrow$ regularity incompatibilities between the fluid- and the solid-phase.
- The boundary only belongs to $W^{2,2}$ and is not Lipschitz!

Strategy

- Understand sequential compactness of weak solutions.
- Construct them by means of a multi-layer approximation scheme inspired by Feireisl et al.
- Apply a fixed point argument on the basic level taking into account a coupling of the system $(\eta \mapsto (\mathbf{u}, \varrho)$ vs. $(\zeta, \mathbf{v}) \mapsto (\eta, \mathbf{u}))$.
- For the fixed point to work linearise equation (replace $\mathbf{u}$ by $\mathbf{v}$ in the convective terms).

Sequential compactness (1)

Assume there a sequence of solutions $(\eta_n, \mathbf{u}_n, \varrho_n)$ which enjoys suitable regularity properties and satisfies the energy estimate uniformly in n.

How can we pass to the limit in the equation?

- passage to the limit in the convective terms $\varrho_n \mathbf{u}_n$ and $\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n$ by local arguments and global integrability
  No problems with the moving boundary!
  Much easier than incompressible case!
- Main problem (typical for compressible Navier–Stokes):
  Passing to the limit in the nonlinear pressure $p(\varrho_n) = a\varrho_n^\gamma$.

Higher integrability of the pressure (1)

- A priori $p(\varrho_n)$ only bounded in $L^\infty(L^1)$
  $\Rightarrow$ Concentrations possible!
- Improve integrability by "computing the pressure": Use globally Bogovskiĭ-operator $\approx \mathrm{div}^{-1}$ or locally $\Delta^{-1}\mathrm{div}$.
- Bogovskiĭ-operator requires Lipschitz boundary $\Rightarrow$ standard approach only gives higher integrability locally.
- How to exclude concentrations at the boundary?

Higher integrability of the pressure (2)

Lemma

There is a measurable set $A_\kappa \Subset I \times \Omega_{\eta_n}$ such that for all $n \geq n_0$

$$\int_{I\times\mathbb{R}^3 \setminus A_\kappa} a\varrho_n^\gamma \chi_{\Omega_{\eta_n}} \, \mathrm{d}x \, \mathrm{d}t \leq \kappa.$$

Construct test-function φ_n such that:

- zero boundary conditions on $\partial\Omega_{\eta_n}$;
- $\operatorname{div} \varphi_n \geq K_\kappa$ close to the boundary;
- Critical term $\int_I \int_{\Omega_{\eta_n}} \varrho_n \mathbf{u}_n \, \partial_t \varphi_n \, \mathrm{d}x \, \mathrm{d}t$ with $\partial_t \varphi_n \sim \partial_t \eta_n = \mathbf{u}_n \circ \mathbf{\Psi}_n|_{\partial\Omega_{\eta_n}} \in L^2(L^{4^-})$;
- Since $\varrho_n \mathbf{u}_n \in L^2(L^{\frac{6\gamma}{\gamma+6}})$ we need $\gamma > \frac{12}{7}$.

Strong convergence of the pressure (1)

Effective viscous flux identity (Lions '93) with cut-off T_k

$$\int_{I\times\Omega_{\eta_n}} \left(a\varrho_n^\gamma - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_n\right) T_k(\varrho_n)\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_{I\times\Omega_\eta} \left(\overline{a\varrho^\gamma} - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}\right) \overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t.$$

- Locally by standard method (commutator, div-curl-lemma);
- Globally by uniform integrability;
- Needs renormalized continuity equation to conclude (note that $\varrho \mapsto \varrho^\gamma$ is monotone);
- DiPerna-Lions theory does not apply for small γ.

Strong convergence of the pressure (2)

Oscillation defect measures (Feireisl '01)

$$\limsup_{n\to\infty} \int_{I\times\mathbb{R}^3} \left| T_k(\varrho_n) - T_k(\varrho) \right|^{\gamma+1} \mathrm{d}x\,\mathrm{d}t \le C.$$

- By standard method (using effective viscous flux identity);
- Implies renormalized continuity equation in the limit

$$0 = \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \theta(\varrho)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\theta(\varrho)\partial_t\psi + \theta(\varrho)\mathbf{u}\cdot\nabla\psi \Big)\,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\Omega_\eta} \left(\varrho\theta'(\varrho) - \theta(\varrho) \right) \operatorname{div}\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t$$

- Extension to $\mathbb{R}^3$ requires no-slip boundary conditions ("no slip boundary conditions w.r.t. to the moving shell").

Four layer approximation scheme

- Artificial pressure (δ-layer): replace $p(\varrho) = a\varrho^\gamma$ by $p_\delta(\varrho) = a\varrho^\gamma + \delta\varrho^\beta$ where β is chosen large enough.
- Artificial viscosity (ε-layer): add $\varepsilon\Delta\varrho$ to the right-hand side of the continuity equation.
- Regularization of the boundary (κ-layer): Replace the underlying domain Ω_η by Ω_{η_κ} where η_κ is a suitable regularization of η. Accordingly, the convective terms and the pressure have to be regularized as well.
- Finite-dimensional approximation (N-layer): the momentum equation has to be solved by means of a Galerkin-approximation.

The first two layers are common in the theory of compressible Navier–Stokes equations, see Feireisl et al. Third layer is needed due to the low regularity of η.

The last layer

- Function space depends on the solution itself $\Rightarrow$ Galerkin approximation not possible.
- Lengeler-Růžička: Apply a fixed point argument in η and $\mathbf{u}$ for a linearised problem (replace $\varrho\mathbf{u}\otimes\mathbf{u}$ by $\varrho\mathbf{u}\otimes\mathbf{v}$ and $\varrho\mathbf{u}$ by $\varrho\mathbf{v}$ for $\mathbf{v}$ given).
- It is crucial for our fixed point argument that the momentum equation is linear in $\mathbf{u}$.

For $(\zeta,\mathbf{v})$ given we solve the system on the domain Ω_ζ. The domain still varies in time but is independent of the solution. Note here, that ϱ is independent of $\mathbf{u}$.

Fixed point argument

Show that the map $(\zeta, \mathbf{v}) \mapsto (\eta, \mathbf{u})$ has a fixed point:

- Uniqueness not known $\Rightarrow$ set-valued mapping;
- Needs compactness of $\mathbf{u}_n$ in $L^2(I \times \mathbb{R}^3)$;
- Compactness of ϱ_n easy as $\varepsilon > 0$;
- We can show (as for convective term $\varrho \mathbf{u} \otimes \mathbf{u}$)

$$\int_I \int_{\mathbb{R}^3} \varrho_n |\mathbf{u}_n|^2 \, \mathrm{d}x \, \mathrm{d}t \longrightarrow \int_I \int_{\mathbb{R}^3} \varrho |\mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t$$

- What happens in the vacuum? $\Rightarrow$ replace $\partial_t(\varrho \mathbf{u})$ by $\partial_t((\varrho + \kappa)\mathbf{u})$ in the momentum equation!

Open problems

- Local strong solutions for both compressible and incompressible model;
- Global strong solutions for incompressible 2D;
- Navier–Stokes–Fourier equations for compressible, heat conducting fluids (in progress): sequential compactness seems fine, problems with the fixed point (regularity incompabilities).

- D. Breit & S. Schwarzacher: *Compressible fluids interacting with a linear-elastic shell.* **Arch. Rational Mech. Anal.** 228, 495–562. (2018)
- E. Feireisl, A. Novotný & H. Petzeltová: *On the existence of globally defined weak solutions to the Navier–Stokes equations of compressible isentropic fluids.* **J. Math. Fluid. Mech.** 3, 358–392. (2001)
- D. Lengeler & M. Růžička: *Weak Solutions for an Incompressible Newtonian Fluid Interacting with a Koiter Type Shell.* **Arch. Rational Mech. Anal.** 211, 205–255. (2014)