Convergence rates for the numerical approximation of the 2D stochastic Navier–Stokes equations

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Stochastic Navier–Stokes equations

Velocity field $\mathbf{u}$ and pressure π on $Q = (0, T) \times \mathbb{T}^2$

$$\begin{cases} \mathrm{d}\mathbf{u} = [\mu\Delta\mathbf{v} - \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) - \nabla\pi]\,\mathrm{d}t + \mathbb{G}\,\mathrm{d}W & \text{in } Q, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } Q. \end{cases}$$

- Cylindricial Wiener process W on a separable Hilbert space with basis $(e_k)_{k\geq 0}$ (e.g. $L^2(\mathbb{T}^2)$): $W = \sum_{k\geq 0} \beta_k e_k$;
- Coefficient $\mathbb{G}$ is a Hilbert-Schmidt operator;
- Multiplicative noise $\mathbb{G} = \mathbb{G}(\mathbf{u})$ has to be Lipschitz;
- Defined on filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$;
- Initial datum $\mathbf{u}_0$ can be random.

Stochastic perturbations

1. It can be understood as turbulence in the fluid motion.
   - Mikulevicius and Rozovskii (2004): Dynamics of fluid particles contain turbulent part $\boldsymbol{\sigma} \circ \mathrm{d}W$ (Kraichnan's turbulence model)

$$\mathrm{d}\mathbf{u} = \ldots + [\boldsymbol{\sigma}\nabla\mathbf{u} - \nabla\tilde{\pi} + \mathbf{g}(\mathbf{u})] \circ \mathrm{d}W;$$

   - Birner (2013): uses the ansatz $(e_k(x) = e^{2\pi t k \cdot x})$

$$\mathrm{d}\mathbf{u} = \ldots + \sum_{k\in\mathbb{Z}^3} c_k^{1/2} e_k \mathrm{d}W_k + \mathbf{v} \sum_{k\in\mathbb{Z}^3,\,|k|\leq m} \int_{-\infty}^{\infty} h_k \overline{N}^k \,\mathrm{d}z\,\mathrm{d}t$$

   to derive Kolmogorov–Obukhov Statistical Theory of Turbulence.
2. Can be interpreted as a perturbation from the physical model.
3. Apart from the force $\mathbf{f}$, which we are observing, there are further quantities with (usually small) influence on the motion.

Weak formulation

Find $\mathbf{u}$ such that for all $\varphi \in C^\infty_{\text{div}}(\mathbb{T}^2)$ and all $t \in [0, T]$

$$\int_{\mathbb{T}^2} \mathbf{u}(t) \cdot \varphi \, \mathrm{d}x = \int_{\mathbb{T}^2} \mathbf{u}_0 \cdot \varphi \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^2} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}\sigma - \mu \int_0^t \int_{\mathbb{T}^2} \nabla \mathbf{u} : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}\sigma + \int_{\mathbb{T}^2} \varphi \cdot \int_0^t \mathbb{G} \, \mathrm{d}W \, \mathrm{d}x.$$

- Pressure disappears in weak formulation;
- By Itô-isometry if $\mathbb{G} \in L_2(L^2_x; L^2_x)$

$$\mathbb{E} \left\| \int_0^T \mathbb{G} \, \mathrm{d}W \right\|^2_{L^2_x} = \mathbb{E} \int_0^T \|\mathbb{G}\|^2_{L_2(L^2_x; L^2_x)} \, \mathrm{d}t = \sum_{k \geq 0} \mathbb{E} \int_0^T \|\mathbb{G} e_k\|^2_{L^2_x} \, \mathrm{d}t.$$

Concept of solution

A stochastically strong solution

is an $(\mathfrak{F}_t)$-adapted stochastic process $\mathbf{u}$ with

$$\mathbf{u} \in C([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^2)) \cap L^2(0,T; W^{1,2}_{\mathrm{div}}(\mathbb{T}^2))$$

$\mathbb{P}$-a.s. which solves the momentum equation in the weak sense.

- Exists on a given stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$;
- W is a given $(\mathfrak{F}_t)$-cylindrical Wiener process;
- We have $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ $\mathbb{P}$-a.s. where $\mathbf{u}_0$ is $\mathfrak{F}_0$-measurable with $\mathbf{u}_0 \in L^2(\Omega; L^2_{\mathrm{div}}(\mathbb{T}^2))$;
- First results by Capiński and Capiński-Cutland (1991/1993).

Qualitative properties

Find $\mathbf{u}$ such that for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^2)$ and all $t \in [0, T]$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\int_{\mathbb{T}^2}|\nabla\mathbf{u}|^2+\int_0^T\int_{\mathbb{T}^2}|\nabla^2\mathbf{u}|^2\Big]^r\le c_r\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^2_{W^{1,2}_x}\Big]^r.$$

- "Test" by $\Delta\mathbf{u}$ (apply Itô's formula to $t\mapsto\|\mathbf{u}\|^2_{W^{1,2}_x}$);
- Higher order estimates: $\mathbf{u}_0\in W^{k,2}_x\Rightarrow$ $\mathbf{u}\in L^\infty_t(W^{k,2}_x)\cap L^2_t(W^{k+1,2}_x)$;
- Dirichlet-problem: not known if $\mathbb{E}[\|\mathbf{u}(T)\|^2_{W^{1,2}_x}]<\infty$ for ANY finite T (only for a random T which is large).

Finite-element spaces

$\mathscr{T}_h$ is quasi-uniform subdivision of $\mathbb{T}^2$ into triangles $\mathcal{S}$, $i, j \in \mathbb{N}$

$$V^h(\mathbb{T}^2) := \{\mathbf{v}_h \in W^{1,2}(\mathbb{T}^2) \,:\, \mathbf{v}_h|_{\mathcal{S}} \in \mathscr{P}_i(\mathcal{S}) \ \forall \mathcal{S} \in \mathscr{T}_h\},$$
$$P^h(\mathbb{T}^2) := \{\pi_h \in L^2(\mathbb{T}^2) \,:\, \pi_h|_{\mathcal{S}} \in \mathscr{P}_j(\mathcal{S}) \ \forall \mathcal{S} \in \mathscr{T}_h\}.$$

- $V^h(\mathbb{T}^2)$ and $P^h(\mathbb{T}^2)$ linked by inf-sup condition:

$$\sup_{\mathbf{v}_h \in V^h(\mathbb{T}^2)} \frac{\int_{\mathbb{T}^2} \operatorname{div} \mathbf{v}_h \, \pi_h \, \mathrm{d}x}{\|\nabla \mathbf{v}_h\|_{L^2_x}} \geq C \, \|\pi_h\|_{L^2_x} \quad \forall \pi_h \in P^h(\mathbb{T}^2);$$

- Discretely solenoidal finite element functions by

$$V^h_{\mathrm{div}}(\mathbb{T}^2) := \left\{ \mathbf{v}_h \in V^h(\mathbb{T}^2) : \int_{\mathbb{T}^2} \operatorname{div} \mathbf{v}_h \ \pi_h \, \mathrm{d}x = 0 \, \forall \pi_h \in P^h(\mathbb{T}^2) \right\}.$$

The algorithm

Find r.v. $\mathbf{u}_{h,m}$ with values in $V^h_{\mathrm{div}}(\mathbb{T}^2)$ s.t. for all $\varphi \in V^h_{\mathrm{div}}(\mathbb{T}^2)$

$$\int_{\mathbb{T}^2} \mathbf{u}_{h,m} \cdot \varphi \,\mathrm{d}x + \Delta t \int_{\mathbb{T}^2} \big((\nabla \mathbf{u}_{h,m}) \mathbf{u}_{h,m-1} + (\mathrm{div}\, \mathbf{u}_{h,m-1}) \mathbf{u}_{h,m} \big) \cdot \varphi \,\mathrm{d}x$$
$$+ \mu \,\Delta t \int_{\mathbb{T}^2} \nabla \mathbf{u}_{h,m} : \nabla \varphi \,\mathrm{d}x = \int_{\mathbb{T}^2} \mathbf{u}_{h,m-1} \cdot \varphi \,\mathrm{d}x$$
$$+ \int_{\mathbb{T}^2} \mathbb{G}(\mathbf{u}_{h,m-1}) \,\Delta_m W \cdot \varphi \,\mathrm{d}x \,.$$

- Initial datum $\mathbf{u}_{h,0} \in V^h_{\mathrm{div}}(\mathbb{T}^2)$ given (e.g. $\mathbf{u}_{h,0} = \Pi_h \mathbf{u}_0$);
- Here $\Delta_m W = W(t_m) - W(t_{m-1})$ where $t_m = m\frac{T}{M}$;
- First analysed by Carelli-Prohl (2012).

Error estimate

Breit-Dodgson (2019): for any $\alpha < \frac{1}{2}$

$$\mathbb{E}\left[\mathbf{1}_{\Omega^\varepsilon}\left(\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{u}_{h,m}\|^2_{L^2_x}+\sum_{m=1}^{M}\Delta t\|\nabla\mathbf{u}(t_m)-\nabla\mathbf{u}_{h,m}\|^2_{L^2_x}\right)\right] \le c\left(h^2+(\Delta t)^{2\alpha}\right).$$

- Control first error to time-discrete solution $\mathbf{u}_m$;
- $\mathbf{1}_{\Omega^\varepsilon}$ controls blow-up of $\max_m \|\nabla\mathbf{u}_m\|_{L^2_x}$ and $\max_m \|\mathbf{u}_{h,m}\|_{L^2_x}$ with bound $\log(\Delta t)^{-\varepsilon}$.
- Convergence in probability with rates (almost) 1/2 and 1.
- Carelli-Prohl (2012): same estimate with $\left(h^2+\frac{h^2}{\Delta t}+(\Delta t)^{\alpha}\right)$.

Assumptions Breit-Dodgson

Convergence rate in time (almost) 1/2 provided

- $\mathbb{E}\left[\|\mathbf{u}\|^4_{C^\alpha([0,T];L^4_x)}\right] < \infty$: if $\mathbb{E}\left[\|\mathbf{u}_0\|^8_{W^{1,2}_x}\right] < \infty$ and $\mathbb{G}(\mathbf{u})$ has linear growth on $W^{1,2}_x$;
- $\mathbb{E}\left[\|\mathbf{u}\|^2_{C^\alpha([0,T];W^{1,2}_x)}\right] < \infty$: if $\mathbb{E}\left[\|\mathbf{u}_0\|^4_{W^{2,2}_x}\right] + \mathbb{E}\left[\|\mathbf{u}_0\|^{20}_{W^{1,2}_x}\right] < \infty$ and $\mathbb{G}(\mathbf{u})$ has linear growth on $W^{2,2}_x$;
- $P^h(\mathbb{T}^2)$ with elements locally from $\mathscr{P}_j$ for $j \geq 1$.

Assumptions Carelli-Prohl

Convergence rate in time (almost) $1/4$ provided

- $\mathbb{E}\left[\|\mathbf{u}\|^4_{C^\alpha([0,T];L^4_x)}\right] < \infty$ and $\mathbb{E}\left[\|\mathbf{u}\|^4_{C^{\frac{\alpha}{2}}([0,T];W^{1,2}_x)}\right] < \infty$: if $\mathbb{E}\left[\|\mathbf{u}_0\|^8_{W^{1,2}_x}\right] < \infty$ and $\mathbb{G}(\mathbf{u})$ has linear growth on $W^{1,2}_x$;
- $P^h(\mathbb{T}^2)$ with elements locally from $\mathscr{P}_j$ for $j \geq 0$.

Carelli-Prohl: the term $\frac{h^2}{\Delta t}$ disappears provided additionally

- Noise is solenoidal, that is $\operatorname{div}\mathbb{G} = 0$;
- Exactly divergence-free elements, that is $V^h(\mathbb{T}^2) \subset W^{1,2}_{\mathrm{div}}(\mathbb{T}^2)$.

The algorithm

Find r.v. $\mathbf{u}_m$ with values in $W^{1,2}_{\mathrm{div}}(\mathbb{T}^2)$ s.t. for all $\varphi \in W^{1,2}_{\mathrm{div}}(\mathbb{T}^2)$

$$\begin{aligned}\int_{\mathbb{T}^2} \mathbf{u}_m \cdot \varphi \,\mathrm{d}x + \Delta t \int_{\mathbb{T}^2} (\nabla \mathbf{u}_m)\mathbf{u}_{m-1} \cdot \varphi \,\mathrm{d}x \\ + \mu\, \Delta t \int_{\mathbb{T}^2} \nabla \mathbf{u}_m : \nabla \varphi \,\mathrm{d}x = \int_{\mathbb{T}^2} \mathbf{u}_{m-1} \cdot \varphi \,\mathrm{d}x \\ + \int_{\mathbb{T}^2} \mathbb{G}(\mathbf{u}_{m-1})\, \Delta_m W \cdot \varphi \,\mathrm{d}x\,.\end{aligned}$$

- Initial datum $\mathbf{u}_0 \in W^{1,2}_{\mathrm{div}}(\mathbb{T}^2)$.
- Here $\Delta_m W = W(t_m) - W(t_{m-1})$ where $t_m = m\frac{T}{M}$.

Uniform bounds

Brzeźniak-Carelli-Prohl (2013): if $\mathbf{u}_0 \in L^{2^q}(\Omega; W^{1,2}_{\mathrm{div}}(\mathbb{T}^2))$

$$\mathbb{E}\left[\max_{1\le m\le M} \|\mathbf{u}_m\|^{2^q}_{W^{1,2}_x} + \Delta t \sum_{k=1}^{M} \|\mathbf{u}_m\|^{2^q-2}_{W^{1,2}_x} \|\nabla^2 \mathbf{u}_m\|^2_{L^2_x}\right] \le c,$$

$$\mathbb{E}\left[\sum_{k=1}^{M} \|\mathbf{u}_m - \mathbf{u}_{m-1}\|^2_{W^{1,2}_x} \|\nabla \mathbf{u}_m\|^2_{L^2_x}\right] \le c,$$

$$\mathbb{E}\left[\left(\sum_{k=1}^{M} \|\mathbf{u}_m - \mathbf{u}_{m-1}\|^2_{W^{1,2}_x}\right)^4 + \left(\Delta t \sum_{k=1}^{M} \|\nabla \mathbf{u}_m\|^2_{L^2_x}\right)^4\right] \le c.$$

Error estimate

Breit-Dodgson (2019): for any $\alpha < \frac{1}{2}$

$$\mathbb{E}\left[\mathbf{1}_{\Omega^{\varepsilon}_{\Delta t}}\left(\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{u}_m\|^2_{L^2_x}+\sum_{m=1}^{M}\Delta t\|\nabla\mathbf{u}(t_m)-\nabla\mathbf{u}_m\|^2_{L^2_x}\right)\right] \le c\,(\Delta t)^{2\alpha}.$$

- $\operatorname{div}\mathbf{u}_m = 0$, so pressure is not seen;
- $\mathbf{1}_{\Omega^{\varepsilon}_{\Delta t}}$ controls blow-up of $\max_m \|\nabla\mathbf{u}_m\|_{L^2_x}$;
- Carelli-Prohl (2012): same estimate with $(\Delta t)^{\alpha}$. Difference:

$$\mathbb{E}\left[\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\int_{\mathbb{T}^2}\left|\nabla\mathbf{u}(t_m)-\nabla\mathbf{u}(t)\right|^2\mathrm{d}x\,\mathrm{d}t\right] \le c\,(\Delta t)^{2\alpha}.$$

Time-discrete pressure (1)

There is r.v. π_m with values in $\in L^2(\mathbb{T}^2)$ s.t. for all $\varphi \in W^{1,2}(\mathbb{T}^2)$

$$\int_{\mathbb{T}^2} \mathbf{u}_m \cdot \varphi \, \mathrm{d}x + \Delta t \int_{\mathbb{T}^2} (\nabla \mathbf{u}_m)\mathbf{u}_{m-1} \cdot \varphi \, \mathrm{d}x + \mu \, \Delta t \int_{\mathbb{T}^2} \nabla \mathbf{u}_m : \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{T}^2} \mathbf{u}_{m-1} \cdot \varphi \, \mathrm{d}x + \Delta t \int_{\mathbb{T}^2} \pi_m \, \mathrm{div}\, \varphi \, \mathrm{d}x + \int_{\mathbb{T}^2} \mathbb{G}(\mathbf{u}_{m-1}) \, \Delta_m W \cdot \varphi \, \mathrm{d}x \,.$$

Apply formerly div to get

$$\Delta \pi_m \Delta t = \mathrm{div}\,\mathrm{div}\,\big(\mathbf{u}_m \otimes \mathbf{u}_m\big) \Delta t + \mathrm{div}\,\big(\mathbb{G}(\mathbf{u}_{m-1}) \, \Delta_m W\big)$$

determine regularity of $\nabla \pi_m$.

Assumptions Carelli-Prohl

General case:

$$\Delta t \sum_{m=1}^{M} \mathbb{E}\|\nabla \pi_m\|_{L_x^2}^2 \sim (\Delta t)^{-1} \sum_{m=1}^{M} \mathbb{E}\|\mathbb{G}(\mathbf{u}_{m-1})\,\Delta_m W\|_{L_x^2}^2$$

$$\sim (\Delta t)^{-1} \sum_{m=1}^{M} \Delta t \sim (\Delta t)^{-1}.$$

Solenoidal noise ($\operatorname{div} \mathbb{G}(\mathbf{u}) = 0$):

$$\Delta t \sum_{m=1}^{M} \mathbb{E}\|\nabla \pi_m\|_{L_x^2}^2 \sim \Delta t \sum_{m=1}^{M} \mathbb{E}\|(\nabla \mathbf{u}_m)\mathbf{u}_m\|_{L_x^2}^2$$

$$\sim \Delta t \sum_{m=1}^{M} \mathbb{E}\|\mathbf{u}_m\|_{L_x^2}\|\nabla \mathbf{u}_m\|_{L_x^2}^2\|\nabla^2 \mathbf{u}_m\|_{L_x^2} \leq c.$$

Equation for the error

The error $\mathbf{e}_{h,m} = \mathbf{u}_m - \mathbf{u}_{h,m}$ satisfies for all $\varphi \in V^h_{\text{div}}(\mathbb{T}^2)$

$$\int_{\mathbb{T}^2} \mathbf{e}_{h,m} \cdot \varphi \,\mathrm{d}x + \mu\,\Delta t \int_{\mathbb{T}^2} \nabla \mathbf{e}_{h,m} : \nabla \varphi \,\mathrm{d}x = \int_{\mathbb{T}^2} \mathbf{e}_{h,m-1} \cdot \varphi \,\mathrm{d}x$$
$$+ \cdots + \Delta t \int_{\mathbb{T}^2} \pi_m \,\mathrm{div}\, \varphi \,\mathrm{d}x$$
$$+ \int_{\mathbb{T}^2} \big(\mathbb{G}(\mathbf{u}_{m-1}) - \mathbb{G}(\mathbf{u}_{h,m-1})\big)\, \Delta_m W \cdot \varphi \,\mathrm{d}x\,.$$

- Use test-function $\Pi_h \mathbf{e}_{h,m} = \Pi_h \mathbf{u}_m - \mathbf{u}_{h,m}$;
- Carelli-Prohl: critical term is

$$\Delta t \sum_{m=1}^{M} \|\pi_m - \Pi_h^\pi \pi_m\|^2_{L^2_x} \leq c\, h^2 \Delta t \sum_{m=1}^{M} \|\nabla \pi_m\|^2_{L^2_x}.$$

Stochastic pressure decomposition

The error $\mathbf{e}_{h,m} = \mathbf{u}_m - \mathbf{u}_{h,m}$ satisfies for all $\varphi \in V^h_{\text{div}}(\mathbb{T}^2)$

$$\begin{aligned}\int_{\mathbb{T}^2} \mathbf{e}_{h,m} \cdot \varphi \, \mathrm{d}x + \mu \, \Delta t \int_{\mathbb{T}^2} \nabla \mathbf{e}_{h,m} : \nabla \varphi \, \mathrm{d}x &= \int_{\mathbb{T}^2} \mathbf{e}_{h,m-1} \cdot \varphi \, \mathrm{d}x \\ &+ \cdots + \Delta t \int_{\mathbb{T}^2} \pi_m^{det} \ \operatorname{div} \varphi \, \mathrm{d}x \\ &+ \int_{\mathbb{T}^2} \left(\mathbb{G}(\mathbf{u}_{m-1}) - \mathbb{G}(\mathbf{u}_{h,m-1}) \right) \Delta_m W \cdot \varphi \, \mathrm{d}x \\ &- \int_{\mathbb{T}^2} \nabla \Delta^{-1} \operatorname{div} \mathbb{G}(\mathbf{u}_{m-1})) \, \Delta_m W \cdot \varphi \, \mathrm{d}x\end{aligned}$$

Deterministic pressure $\pi_m^{det} = \Delta^{-1} \operatorname{div} \operatorname{div} \left(\mathbf{u}_m \otimes \mathbf{u}_m \right)$.

Stochastic pressure estimate (1)

Testing with $\varphi = \Pi_h \mathbf{e}_{h,m}$

$$\begin{aligned}
\mathscr{M}_{m,2} &= \sum_{n=1}^{m} \int_{\mathbb{T}^2} \int_{t_{n-1}}^{t_n} \big(\mathrm{Id} - \Pi_h^{\pi}\big) \Delta^{-1} \operatorname{div} \mathbb{G}(\mathbf{u}_{n-1}) \, \mathrm{d}W \ \operatorname{div} \Pi_h \mathbf{e}_{h,n} \, \mathrm{d}x \\
&= \sum_{n=1}^{m} \int_{\mathbb{T}^2} \int_{t_{n-1}}^{t_n} \big(\mathrm{Id} - \Pi_h^{\pi}\big) \Delta^{-1} \operatorname{div} \mathbb{G}(\mathbf{u}_{n-1}) \, \mathrm{d}W \ \operatorname{div} \Pi_h \mathbf{e}_{h,n-1} \, \mathrm{d}x \\
&+ \sum_{n=1}^{m} \int_{\mathbb{T}^2} \int_{t_{n-1}}^{t_n} \big(\mathrm{Id} - \Pi_h^{\pi}\big) \Delta^{-1} \operatorname{div} \mathbb{G}(\mathbf{u}_{n-1}) \, \mathrm{d}W \ \operatorname{div} \Pi_h (\mathbf{e}_{h,n} - \mathbf{e}_{h,n-1}) \, \mathrm{d}x \\
&=: \mathscr{M}_{m,2}^1 + \mathscr{M}_{m,2}^2 .
\end{aligned}$$

Stochastic pressure estimate (2)

$$
\begin{aligned}
&\mathbb{E}\left[\max_{1\le m\le M}\left|\mathscr{M}^1_{m,2}\right|\right]\\
&\le\; c\,\mathbb{E}\left[\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\left\|\left(\mathrm{Id}-\Pi_h^{\pi}\right)\Delta^{-1}\operatorname{div}\Phi(\mathbf{u}_{n-1})\right\|_{L_2}^2\left\|\nabla\Pi_h\mathbf{e}_{h,n-1}\right\|_{L_x^2}^2\right]^{\frac{1}{2}}\\
&\le\; c\,h^2\,\mathbb{E}\left[\max_n\left\|\nabla^2\Delta^{-1}\operatorname{div}\Phi(\mathbf{u}_{n-1})\right\|_{L_2}\left(\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\left\|\nabla\Pi_h\mathbf{e}_{h,n-1}\right\|_{L_x^2}^2\right)^{\frac{1}{2}}\right]\\
&\le\; c_\kappa h^4\,\mathbb{E}\left[\max_n\left\|\nabla\mathbf{u}_{n-1}\right\|_{L_x^2}^2\right]+\kappa\,\mathbb{E}\left[\Delta t\sum_{n=1}^{M}\left\|\nabla\mathbf{e}_{h,n-1}\right\|_{L_x^2}^2\right].
\end{aligned}
$$

Stochastic pressure estimate (3)

$$
\begin{aligned}
&\leq \kappa h^2\, \mathbb{E}\Bigg[\sum_{n=1}^{M} \big\|\nabla\big(\Pi_h \mathbf{e}_{h,n} - \Pi_h \mathbf{e}_{h,n-1}\big)\big\|_{L^2_x}^2\Bigg] \\
&+ c_\kappa h^{-2}\, \mathbb{E}\Bigg[\sum_{n=1}^{M} \Bigg\| \int_{t_{n-1}}^{t_n} \big(\mathrm{Id} - \Pi_h^{\pi}\big)\Delta^{-1}\operatorname{div} \Phi(\mathbf{u}_{n-1})\, \mathrm{d}W \Bigg\|_{L^2_x}^2\Bigg] \\
&\leq c\,\kappa\, \mathbb{E}\Bigg[\sum_{n=1}^{M} \big\|\Pi_h \mathbf{e}_{h,n} - \Pi_h \mathbf{e}_{h,n-1}\big\|_{L^2_x}^2\Bigg] \\
&+ c_\kappa h^{-2}\, \mathbb{E}\Bigg[\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \big\|\big(\mathrm{Id} - \Pi_h^{\pi}\big)\Delta^{-1}\operatorname{div} \Phi(\mathbf{u}_{n-1})\big\|_{L_2(\mathfrak{U};L^2_x)}^2\, \mathrm{d}t\Bigg] \\
&\leq \cdots + c_\kappa h^2\, \mathbb{E}\Bigg[\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \big\|\nabla^2\Delta^{-1}\operatorname{div} \Phi(\mathbf{u}_{n-1})\big\|_{L_2(\mathfrak{U};L^2_x)}^2\, \mathrm{d}t\Bigg].
\end{aligned}
$$