Singular limits for compressible fluids with stochastic forcing

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Navier-Stokes equations

Find velocity $\mathbf{u} : Q \to \mathbb{R}^d$ and density $\varrho : Q \to \mathbb{R}$ satisfying the

system of partial differential equations

$$
\begin{cases}
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div} \mathbf{S} - a \nabla \varrho^\gamma & \text{in } Q, \\
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 & \text{in } Q, \\
\mathbf{u} = 0 & \text{on } \partial G, \\
\varrho(0, \cdot) = \varrho_0 & \text{in } G, \\
\varrho(0, \cdot)\mathbf{u}(0, \cdot) = \mathbf{q}_0 & \text{in } G,
\end{cases}
$$

- $Q := (0, T) \times G$ with $G \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and $T > 0$;
- $\mathbf{S} : Q \to \mathbb{R}^{d \times d}$ is given by Newton's law

$$\mathbf{S} = \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \operatorname{div} \mathbf{u}\, I \right) + \eta \operatorname{div} \mathbf{u}\, I.$$

Momentum equation

Weak formulation of the momentum equation:

find $\mathbf{v}$, ϱ such that for all $\varphi \in C_0^\infty(Q)$

$$\int_Q \varrho \mathbf{u} \cdot \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t = \mu \int_Q \nabla \mathbf{u} : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t + (\eta + \mu) \int_Q \operatorname{div} \mathbf{u} \operatorname{div} \varphi$$
$$- \int_Q (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_Q a \varrho^\gamma \operatorname{div} \varphi \, \mathrm{d}x \, \mathrm{d}t$$

- Function space for weak solutions

$$\mathbf{v} \in L^2(0, T; W_0^{1,2}(G)),$$
$$\varrho \in C_w([0, T]; L^\gamma(G)).$$

Continuity equation

Renormalized formulation of the continuity equation (DiPerna-Lions, '89):

ϱ satisfies for all $\psi \in C_0^\infty(Q)$

$$\int_Q b(\varrho)\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t = \int_Q b(\varrho)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_Q \big(b'(\varrho)\varrho - b(\varrho)\big)\,\mathrm{div}\,\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t$$

- $b \in C^1(\mathbb{R})$ with $b'(z) = 0$ for all $z \geq M(b)$;
- ϱ solves weak formulation too ($b(z) = z$).

Known results

- Existence of weak solutions for $\gamma \geq \frac{9}{5}$ by Lions (1998);
- Existence of weak solutions for $\gamma > \frac{3}{2}$ by Feireisl, Novotný, Petzeltová (2001);
- We have

$$\varrho \in L^{\frac{5}{3}\gamma - 1}(Q), \quad \mathbf{u} \in L^2(0, T; W_0^{1,2}(G)),$$
$$\varrho \mathbf{u} \in C_w([0, T]; L^{\frac{2\gamma}{\gamma+1}}(G)),$$
$$\varrho \mathbf{u} \otimes \mathbf{u} \in L^2(0, T; L^{\frac{6\gamma}{4\gamma+3}}(G)).$$

Stochastic Navier-Stokes equations

Velocity field $\mathbf{v}$ and density ϱ on $Q = (0,T) \times \mathbb{T}^3$

$$\begin{cases} \mathrm{d}(\varrho \mathbf{u}) = [\mu \Delta \mathbf{u} - \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + a \nabla \varrho^\gamma]\,\mathrm{d}t + \Phi\,\mathrm{d}W & \text{in } Q, \\ \mathrm{d}\varrho = -\,\mathrm{div}(\varrho \mathbf{u})\,\mathrm{d}t & \text{in } Q, \\ \varrho(0,\cdot) = \varrho_0 & \text{in } \mathbb{T}^3, \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{q}_0 & \text{in } \mathbb{T}^3. \end{cases}$$

- Momentum equation in the weak sense: $\int_{\mathbb{T}^3} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}\,\mathrm{d}x = \ldots$ for all $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3)$;
- Mass equation in the renormalizes sense: $\mathrm{d}b(\varrho) = \ldots$ for all $b \in C^1(\mathbb{R})$ with $b'(z) = 0$ for all $z \geq M_b$.

Known Results

- If we have no uniqueness for the deterministic problem we expect martingale solutions, i.e. there is a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}})$ and a Brownian motion $\tilde{W}$ on it such that a weak solutions $(\tilde{\varrho}, \tilde{\mathbf{u}})$...
- Many literature for stochastic Navier Stokes equations starting with Bensoussan-Temam in 1973;
- Knowledge about compressible fluids very limited: only the case $\Phi = \Phi(\varrho) = \varrho$ is considered (Tornatore for $d = 2$ in 2000; Feireisl, Maslowski, Novotný for $d = 3$ in 2013).

Main result

Theorem (Breit & Hofmanová, to appear in IUMJ)

Under appropriate assumptions on the initial law Λ there is a finite energy weak martingale solution to the stochastic compressible Navier-Stokes equations provided $\gamma > \frac{3}{2}$ substract to periodic boundary conditions.

- Adaption of theory by Feireisl et all to the stochastic setting;
- Wiener process $W = \sum_k \beta_k \mathbf{e}_k$, $(\mathbf{e}_k)$ ONB of HS $\mathfrak{U}$;
- General noise $\varPhi = \varPhi(\varrho, \varrho\mathbf{u}) \in L^2(0, T; L_2(\mathfrak{U}; W^{-l,2}(\mathbb{T}^3)))$ s.t.

$$\begin{aligned} \varPhi(\varrho, \varrho\mathbf{u})\mathbf{e}_k &= g_k(\cdot, \varrho, \varrho\mathbf{u}), \\ g_k(\varrho, \mathbf{q}) &= h_{1,k}(\cdot, \varrho) + h_{2,k}(\cdot, \mathbf{q}). \end{aligned}$$

Concept of solution (1)

A finite energy weak martingale solution is a quantity

$$((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \varrho, \mathbf{u}, W).$$

- $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis,
- W is an $(\mathscr{F}_t)$-cylindrical Wiener process,
- $\varrho \geq 0$ is $(\mathscr{F}_t)$-adapted and $\varrho \in C_w([0, T]; L^\gamma(\mathbb{T}^3))$ a.s.,
- $\mathbf{u}$ is $(\mathscr{F}_t)$-adapted and $\mathbf{u} \in L^2(0, T; W^{1,2}(\mathbb{T}^3))$ a.s.,
- the momentum $\varrho\mathbf{u} \in C_w([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))$ a.s.,
- $\Lambda = \mathbb{P} \circ \big(\varrho(0), \varrho\mathbf{u}(0)\big)^{-1}$.

Concept of solution (2)

- $(\varrho, \mathbf{u})$ solves the momentum equation in the weak sense $\mathbb{P}$-a.s.
- $(\varrho, \mathbf{u})$ solves the continuity equation in the renormalized sense $\mathbb{P}$-a.s.
- We have the energy inequality

$$\mathrm{d}\int_{\mathbb{T}^3}\left(\frac{1}{2}\varrho(t)\big|\mathbf{u}(t)\big|^2+\frac{a}{\gamma-1}\varrho^{\gamma}(t)\right)\mathrm{d}x\,\mathrm{d}t+\mu\int_{\mathbb{T}^3}|\nabla\mathbf{u}|^2\,\mathrm{d}x$$
$$\leq \mathrm{d}M+\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\frac{|g_k(\varrho,\varrho\mathbf{u})|^2}{\varrho}\,\mathrm{d}x\,\mathrm{d}t$$

for some real-valued $(\mathscr{F}_t)$-martingale M.

Strategy of the proof

- Four layer approximation;
- Add artificial pressure $\delta\varrho^{\beta}$ for $\beta > 4$;
- Add artificial viscosity $\varepsilon\Delta\varrho$ in continuity equation;
- Solve problem with artificial viscosity and pressure via Faedo-Galerkin approximation;
- Faedo-Galerkin approximation via stopping time, then stopping time $\to T$.
- Use Jakubowski-Skorokhod representations theorem for compactness; applies to quasi-polish spaces (e.g. Banach spaces with weak topology).

Formulation of the problem

System of stochastic partial differential equations in Q

$$\left\{\begin{aligned} \mathrm{d}(\varrho\mathbf{u}) &= [\mu\Delta\mathbf{u} - \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \tfrac{1}{\varepsilon^2}\nabla\varrho^\gamma]\,\mathrm{d}t + \Phi(\varrho,\varrho\mathbf{u})\,\mathrm{d}W \\ \mathrm{d}\varrho &= -\,\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}t \\ \varrho(0,\cdot) &= \varrho_0 \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) &= \mathbf{q}_0 \end{aligned}\right. \tag{3.1}$$

- Scaling arguments (large time-scales, small velocity).
- Suppose that ϱ_0 is constant or converges to a constant.
- What happens if $\varepsilon \to 0$?

Assumptions

Let u has initial law Λ and solves in $C^\infty_{\mathrm{div}}(\mathbb{T}^3)'$

$$\begin{cases} \mathrm{d}\mathbf{u} = [\mu\Delta\mathbf{u} - \mathrm{div}(\mathbf{u}\otimes\mathbf{u})]\,\mathrm{d}t + \Psi(\mathbf{u})\,\mathrm{d}W & \text{in } Q, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } Q, \end{cases} \tag{3.2}$$

- Here $\Psi(\mathbf{u}) = \mathcal{P}_H\Phi(1,\mathbf{u})$ with Helmholtz-projection $\mathcal{P}_H$.
- Let Λ be a given Borel probability measure on $L^2(\mathbb{T}^3)$.
- Let Λ_ε be a Borel probability measure on $L^\gamma(\mathbb{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$
- The marginal law of Λ_ε corresponding to the second component converges to Λ weakly as a measures on $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$.

Main result (d=3)

Theorem (Breit, Feireisl, Hofmanová)

If $\big((\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon), \mathbb{P}^\varepsilon), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon\big)$ is a finite energy weak martingale solution to (3.1) with the initial law Λ_ε, $\varepsilon \in (0,1)$, then

$$\varrho_\varepsilon \to 1 \quad \text{in law on} \quad L^\infty(0,T;L^\gamma(\mathbb{T}^3)),$$
$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{in law on} \quad \big(L^2(0,T;W^{1,2}(\mathbb{T}^3)), w\big),$$

where $\mathbf{u}$ is a weak martingale solution to (3.2) with initial law Λ.

- Find suitable uniform bounds independent of ε.
- Analysis of rapidly oscillating *acoustic waves*.
- Deterministic result by Lions-Masmoudi in 1998/'99.

Main result (d=2)

Theorem (Breit, Feireisl, Hofmanová)

If $\big((\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon), \mathbb{P}^\varepsilon), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon\big)$ is a finite energy weak martingale solution to (3.1) with the initial law Λ_ε, $\varepsilon \in (0,1)$, then

$$\begin{aligned} \varrho_\varepsilon \to 1 \quad &\text{in} \quad L^\infty(0,T; L^\gamma(\mathbb{T}^2)) \quad \mathbb{P}\text{-a.s.}, \\ \mathbf{u}_\varepsilon \to \mathbf{u} \quad &\text{in} \quad \big(L^2(0,T; W^{1,2}(\mathbb{T}^2)), w\big) \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where $\mathbf{u}$ is the pathwise solution to (3.2) with initial condition $\mathbf{u}_0$.

- The result is based on uniqueness for (3.2) if $d = 2$.
- Also holds if $d = 3$ locally in time.

Formulation of the problem

system of stochastic partial differential equations

$$\begin{cases} \mathrm{d}(\varrho\mathbf{u}) = [\varepsilon\Delta\mathbf{u} - \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla\varrho^\gamma]\,\mathrm{d}t + \Phi\,\mathrm{d}W \\ \mathrm{d}\varrho = -\,\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}t \\ \varrho(0,\cdot) = \varrho_0 \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{q}_0 \end{cases} \tag{4.3}$$

- Inviscid limit: viscosity tends to zero.
- Suppose that ϱ_0 is constant or converges to a constant.
- What happens if $\varepsilon \to 0$?

Assumptions

Let $\mathbf{u}$ has initial law Λ and solves in $C^\infty_{\mathrm{div}}(\mathbb{T}^3)'$

$$\begin{cases} \mathrm{d}\mathbf{u} = -\operatorname{div}(\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}t + \Psi(\mathbf{u})\,\mathrm{d}W & \text{in } Q, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } Q, \end{cases} \tag{4.4}$$

- Here $\Psi(\mathbf{u}) = \mathcal{P}_H\Phi(1,\mathbf{u})$ with Helmholtz-projection $\mathcal{P}_H$.
- Let Λ be a given Borel probability measure on $L^2(\mathbb{T}^3)$.
- Let Λ_ε be a Borel probability measure on $L^\gamma(\mathbb{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$
- The marginal law of Λ_ε corresponding to the second component converges to Λ weakly as a measures on $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$.

Main result

Theorem (Breit, Feireisl, Hofmanová)

If $\big((\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon), \mathbb{P}^\varepsilon), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon\big)$ is a finite energy weak martingale solution to (4.3) with the initial law Λ_ε, $\varepsilon \in (0,1)$, then

$$\mathbb{E}\int_0^{T\wedge\tau} \|\mathbf{u}_\varepsilon - \mathbf{u}\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}t \longrightarrow 0,$$

$$\mathbb{E}\int_0^{T\wedge\tau} \|\varrho_\varepsilon - 1\|^\gamma_{L^\gamma(\mathbb{T}^3)}\,\mathrm{d}t \longrightarrow 0,$$

where $\mathbf{u}$ is the pathwise solution to (4.4) with initial condition $\mathbf{u}_0$.

- Requires smooth solutions to (4.4): up to stopping time τ.
- Works only for periodic boundary, otherwise boundary layers.

Relative energy inequality

Quantifies “distance” between weak solution $(\varrho, \mathbf{u})$ and

some smooth functions r and $\mathbf{U}$

$$\mathrm{d}\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\Big) + \int_{\mathbb{T}^3} \big(\mathbf{S}(\nabla\mathbf{u}) - \mathbf{S}(\nabla\mathbf{U})\big) : (\nabla\mathbf{u} - \nabla\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \mathrm{d}M + \mathcal{R}\Big([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\Big)\,\mathrm{d}t.$$

- Relative energy functional $\mathcal{E}$

$$\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\Big) = \int_{\mathbb{T}^3} \left[\varrho|\mathbf{u}-\mathbf{U}|^2 + a\frac{\varrho^\gamma - r^\gamma - (\gamma-1)r^{\gamma-1}(\varrho - r)}{\gamma-1}\right]\mathrm{d}x\,.$$

- Remainder $\mathcal{R}$ simplifies if $(r, \mathbf{U})$ solves...

Pathwise weak-strong uniqueness

Theorem (Breit, Feireisl, Hofmanová)

Let $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \varrho, \mathbf{u}, W]$ be a finite energy weak martingale solution and let $(\tilde{\varrho}, \tilde{\mathbf{u}})$ and a stopping time τ be a strong solution of the same problem defined on the same stochastic basis with the same Wiener process and the same initial data. Then
$\varrho(\cdot \wedge \tau) = \tilde{\varrho}(\cdot \wedge \tau)$ and $\varrho\mathbf{u}(\cdot \wedge \tau) = \tilde{\varrho}\tilde{\mathbf{u}}(\cdot \wedge \tau)$ a.s.

- If a strong solution exists it is unique!
- Only holds up to stopping time τ.
- Requires certain integrability of $\nabla\tilde{\mathbf{u}}$ and $\nabla^2\tilde{\mathbf{u}}$.

Weak-strong uniqueness in law

What if $(\varrho, \mathbf{u})$ and $(\tilde{\varrho}, \tilde{\mathbf{u}})$ are defined on different probability spaces?

Theorem (Breit, Feireisl, Hofmanová): Let

$$\Big[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \varrho, \mathbf{u}, W\Big], \quad \Big[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W}\Big]$$

be a finite energy weak martingale solution and a strong solution, respectively. If they have the same initial law then

$$\mathbb{P} \circ (\varrho, \varrho\mathbf{u})^{-1} = \tilde{\mathbb{P}} \circ (\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}})^{-1}.$$

- If a strong solution exists its law is unique!
- Similar to Yamada–Watanabe type results for SDEs.