Pointwise gradient estimates for the p-Laplacian

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Outline

1. Introduction to p-Laplacian
2. Oscilations & Potential estimates
3. Pointwise estimates
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p-Laplacian system (1)

Find a velocity field $\mathbf{u} : \Omega \to \mathbb{R}^N$ satisfying the following

system of partial differential equations

$$\begin{cases} \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \mathbf{f} = \operatorname{div}\mathbf{F} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Assume $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain and $p \in (1, \infty)$;
- Weak solution $\mathbf{u} \in W_0^{1,p}(\Omega)$ exists provided $\mathbf{F} \in L^{p'}(\Omega)$;
- How does the regularity of $\mathbf{F}$ transfers to $\mathbf{u}$?

p-Laplacian system (2)

Weak formulation for $\mathbf{u} \in W_0^{1,p}(\Omega)$ and all $\varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u} : \nabla \varphi \, \mathrm{d}x = - \int_\Omega \mathbf{f} \cdot \varphi \, \mathrm{d}x = \int_\Omega \mathbf{F} : \nabla \varphi \, \mathrm{d}x \, .$$

- Existence of weak solutions by monotone operator theory or by minimizing

$$\mathcal{J}[\mathbf{u}] := \frac{1}{p} \int_\Omega |\nabla \mathbf{u}|^p \, \mathrm{d}x - \int_\Omega \mathbf{F} : \nabla \mathbf{u} \, \mathrm{d}x \, .$$

Classical results

Regularity theory for p-harmonic functions

$$\operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = 0 \quad \text{in} \quad \Omega.$$

- Ural'tseva ('68), $N = 1$: $u \in C^{1,\alpha}(\Omega)$ for some $\alpha < 1$;
- Uhlenbeck ('77), $N \geq 2$ and $p > 2$: $\mathbf{u} \in C^{1,\alpha}(\Omega)$;
- Acerbi-Fusco/DiBenedetto/Manfredi/Tolksdorff ('83-'87), $N \geq 2$ and $p < 2$: $\mathbf{u} \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$;
- Manfredi-Iwaniec ('89), $n = 2$: optimal values for α.

L^q- Theory

We have for $q \geq p'$

$$\operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \operatorname{div}\mathbf{F},$$
$$\mathbf{F} \in L^q \;\; \Rightarrow \;\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in L^q.$$

- Iwaniec ('83), $N = 1$: by nonlinear Calderon-Zygmund theory;
- DiBenedetto-Manfredi ('93), $N \geq 2$;
- Generalizations: Kinnuen-Zhou ('99/'01);
- Parabolic problems: Acerbi-Mingione ('07).

BMO-Estimates (1)

What happens for $q = \infty$?

$$\mathcal{M}^{\sharp,s} f(x) := \sup_{B \ni x} \left(\fint_B |f - (f)_B|^s dy \right)^{\frac{1}{s}},$$

$$f \in \mathrm{BMO} :\Leftrightarrow \|\mathcal{M}^{\sharp} f\|_\infty < \infty.$$

- We have $L^\infty \subsetneq \mathrm{BMO} \subsetneq \cap_q L^q$;
- Singular integrals are continuous on BMO;
- Campanato spaces C^α as weighted BMO_ω-spaces:

$$\mathcal{M}^{\sharp}_\omega f(x) := \sup_{B \ni x} \frac{1}{\omega(r_B)} \fint_B |f - (f)_B| dy, \quad \omega(r) = r^\alpha.$$

BMO-Estimates (2)

What happens for $q = \infty$?

$$\operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \operatorname{div}\mathbf{F},$$
$$\mathbf{F} \in \mathrm{BMO} \ \Rightarrow \ |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathrm{BMO}.$$

- DiBenedetto-Manfredi ('93), $p > 2$:
  $\mathbf{F} \in \mathrm{BMO} \ \Rightarrow \ \nabla \mathbf{u} \in \mathrm{BMO}$;
- Diening-Kaplický-Schwarzacher ('12), $p > 1$:
  $\mathbf{F} \in \mathrm{BMO}_\omega \ \Rightarrow \ |\nabla \mathbf{u}|^{p-2}\mathbf{u} \in \mathrm{BMO}_\omega$;
- Parabolic versions by Schwarzacher ('14).

Continuous gradients

Minimal assumption for bounded gradients

$$\mathrm{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \mathbf{f},$$
$$\mathbf{f} \in L^{n,1} \;\Rightarrow\; \nabla \mathbf{u} \in C^0.$$

- Lorentz-space $L^{n,1}$ with $L^{n+\varepsilon} \subsetneq L^{n,1} \subsetneq L^n$;
- Stein ('81) $p = 2$, $\mathbf{f} \in L^{n,1}$ is optimal: Cianchi ('92);
- $p > 1$ by Maz'ya-Cianchi/Kuusi-Mingione ('14).

Potential estimates

Let μ be a Radon measure

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f = \mu,$$

$$|\nabla u(x)|^{p-1} \leq c\int_0^R \frac{|\mu|(B_\rho(x))}{\rho^{n-1}}\frac{d\rho}{\rho} + \left(\fint_{B_R(x)} |\nabla u|\, dy\right)^{p-1}.$$

- Kuusi-Mingione ('13): linear potentials;
- Kilpelainen-Malý ('92/'94): Nonlinear potential estimates;
- Parabolic versions by Kuusi-Mingione ('14).

Main result

Breit-Cianchi-Diening-Kuusi-Schwarzacher: $\forall x \in B_R$

$$M_R^\sharp(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})(x) \le c\, M_R^{\sharp,p'}(\mathbf{F})(x) + c\left(\fint_{B_{2R}} \left||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} - (|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})_{B_{2R}}\right|^{p'} dy \right)^{\frac{1}{p'}}.$$

- Note that $\|\mathcal{M}^\sharp(f)\|_\infty \sim \|\mathcal{M}^{\sharp,s}(f)\|_\infty$;
- For $q > s$ we have $\|\mathcal{M}^{\sharp,s}(f)\|_q \sim \|f\|_q$;
- Also for weighted $\mathcal{M}^\sharp_\omega$.

Known estimates

Recovering of known local results:

- $\mathbf{F} \in L^q \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in L^q$ for $q > p'$;
- $\mathbf{F} \in \mathrm{BMO} \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathrm{BMO}$;
- $\mathbf{F} \in \mathrm{VMO} \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathrm{VMO}$;
- $\mathbf{F} \in C^\alpha \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in C^\alpha$.

Main tool: continuity of $\mathcal{M}^\sharp$.

Rearrangement invariant function spaces (1)

Let $\|\cdot\|_{X(\mathbb{R}^n)}$ and $\|\cdot\|_{Y(\mathbb{R}^n)}$ be rearrangement-invariant functionals satisfying (3.1) and (3.2).

Let $\mathbf{u}$ be a local weak solution on $\mathbb{R}^n$, then

$$\||\nabla \mathbf{u}|^{p-1}\|_{Y(\mathbb{R}^n)} \leq c' \|c' \mathbf{F}\|_{X(\mathbb{R}^n)}.$$

For every nonnegative function $\varphi \in X(0,\infty)$ we have

$$\left\| \frac{1}{s}\int_0^s \varphi(r)dr \right\|_{X^{\frac{1}{p'}}(0,\infty)} \leq c\|c\varphi\|_{X^{\frac{1}{p'}}(0,\infty)}, \tag{3.1}$$

$$\left\| \int_s^\infty \varphi(r)\frac{dr}{r} \right\|_{Y(0,\infty)} \leq c\|c\varphi\|_{X(0,\infty)}. \tag{3.2}$$

Rearrangement invariant function spaces (2)

- Lebesgue spaces: for $q > p'$

$$\mathbf{F} \in L^q \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in L^q;$$

- Orlicz-spaces: for Ψ, Φ satisfying...

$$\mathbf{F} \in L^\Psi \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in L^\Phi,$$
$$\mathbf{F} \in \mathrm{Exp}^\gamma \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in \mathrm{Exp}^{\gamma/(\gamma+1)};$$

- Lorentz spaces: for $q > p'$ and $r \in [1, \infty]$

$$\mathbf{F} \in L^{q,r} \;\Rightarrow\; |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in L^{q,r}.$$

Potential estimates (1)

Assume that $\mathbf{F} \in L^{p'}_{\mathrm{loc}}(\Omega)$: $\forall x \in B_R$, $B_R(x) \subset \Omega$

$$|\nabla \mathbf{u}(x)|^{p-1} \leq c \int_0^R \left(\fint_{B_r(x)} \left(\frac{|\mathbf{F}(y) - (\mathbf{F})_{B_r(x)}|}{r} \right)^{p'} dy \right)^{\frac{1}{p'}} dr$$
$$+ c \fint_{B_R(x)} |\nabla \mathbf{u}|^{p-1} \, dy$$

- Lebesgue point of $|\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u}$ whenever the r.h.s. is finite;
- $\nabla \mathbf{u}$ bounded iff $\mathbf{F}$ is Dini-continuous.

Potential estimates (2)

Assume that $\int_0 \frac{\omega(r)}{r}\,dr = \infty$

$$\mathbf{u}(x) = x_2\psi(|x|) \quad \text{for } x \in B_1(0),$$
$$\psi(r) = -\int_r^1 \frac{\omega(r)}{\rho}\,d\rho \quad \text{for } r \in (0,1).$$

We have $\Delta\mathbf{u} = \operatorname{div}\mathbf{F}$ with

$$\nabla\mathbf{u}(x) = \Big(\frac{x_1x_2}{|x|^2}\omega(|x|), \psi(|x|) + \frac{x_2^2}{|x|^2}\omega(|x|)\Big) \notin L^\infty_{\mathrm{loc}}(B_1(0)),$$
$$\mathbf{F}(x) = \Big(\frac{2x_1x_2}{|x|^2}\omega(|x|), \frac{x_2^2 - x_1^2}{|x|^2}\omega(|x|)\Big) \in C^\omega(B_1(0)).$$

The linear case (1)

Poisson equation

$$\begin{cases} \Delta \mathbf{u} = \operatorname{div} \mathbf{F} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

We want to show that for all $\delta > 0$ there is $\theta \in (0,1)$ s.t.

$$\fint_{\theta B} |\nabla \mathbf{u} - (\nabla \mathbf{u})_{\theta B}|^2 \, dy \leq \delta \fint_{B} |\nabla \mathbf{u} - (\nabla \mathbf{u})_{B}|^2 \, dy + c(\delta) \fint_{B} |\mathbf{F} - (\mathbf{F})_{B}|^2 \, dy.$$

The linear case (2)

Compare with the solution **h** to a homogenuous system.

Poisson equation

$$\begin{cases} \Delta \mathbf{h} = 0 & \text{in } B, \\ \mathbf{h} = \mathbf{u} & \text{on } \partial B. \end{cases}$$

Harmonic functions satisfy the Campanato-estimates for $\theta \in (0, \frac{1}{2})$

$$\fint_{\theta B} |\nabla \mathbf{h} - (\nabla \mathbf{h})_{\theta B}|^2 \, dy \leq c\theta^2 \fint_{B} |\nabla \mathbf{h} - (\nabla \mathbf{h})_{B}|^2 \, dy.$$

The linear case (3)

$$
\begin{aligned}
\fint_{\theta B} |\nabla \mathbf{u} - (\nabla \mathbf{u})_{\theta B}|^2 \, dy &\leq \fint_{\theta B} |\nabla \mathbf{h} - (\nabla \mathbf{h})_{\theta B}|^2 \, dy + \fint_{\theta B} |\nabla \mathbf{h} - \nabla \mathbf{u}|^2 \, dy \\
&\leq c\theta^2 \fint_{B} |\nabla \mathbf{h} - (\nabla \mathbf{h})_{B}|^2 \, dy + c\,\theta^{-n} \fint_{B} |\nabla \mathbf{h} - \nabla \mathbf{u}|^2 \, dy \\
&\leq c\theta^2 \fint_{B} |\nabla \mathbf{u} - (\nabla \mathbf{u})_{B}|^2 \, dy + c\,\theta^{-n} \fint_{B} |\nabla \mathbf{h} - \nabla \mathbf{u}|^2 \, dy \\
&\leq c\theta^2 \fint_{B} |\nabla \mathbf{u} - (\nabla \mathbf{u})_{B}|^2 \, dy + c\,\theta^{-n} \fint_{B} |\mathbf{F} - (\mathbf{F})_{B}|^2 \, dy
\end{aligned}
$$

Comparison

Compare with the solution $\mathbf{h}$ to a homogenuous system.

For a ball $B \Subset \Omega$ and a small ε (choosen later), $\mathbf{V}(\xi) = |\xi|^{\frac{p-2}{2}}\xi$

$$(I) \quad \fint_B |\mathbf{V}(\nabla \mathbf{u}) - (\mathbf{V}(\nabla \mathbf{u}))_B|^2\, dy \leq \varepsilon \fint_B |\nabla \mathbf{u}|^p\, dy,$$

$$(II) \quad \fint_B |\nabla \mathbf{u}|^p\, dy \leq \frac{1}{\varepsilon} \fint_B |\mathbf{V}(\nabla \mathbf{u}) - (\mathbf{V}(\nabla \mathbf{u}))_B|^2\, dy.$$

- (I) non-degenerate case: $|\nabla \mathbf{u}| > \delta$;
- (II) degenerate case: $|\nabla \mathbf{u}| \leq \delta$.

Degenerate points

Compare with the solution $\mathbf{h}$ to the nonlinear system.

For a ball $B \Subset \Omega$

$$\operatorname{div}\left(|\nabla \mathbf{h}|^{p-2}\nabla \mathbf{h}\right) = 0 \quad \text{in} \quad B,$$
$$\mathbf{h} = \mathbf{u} \quad \text{on} \quad \partial B.$$

We show for $\varphi_{p',a}(t) := \int_0^t (a+s)^{p'-2} s\, ds$ and $\mathbf{A}(\xi) = |\xi|^{p-2}\xi$

$$\fint_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{h})|^2\, dx \leq c_\delta \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^{p'}\, dx$$
$$+ \delta\, \varphi_{p',|(\mathbf{A}(\nabla \mathbf{u}))_{2B}|}\left(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - (\mathbf{A}(\nabla \mathbf{u}))_{2B}|\, dx\right).$$

Non-degenerate points

For $\mathbf{A}(\xi) = |\xi|^{p-2}\xi$ and $\mathbf{Q} = \fint_B \nabla \mathbf{u}\, dx$, linear comparison

$$\begin{aligned} \operatorname{div}\big(D\mathbf{A}(\mathbf{Q})\nabla \mathbf{h}\big) &= 0 \quad \text{in} \quad B, \\ \mathbf{h} = \mathbf{u} &\quad \text{on} \quad \partial B. \end{aligned}$$

We show for $1 < s < q < \min(p, p')$

$$|\mathbf{Q}|^{(p-2)s} \fint_B |\nabla \mathbf{u} - \nabla \mathbf{h}|^s\, dx \le c_\delta \fint_{2B} |\mathbf{F} - \mathbf{F}_0|^s\, dx + \delta \left(\fint_{2B} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}(\mathbf{Q})|^q\, dx \right)^{\frac{s}{q}}.$$

Open problems (1)

Pointwise estimates on the boundary: $\forall x \in \partial\Omega$

$$M_R^\sharp(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})(x) \leq c\, M_R^{\sharp,p'}(\mathbf{F})(x)$$
$$+ c\left(\fint_{B_{2R}\cap\Omega} \left||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} - (|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})_{B_{2R}\cap\Omega}\right|^{p'} dy \right)^{\frac{1}{p'}}.$$

$\Rightarrow$ Global estimates in BMO and $C^\alpha(\overline{\Omega})$.

- Flattening of the boundary by local coordinates;
- reflecting the problem (zero boundary data!);
- Lost of Uhlenbeck-structure.

Open problems (2)

Nonhomogenuous boundary data

$$\begin{cases} \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_0 & \text{on } \partial\Omega. \end{cases}$$

- L^q-theory extends easily by considering $\mathbf{w} := \mathbf{u} - \mathbf{u}_0$ and

  $$\operatorname{div}\left(|\nabla \mathbf{w}|^{p-2}\nabla \mathbf{w}\right) = \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} - |\nabla \mathbf{u}_0|^{p-2}\nabla \mathbf{u}_0\right),$$

  This yields $\|\nabla \mathbf{u}\|_{L^q(\Omega)} \leq c\,\|\nabla \mathbf{u}_0\|_{L^q(\Omega)}$ for $q \geq p'$;
- Can we show $\|\nabla \mathbf{u}\|_{C^\alpha(\overline{\Omega})} \leq c\,\|\nabla \mathbf{u}_0\|_{C^\alpha(\overline{\Omega})}$?

Open problems (3)

Parabolic problems for $Q := (0, T) \times \Omega$

$$\begin{cases} \partial_t \mathbf{u} - \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \operatorname{div} \mathbf{F} & \text{in } Q, \\ \mathbf{u} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \mathbf{u}(0, \cdot) = 0 & \text{in } \Omega \end{cases}$$

- Local C^α-estimates are known;
- Can we show global C^α-estimates?
- Can we show pointwise estimates for parabolic problems?