Semiflow selection for the isentropic Euler system

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18.02.2020

Compressible Euler equations

Find velocity $\mathbf{u} : (0, T) \times \mathbb{T}^N \to \mathbb{R}^N$, density $\varrho : (0, T) \times \mathbb{T}^N \to \mathbb{R}$

satisfying the system of PDEs

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) &= 0, \\ \varrho(0, \cdot) = \varrho_0, \quad \varrho \mathbf{u}(0, \cdot) &= \mathbf{m}_0. \end{aligned}$$

- Periodic boundary conditions for simplicity;
- Also formulation with momentum $\mathbf{m} = \varrho \mathbf{u}$ (vacuum!!);
- Adiabatic pressure law $p(\varrho) = \frac{1}{\mathrm{Ma}^2} \varrho^\gamma$.

Strong solutions

Strong solutions exists locally in time (in $[0, T_{\max})$ with $T_{\max} > 0$)

- Tani (1977), Matsumura-Nishida (1980):
  Existence for initial data in $W^{3,2}$;
- Further results by Agemi (1981), Beirao da Veiga (1981), Ebin (1979), Schochet (1986);
- Global classical solutions do not exists in general (even for smooth initial data) $\rightsquigarrow$ weak solutions;
- Even in 1D global classical solutions are not known to exists (contrast to incompressible Euler!)

Weak solutions

$[\varrho, \mathbf{m}]$ is a weak solution on the time interval $[0, \infty)$ provided

$$\left[\int_{\mathbb{T}^N} \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{T}^N} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla \varphi \right] \ \mathrm{d}x \ \mathrm{d}t \qquad (1.1)$$

$$\left[\int_{\mathbb{T}^N} \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{T}^N} \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla \boldsymbol{\varphi} \right] \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^N} a \varrho^\gamma \operatorname{div} \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t \qquad (1.2)$$

for any $\varphi, \boldsymbol{\varphi} \in C^1_c([0, \infty) \times \mathbb{T}^N)$ and any $\tau > 0$.

Admissible solutions

Total energy as sum of kinetic and internal energy

$$E = \int_{\mathbb{T}^N} \mathcal{E} \, \mathrm{d}x, \quad \mathcal{E} = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho), \quad P(\varrho) = \frac{a}{\gamma - 1} \varrho^{\gamma}.$$

1. Mechanical energy equation

$$\partial_t \mathcal{E} + \operatorname{div} \left(\mathcal{E} u \right) + \operatorname{div} \left(\mathcal{E} P(\varrho) \right) = 0;$$

2. Admissible solutions satisfy energy inequality $\partial_t E \leq 0$;
3. Dafermos (1973): solutions with maximal dissipation, i.e. no other solution exists with $\tilde{E}(t) \leq E(t)$ for all t.

Ill-posedness

Non-uniqueness

There are infinitely many admissible solutions to the isentropic Euler system for initial data...

- De Lellis-Székelyhidi (2010): ...some bounded initial data;
- Chiodaroli (2014):...for any ϱ_0 there is $\mathbf{m}_0$ s.t....
- Feireisl (2014):...for any ϱ_0 there is $\mathbf{u}_0$ s.t....
  admissible but not with maximal dissipation;
- Chiodaroli-Kreml (2014): ... for every Riemann-data s.t....
- Chiodaroli-Kreml-Mácha-Schwarzacher: ...some smooth data;

Dissipative measure-valued solutions

- the Young measure:

$$(t,x) \mapsto \nu_x(t) \in L^\infty_{w^*}((0,\infty)\times\Omega; \mathcal{P}(\mathbb{R}^+\times\mathbb{R}^N));$$

- the kinetic and internal energy concentration defect measures:

$$t \mapsto \mathfrak{C}_{\rm kin}(t), \mathfrak{C}_{\rm int}(t) \in L^\infty_{w^*}(0,\infty; \mathcal{M}^+(\Omega)),$$

- the convective and pressure concentration defect measures:

$$t \mapsto \mathfrak{C}_{\rm conv}(t) \in L^\infty_{w^*}\left(0,\infty; \mathcal{M}^+\left(\Omega\times S^{N-1}\right)\right),$$
$$t \mapsto \mathfrak{C}_{\rm press}(t) \in L^\infty_{w^*}\left(0,\infty; \mathcal{M}^+(\Omega)\right).$$

- Compatibility conditions

$$\mathfrak{C}_{\rm conv}(t,{\rm d}x,{\rm d}\xi) = 2r_x(t,{\rm d}\xi)\otimes\mathfrak{C}_{\rm kin}(t,{\rm d}x),\ \mathfrak{C}_{\rm press} = (\gamma-1)\mathfrak{C}_{\rm int}.$$

The Young measure

Density and momentum are bari-centre

$$\varrho(\tau, x) = \langle \nu_x(\tau); \tilde{\varrho} \rangle \geq 0, \ \mathbf{m}(\tau, x) = \langle \nu_x(\tau); \tilde{\mathbf{m}} \rangle \ \text{ for a.a } x \in \mathbb{T}^N,$$

$$\varrho \in C_{w,\mathrm{loc}}([0, \infty); L^{\gamma}(\mathbb{T}^N)), \ \mathbf{m} \in C_{w,\mathrm{loc}}([0, \infty); L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N; \mathbb{R}^N)).$$

- The energy $E \in \mathrm{BV}_{\mathrm{loc}}([0, \infty); \mathbb{R})$ is non-increasing and

$$E(\tau) = \int_{\mathbb{T}^N} \left\langle \nu_x(\tau); \frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + \frac{a}{\gamma - 1} \tilde{\varrho}^{\gamma} \right\rangle \, \mathrm{d}x$$
$$+ \int_{\mathbb{T}^N} \, \mathrm{d}\mathfrak{C}_{\mathrm{kin}}(\tau) + \int_{\mathbb{T}^N} \, \mathrm{d}\mathfrak{C}_{\mathrm{int}}(\tau).$$

- Energy balance

$$\Big[E\psi \Big]_{t=\tau_1-}^{t=\tau_2+} - \int_{\tau_1}^{\tau_2} E \partial_t \psi \ \mathrm{d}t \leq 0, \ \ E(0-) = E_0.$$

Field equations

- Momentum equation

$$\left[\int_{\mathbb{T}^N} \mathbf{m} \cdot \varphi(\tau, \cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_{\mathbb{T}^N} \left[\mathbf{m} \cdot \partial_t \varphi + \left\langle \nu_x(t); \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle : \nabla_x \varphi \right] \, \mathrm{d}x \, \mathrm{d}t$$
$$+ 2 \int_0^\tau \int_{\mathbb{T}^N} \langle r_x(t); \xi \otimes \xi \rangle : \nabla_x \varphi \ \ \mathrm{d}\mathfrak{C}_{\mathrm{kin}} \, \mathrm{d}t$$
$$+ \int_{\mathbb{T}^N} \langle \nu_x(t); a \tilde{\varrho}^\gamma \rangle \, \mathrm{div} \, \varphi \, \mathrm{d}t + (\gamma - 1) \int_0^\tau \int_{\mathbb{T}^N} \mathrm{div} \, \varphi \ \mathrm{d}\mathfrak{C}_{\mathrm{int}} \, \mathrm{d}t;$$

- Continuity equation

$$\left[\int_{\mathbb{T}^N} \varrho \varphi(\tau, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{T}^N} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t,$$

- Initial conditions $\varrho(0, \cdot) = \varrho_0$ and $\mathbf{m}(0, \cdot) = \mathbf{m}_0$.

Known results

Existence results:

- DiPerna (1985): hyperbolic conservation laws;
- DiPerna-Majda (1987): incompressible Euler equation;
- Neustupa (1993): compressible Euler equation;
- Kröner-Zajaczkowski (1996): complete Euler equations;

Weak-strong uniqueness:

- Brennier-DeLellis-Székelyhidi (2011): incompressible Euler equation;
- Gwiazda-Świerczewska-Wiedemann (2015): compressible Euler equation;
- Feireisl-Březina (2018): complete Euler equations.

Semiflow selection theorem (1)

- Phase space

$$X = W^{-\ell,2}(\mathbb{T}^N) \times W^{-\ell,2}(\mathbb{T}^N;\mathbb{R}^N) \times \mathbb{R},$$

- Initial data from

$$D = X \cap \left\{ \varrho_0 \geq 0,\ \int_{\mathbb{T}^N} \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1}\varrho_0^\gamma \right] \,\mathrm{d}x \leq E_0 \right\};$$

- Trajectory space

$$\Omega = C_{\rm loc}([0,\infty); W^{-\ell,2}) \times C_{\rm loc}([0,\infty); W^{-\ell,2}) \times L^1_{\rm loc}(0,\infty).$$

- Solution set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$

$$\left\{ [\varrho, \mathbf{m}, E] \in \Omega \ \middle|\ [\varrho, \mathbf{m}, E] \text{ is DMV sol. starting with } [\varrho_0, \mathbf{m}_0, E_0] \right\}.$$

Semiflow selection theorem (2)

Theorem (Breit-Feireisl-Hofmanová, ARMA, 2020)

The isentropic Euler system admits a semiflow selection U in the class of dissipative measure-valued solutions. Moreover, we have that $U[\varrho_0, \mathbf{m}_0, E_0]$ is maximal for any $[\varrho_0, \mathbf{m}_0, E_0] \in D$.

- A semiflow is a map

$$U : D \to \Omega, \;\; U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0];$$

- The map $U : D \to \Omega$ is Borel measurable;
- We have the semigroup property

$$U[t_1 + t_2, \varrho_0, \mathbf{m}_0, E_0] = U\left[t_2, U[t_1, \varrho_0, \mathbf{m}_0, E_0]\right]$$

for any $[\varrho_0, \mathbf{m}_0, E_0] \in D$ and any $t_1, t_2 \geq 0$.

Required properties (1)

Method by Krylov adapted by Cardona-Kapitanski:

- Multi-valued solution mapping

$$\mathcal{U} : [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega};$$

- Time shift

$$S_T \circ \xi, \ S_T \circ \xi(t) = \xi(T + t), \ t \geq 0;$$

- Continuation

$$\xi_1 \cup_T \xi_2(\tau) = \begin{cases} \xi_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \xi_2(\tau - T) \text{ for } \tau > T. \end{cases}$$

Required properties (2)

- **(A1) Compactness:** For any $[\varrho_0, \mathbf{m}_0, E_0] \in D$, the set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ is a non-empty compact subset of Ω;
- **(A2)** The mapping

$$D \ni [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega}$$

  is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in 2^{Ω};
- **(A3) Shift invariance:** For any $[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(T), E(T-)] \text{ for any } T > 0;$$

- **(A4) Continuation:** If $T > 0$, and

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],$$
$$[\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T-)]$$
$$\Rightarrow [\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

Induction argument (1)

System of functionals

$$I_{\lambda,F}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda t) F(\varrho, \mathbf{m}, E) \ \mathrm{d}t, \ \ \lambda > 0.$$

- Here

$$F : X = W^{-\ell,2}(\Omega) \times W^{-\ell,2}(\Omega; \mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}$$

  is a bounded and continuous functional;
- Semiflow reduction: define $I_{\lambda,F} \circ \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ by

$$\left\{ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \ \middle| I_{\lambda,F}[\varrho, \mathbf{m}, E] \leq I_{\lambda,F}[\tilde{\varrho}, \tilde{\mathbf{m}}, \widetilde{E}] \right\};$$

- Induction argument: $\mathcal{U}$ satisfies (A1) - (A4) $\Rightarrow$ $I_{\lambda,F} \circ \mathcal{U}$ satisfies (A1) - (A4).

Induction argument (2)

Choose countable basis $\{e_n\}_{n=1}^{\infty}$ in $L^2(\mathbb{T}^N)$, $\{\mathbf{w}_m\}_{m=1}^{\infty}$ in $L^2(\mathbb{T}^N;\mathbb{R}^N)$, and a countable set $\{\lambda_k\}_{k=1}^{\infty}$ dense in $(0,\infty)$. We consider a countable family of functionals,

$$I_{k,0,0}[\varrho,\mathbf{m},E] = \int_0^\infty \exp(-\lambda_k t)\beta(E(t))\,\mathrm{d}t,$$

$$I_{k,n,0}[\varrho,\mathbf{m},E] = \int_0^\infty \exp(-\lambda_k t)\beta\left(\int_{\mathbb{T}^N} \varrho e_n\ \mathrm{d}x\right)\,\mathrm{d}t,$$

$$I_{k,0,m}[\varrho,\mathbf{m},E] = \int_0^\infty \exp(-\lambda_k t)\beta\left(\int_{\mathbb{T}^N} \mathbf{m}\cdot\mathbf{w}_m\ \mathrm{d}x\right)\,\mathrm{d}t.$$

By Lerch's theorem $\mathcal{U}^\infty = \cap_{j=1}^{\infty}\mathcal{U}^j$ is a singleton.

Complete Euler equations

Find velocity $\mathbf{u}$, density ϱ and energy $\mathcal{E} : (0, T) \times \mathbb{T}^N \to \mathbb{R}$

satisfying the system of PDEs

$$\begin{aligned} \partial_t \varrho + \operatorname{div} \mathbf{m} &= 0, \\ \partial_t \mathbf{m} + \operatorname{div} \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p &= 0, \\ \partial_t \mathcal{E} + \operatorname{div} \left[(\mathcal{E} + p) \frac{\mathbf{m}}{\varrho} \right] &= 0. \end{aligned}$$

$\mathcal{E}$ is sum of kinetic and internal components ($e =$internal energy),

$$\mathcal{E} = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e.$$

Constitutive relations

- Caloric equation of state in the form

$$(\gamma - 1)\varrho e = p, \text{ where } \gamma > 1 \text{ is the adiabatic constant;}$$

- Temperature ϑ by Boyle–Mariotte thermal equation of state:

$$p = \varrho\vartheta \text{ yielding } e = c_v\vartheta, \; c_v = \frac{1}{\gamma - 1};$$

- Pressure p and internal energy e can be written in the form

$$p = p(\varrho, s) = \varrho^{\gamma} \exp\left(\frac{s}{c_v}\right), \; e = e(\varrho, s) = \frac{1}{\gamma - 1}\varrho^{\gamma-1} \exp\left(\frac{s}{c_v}\right).$$

Entropy balance

The Second law of thermodynamics is enforced through

the entropy balance equation

$$\partial_t(\varrho s) + \operatorname{div}(s\mathbf{m}) = 0 \text{ or } \partial_t s + \left(\frac{\mathbf{m}}{\varrho}\right) \cdot \nabla s = 0.$$

- The entropy s is given as

$$s(\varrho, \vartheta) = \log(\vartheta^{c_v}) - \log(\varrho);$$

- In the weak form entropy inequality

$$\partial_t(\varrho s) + \operatorname{div}(s\mathbf{m}) \geq 0$$

in the sense of distributions.

Maximal dissipation

Total entropy $S = \varrho s$ satisfies

the entropy inequality

$$\partial_t(\varrho S) + \operatorname{div}\left(S\frac{\mathbf{m}}{\varrho}\right) \geq 0.$$

- Maximal dissipation defined via the entropy production rate

$$\sigma(\tau) = \int_{\mathbb{T}^N} \left(S(\tau) - S_0\right) \mathrm{d}x;$$

- A *maximal dissipative solution* is maximal wrt σ.

Semiflow selection theorem (2)

Theorem (Breit-Feireisl-Hofmanová, CMP, online first)

The isentropic Euler system admits a semiflow selection U in the class of dissipative measure-valued solutions. Moreover, we have that $U[\varrho_0, \mathbf{m}_0, S_0, E_0]$ is maximal for any $[\varrho_0, \mathbf{m}_0, S_0, E_0] \in D$.

- S only belongs to $\mathrm{BV}_{loc}([0,\infty); W^{-\ell,2}(\mathbb{T}^N))$;
- *Total energy* is a constant of motion:

$$\int_{\mathbb{T}^N} \left\langle \mathcal{V}_{t,x}; \frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + c_v \tilde{\varrho}^\gamma \exp\left(\frac{\tilde{S}}{c_v \tilde{\varrho}}\right) \right\rangle \,\mathrm{d}x + \int_{\mathbb{T}^N} (\mathrm{d}\mathfrak{C}_{\mathrm{kin}} + \mathrm{d}\mathfrak{C}_{\mathrm{int}}).$$

D. Breit, E. Feireisl & M. Hofmanová: *Solution semiflow for the isentropic Euler system.* **Arch. Rational Mech. Anal.** 235, 167–194. (2020)

D. Breit, E. Feireisl & M. Hofmanová: *Dissipative solutions and semiflow selection for the complete Euler system.* **Commun. Math. Phys.** DOI:10.1007/s00220-019-03662-7