Sharp conditions for Korn inequalities in Orlicz spaces

Dominic Breit
Andrea Cianchi & Lars Diening

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Outline

Korn's inequality

For all $\mathbf{u} \in W_0^{1,2}(\Omega)$ there holds

$$\int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}x \le 2 \int_\Omega |\varepsilon(\mathbf{u})|^2 \, \mathrm{d}x \, .$$

- Ω denotes a domain in $\mathbb{R}^d$;
- $\varepsilon(\mathbf{u}) := \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$ is the symmetric gradient;
- proof: for $\mathbf{u} \in C_0^\infty(\Omega)$ integration by parts, approximation for the general case.

Korn's inequality in L^p

For all $\mathbf{u} \in W_0^{1,p}(\Omega)$ it holds

$$\int_\Omega |\nabla \mathbf{u}|^p \,\mathrm{d}x \leq c(p,\Omega) \int_\Omega |\varepsilon(\mathbf{u})|^p \,\mathrm{d}x \,.$$

- Valid for $1 < p < \infty$: proofs by Gobert (62/71), Nečas (65), Mosolov-Mjasnikov (72), Temam (85).
- false in the limit cases $p = 1$ (Ornstein, 64) and $p = \infty$ (Leeuw-Mirkil, 64).

Non-linear Stokes systems

The p-Stokes problem is equivalent to

minimize the energy

$$J[\mathbf{u}] = \frac{1}{p}\int_\Omega |\varepsilon(\mathbf{u})|^p \,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x, \quad \operatorname{div}\mathbf{u} = 0.$$

- The constitutive law is given by ($\mathbf{S}$= stress deviator)

$$\mathbf{S} = |\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u});$$

- Korn's inequality yields coercivity of J on $W_0^{1,p}(\Omega)$ and (partial) regularity of solutions.

Negative norms (1)

The proof follows from Nečas' negative norm theorem.

For all $\mathbf{u} \in W_0^{1,p}(\Omega)$ there holds

$$\|\nabla \mathbf{u}\|_p \leq c\|\nabla^2 \mathbf{u}\|_{-1,p} \leq c\|\nabla \varepsilon(\mathbf{u})\|_{-1,p} \leq c\|\varepsilon(\mathbf{u})\|_p.$$

- Based on the estimation $|\nabla^2 \mathbf{u}| \leq c|\nabla \varepsilon(\mathbf{u})|$;
- we have

$$\|v\|_{-1,p} := \sup_{\varphi \in W_0^{1,p'}, \|\varphi\| \leq 1} \left| \int_\Omega v\varphi \, \mathrm{d}x \right|.$$

Negative norms (2)

Proof of Nečas' negative norm theorem.

For all $u \in W_0^{1,p}(\Omega)$ there holds for $u_\Omega = |\Omega|^{-1} \int_\Omega u \, \mathrm{d}x$

$$c\|u - u_\Omega\|_p \leq \ \|\nabla u\|_{-1,p} \leq 2\|u - u_\Omega\|_p$$

- Based on the continuity of $\nabla \operatorname{Bog}_\Omega : L^q \to L^q$;
- Using $\operatorname{div} \operatorname{Bog}_\Omega f = f$ and $\operatorname{Bog} f \in W_0^{1,q}(\Omega)$ we have

$$\sup_{\|\varphi\|_{p'} \leq 1} \int_\Omega u \, (\varphi - \varphi_\Omega) \, \mathrm{d}x = \sup_{\|\varphi\|_{p'} \leq 1} \int_\Omega u \ \operatorname{div} \operatorname{Bog}_\Omega(\varphi - \varphi_\Omega) \, \mathrm{d}x \, .$$

Singular integrals (1)

The proof follows from continuity of singular integral operators.

Reshetnyak, '70: $\mathbf{v} \in C^\infty(\overline{\Omega})$ can be written as

$$\mathbf{v}(x) = \mathcal{R}_\Omega \mathbf{v}(x) + \mathscr{L}(\varepsilon(\mathbf{v}))(x)$$

- $\mathcal{R}_\Omega \mathbf{v}$ is a projection of $\mathbf{v}$ into the space $\mathcal{K}_\Omega$ of rigid motions;
- $\mathscr{L}$ is a singular integral operator given by

$$\mathscr{L}(\varphi) = \mathcal{S}(\varphi) + \mathcal{T}(\varphi),\ \mathcal{T}^i(\varphi) = \int_\Omega \boldsymbol{\theta}^i(x,z) : \varphi(z)dz,$$

$$\mathcal{S}^i(\varphi)(x) = \int_\Omega \frac{\mathbf{G}^i(x,e)}{|x-z|^{d-1}} : \varphi(z)dz$$

Singular integrals (2)

The proof follows from continuity of singular integral operators.

Reshetnyak, '70: $\mathbf{v} \in C^\infty(\overline{\Omega})$ can be written as

$$\begin{aligned}\nabla \mathbf{v}(x) &= \nabla \mathscr{L}(\varepsilon(\mathbf{v}))(x) + \dots \\ &= \int_\Omega \frac{\frac{x-z}{|x-z|}\mathbf{G}^i(x,e)}{|x-z|^d} : \varphi(z)dz + \dots\end{aligned}$$

- $\nabla \mathbf{v}$ behaves like a Calderon-Zygmund singular integral operator in $\varepsilon(\mathbf{v})$;
- Singular integral operator are continuous from $L^p \to L^p$ for $1 < p < \infty$.

Young-functions

Let $A : [0, \infty) \to [0, \infty]$ be convex, left-continuous and non-trivial.

There is $a : [0, \infty) \to [0, \infty]$ such that

$$A(s) = \int_0^s a(r)\mathrm{d}r$$

- Examples: $A(s) = s^p$, $A(t) = s\ln(1 + s)$, $A(s) = s$;
- Examples: $A(s) = s^p$, $A(t) = s(e^s - 1)$, $A(s) = \infty\chi_{(1,\infty)}$;
- A is called N-function if in addition strictly increasing and

$$\lim_{t\to 0} \frac{A(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{A(t)}{t} = \infty.$$

Orlicz spaces

The Orlicz space $L^A(\Omega)$ is the set of all measurable functions

$u : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega A(\lambda|u|)\,\mathrm{d}x < \infty \quad \text{for some} \quad \lambda > 0.$$

- $L^A(\Omega)$ is a Banach space together with the Luxemburg norm

$$\|u\|_A := \inf\left\{k : \int_\Omega A\Big(\frac{|u|}{k}\Big)\,\mathrm{d}x \le 1\right\}.$$

Orlicz-Sobolev spaces

The Orlicz-Sobolev space $W^{1,A}(\Omega)$ is set of all functions

$u : \Omega \to \mathbb{R}$ satisfying

$$u \in L^A(\Omega), \quad \nabla u \in L^A(\Omega).$$

- $W^{1,A}(\Omega)$ is a Banach space together with the norm

$$\|u\|_{1,A} := \|u\|_A + \|\nabla u\|_A.$$

- $W^{1,A}(\Omega) := W_0^{1,1} \cap W^{1,A}(\Omega).$

Δ_2-condition

Let A be a Young-function. A fulfils a

Δ_2-condition iff

$$A(2t) \leq K\, A(t) \quad \text{for all} \quad t \geq 0.$$

- Extreme growth conditions excluded, e.g., $A(s) = s(e^s - 1)$;
- Smooth functions are dense in $W^{1,A}(\Omega)$ iff $A \in \Delta_2$ and $W_0^{1,A}(\Omega)$= closure of $C_0^\infty(\Omega)$ in $W^{1,A}(\Omega)$ if Δ_2.

∇_2-condition

Let A be a Young-function, we define the

conjugate Young-function

$$\tilde{A}(t) := \sup_{s \geq 0}[st - A(s)] = \int_0^s a^{-1}(r)\mathrm{d}r.$$

- $A(s) = \frac{1}{p}s^p, \tilde{A}(s) = \frac{1}{p'}s^{p'}; B(s) = s\ln(1+s), \tilde{B}(s) = s(e^s - 1);$
- A fulfils a ∇_2-condition iff $\tilde{A}$ satisfies a Δ_2-condition.
- Cases close to linear growth excluded, e.g., $A(s) = s\ln(1+s)$;
- $L^A(\Omega)$ is reflexive if and only if $A \in \Delta_2 \cap \nabla_2$.

Korn's inequality in L^A

Let A be a N-function satisfying Δ_2 and ∇_2 then

for all $\mathbf{u} \in W_0^{1,A}(\Omega)$ there holds

$$\int_\Omega A(|\nabla \mathbf{u}|)\,\mathrm{d}x \le c(\Delta_2(A), \nabla_2(A), \Omega) \int_\Omega A(|\varepsilon(\mathbf{u})|)\,\mathrm{d}x\,.$$

- Proof in Diening-Růžička-Schumacher ('10) and Fuchs ('10);
- Application: general constitutive law

$$\mathbf{S} = \nu\left(|\varepsilon(\mathbf{u})|\right)\varepsilon(\mathbf{u}), \quad \nu(t) = \frac{\varphi'(t)}{t}.$$

Korn's inequality in L^A (2)

Let A be a N-function satisfying Δ_2 and ∇_2 then

for all $\mathbf{u} \in W^{1,A}(\Omega)$ there holds

$$\int_\Omega A(|\nabla \mathbf{u} - (\nabla \mathbf{u})_\Omega|)\,\mathrm{d}x \leq c \int_\Omega A(|\varepsilon(\mathbf{u}) - (\varepsilon(\mathbf{u}))_\Omega|)\,\mathrm{d}x\,.$$

- proof via a version of Necas negative norm theorem in Orlicz spaces using $|\nabla^2 \mathbf{u}| \leq c|\nabla \varepsilon(\mathbf{u})|$.

Prandtl-Eyring fluids (1)

Eyring suggested in 1936 the constitutive law

$$\mathbf{S}(\varepsilon(\mathbf{v})) = \nu_0 \frac{\operatorname{arsinh}(\lambda|\varepsilon(\mathbf{v})|)}{\lambda|\varepsilon(\mathbf{v})|} \approx \frac{\log(1+|\varepsilon(\mathbf{v})|)}{|\varepsilon(\mathbf{v})|}\varepsilon(\mathbf{v})$$

- The natural function space (for $\varepsilon(\mathbf{u})$) is the Orlicz space generated by $A(t) = t\ln(1+t)$;
- The study of slow flows is completed, existence and (partial) regularity results are due to Fuchs and Seregin in '99;
- Existence of weak solutions in 2D is proved by Breit-Diening-Fuchs (JDE, '12).

Necessity of Δ_2 and ∇_2

Theorem (Breit-Diening (JMFM, '12)

Let A be an N-function such that there exists a constant $K > 0$ such that

$$\|\nabla \mathbf{u}\|_A \leq K \|\varepsilon(\mathbf{u})\|_A$$

for all $\mathbf{u} \in C_0^\infty(\mathbb{R}^d)$. Then A satisfies the Δ_2 and the ∇_2 condition.

- Korn fails for $A(s) = s \ln(1+s)$ and for $A(s) = s(e^s - 1)$;
- New tools necessary for Prandtl-Eyring model.

Balance conditions

Let A and B be two Young functions

satisfying the balance conditions

$$t \int_0^t \frac{B(s)}{s^2}\, ds \le A(ct) \qquad \text{for } t \ge 0, \tag{4.1}$$

$$t \int_0^t \frac{\widetilde{A}(s)}{s^2}\, ds \le \widetilde{B}(ct) \qquad \text{for } t \ge 0, \tag{4.2}$$

- If $A \in \nabla_2$ then (4.1) holds with $B = A$;
- If $A \in \Delta_2$ then (4.2) holds with $B = A$;
- $A(s) = s \ln(1+s)$, $B(s) = s$;
- $A(s) = \infty \chi_{(1,\infty)}$, $B(s) = s(e^s - 1)$.

Korn's inequality: general case (1)

Let A and B be two Young functions satisfying (4.1) and (4.2)

for all $\mathbf{u} \in C_0^\infty(\Omega)$ there holds (Cianchi, '15)

$$\int_\Omega A(|\nabla \mathbf{u}|)\,\mathrm{d}x \leq \int_\Omega B(C|\varepsilon(\mathbf{u})|)\,\mathrm{d}x\,.$$

- Recovers all known Korn-inequalities;
- If $\varepsilon(\mathbf{u}) \in L^{s\ln(1+s)}(\Omega)$ than $\nabla \mathbf{u} \in L^1(\Omega)$;
- If $\varepsilon(\mathbf{u}) \in L^\infty(\Omega)$ than $\nabla \mathbf{u} \in L^{s(e^s-1)}(\Omega)$.

Korn's inequality: general case (2)

Let A and B be two Young functions satisfying (4.1) and (4.2).

Let T be a singular integral operator such that

$$(Tf)^*(s) \leq C\left(\frac{1}{s}\int_0^s f^*(r)\,dr + \int_s^{|\Omega|} f^*(r)\,\frac{dr}{r}\right) \quad \text{for } s \in (0, |\Omega|).$$

- Here $u^*(s) = \inf\{t \geq 0 : |\{x \in \Omega : |u(x)| > t\}| \leq s\}$.
- Hardy inequalities follow from (4.1) and (4.2) (Cianchi '96)

$$\left\|\frac{1}{s}\int_0^s \varphi(r)\,dr\right\|_B \leq C\|\varphi\|_A, \quad \left\|\int_s^\infty \varphi(r)\,\frac{dr}{r}\right\|_B \leq C\|\varphi\|_A.$$

- Proof also follows from negative norm theorem in Orlicz spaces by Breit-Cianchi (JDE, '15).

Necessity of (4.1) and (4.2)

Theorem (Breit-Cianchi-Diening)

Let A and B be Young functions such that there exists a constant $K > 0$ such that

$$\|\nabla \mathbf{u}\|_B \leq K \|\varepsilon(\mathbf{u})\|_A$$

for all $\mathbf{u} \in C_0^\infty(\mathbb{R}^d)$. Then A and B satisfy (4.1) and (4.2).

- Integrabilities are optimal (in the sense of Orlicz spaces);
- Extension of the counterexample by Breit-Diening.

The condition (4.1) (1)

Lemma (Cianchi, '96)

$$\left\| \int_{\cdot}^{\omega_d} \frac{h(r)}{r}\, dr \right\|_{L^B(0,\omega_d)} \leq c \|h\|_{L^A(0,\omega_d)} \quad \forall h \quad \Rightarrow \quad (4.1).$$

Let h be any nonnegative function in $L^A(0,\omega_d)$. Define

$$\rho(r) = \int_r^1 \frac{h(\omega_d t^d)}{t}\, \mathrm{d}t, \quad \mathbf{u}(x) = \mathbf{Q}\, x \rho(|x|) \quad \text{for } x \in B_1,$$

Let $\mathbf{Q} \in \mathbb{R}^{d\times d}$ be anti-symmetric with $|\mathbf{Q}| = 1$, than

$$\varepsilon(\mathbf{u})(x) = \frac{\mathbf{Q}x \otimes^{\mathrm{sym}} x}{|x|^2} \rho'(|x|)|x|,$$

$$\nabla \mathbf{u}(x) = \mathbf{Q}\rho(|x|) + \frac{\mathbf{Q}x \otimes x}{|x|^2} \rho'(|x|)|x|.$$

The condition (4.1) (2)

$$|\varepsilon(\mathbf{u})|(x) \le |\rho'(|x|)||x| = h(\omega_d|x|),$$
$$\rho(|x|) \le |\nabla \mathbf{u}(x)| + |\rho'(|x|)||x| = |\nabla \mathbf{u}(x)| + h(\omega_d|x|).$$

Due to the embedding $L^A(B_1) \to L^B(B_1)$

$$\Big\| \int_s^{\omega_n} \frac{h(r)}{r}\, dr \Big\|_{L^B(0,\omega_d)} = \Big\| \int_{|x|}^1 \frac{h(\omega_d t)}{t}\, dt \Big\|_{L^B(B_1)} = \|\rho(|x|)\|_{L^B(B_1)}$$
$$\le \|\nabla \mathbf{u}\|_B + \|h(\omega_d|x|^d)\|_B \le C\|\varepsilon(\mathbf{u})\|_A + \|h(\omega_d|x|^d)\|_A$$
$$\le C'\|h(\omega_d|x|^d)\|_{L^A(B_1)} = C'\|h(s)\|_{L^A(0,\omega_d)}.$$

The condition (4.2)

- Using a technique developed by Conti, Faraco and Maggi for L^1-counterexamples:

$$\int_{(0,1)^d} \Phi(\nabla \mathbf{u}_i)\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^{d\times d}} \Phi(\mathbf{F})\,\mathrm{d}\nu(\mathbf{F});$$

  where ν is a finite sum of Dirac measures, hence all integrals can easily computed;
- for all such ν there exists a sequence $\mathbf{u}_i \in W_0^{1,\infty}$ such that the convergence above is true for every $\Phi \in C(\mathbb{R}^{d\times d})$.
- Also holds if $\Phi(\mathbf{F}) = A(|\mathbf{F}|)$ for some Young function A.