Stationary solutions to the stochastic compressible Navier–Stokes equations

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Navier-Stokes equations

Find velocity $\mathbf{u} : Q \to \mathbb{R}^d$ and density $\varrho : Q \to \mathbb{R}$ satisfying the

system of partial differential equations

$$\begin{cases} \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) = \operatorname{div}\mathbf{S} - a\nabla\varrho^\gamma & \text{in } Q, \\ \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 & \text{in } Q, \\ \mathbf{u} = 0 & \text{on } \partial\mathcal{O}, \\ \varrho(0,\cdot) = \varrho_0 & \text{in } \mathcal{O}, \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{q}_0 & \text{in } \mathcal{O}, \end{cases}$$

- $Q := (0,T) \times \mathcal{O}$ with $\mathcal{O} \subset \mathbb{R}^d$, $d \in \{2,3\}$, and $T > 0$;
- $\mathbf{S} : Q \to \mathbb{R}^{d\times d}$ is given by Newton's law

$$\mathbf{S} = \mu\Big(\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2} - \tfrac{1}{3}\operatorname{div}\mathbf{u}\, I\Big) + \big(\eta + \tfrac{2}{3}\big)\operatorname{div}\mathbf{u}\, I.$$

Momentum equation

Weak formulation of the momentum equation:

find $\mathbf{v}, \varrho$ such that for all $\varphi \in C_0^\infty(Q)$

$$\int_Q \varrho \mathbf{u} \cdot \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t = \mu \int_Q \nabla \mathbf{u} : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t + (\eta + \mu) \int_Q \operatorname{div} \mathbf{u} \operatorname{div} \varphi$$
$$- \int_Q (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_Q a \varrho^\gamma \operatorname{div} \varphi \, \mathrm{d}x \, \mathrm{d}t$$

- Function space for weak solutions

$$\mathbf{v} \in L^2(0, T; W_0^{1,2}(\mathcal{O})),$$
$$\varrho \in C_w([0, T]; L^\gamma(\mathcal{O})),$$
$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0, T; L^2(\mathcal{O})).$$

Continuity equation

Renormalized formulation of the continuity equation (DiPerna-Lions, '89):

ϱ satisfies for all $\psi \in C_0^\infty(Q)$

$$\int_Q b(\varrho)\partial_t \psi \,\mathrm{d}x\,\mathrm{d}t = \int_Q b(\varrho)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_Q \big(b'(\varrho)\varrho - b(\varrho)\big)\,\mathrm{div}\,\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t$$

- $b \in C^1(\mathbb{R})$ with $b'(z) = 0$ for all $z \geq M(b)$;
- ϱ solves weak formulation too $(b(z) = z)$.

Known results

- Existence of weak solutions for $\gamma \geq \frac{9}{5}$ by Lions (1998);
- Existence of weak solutions for $\gamma > \frac{3}{2}$ by Feireisl, Novotný, Petzeltová (2001);
- We have

$$\varrho \in L^{\frac{5}{3}\gamma-1}(Q), \quad \mathbf{u} \in L^2(0,T;W_0^{1,2}(\mathcal{O})),$$

$$\varrho\mathbf{u} \in C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})),$$

$$\varrho\mathbf{u}\otimes\mathbf{u} \in L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\mathcal{O})).$$

Stochastic Navier-Stokes equations

Velocity field $\mathbf{v}$ and density ϱ on $Q = (0, T) \times \mathbb{T}^3$

$$\begin{cases} \mathrm{d}(\varrho\mathbf{u}) = [\mu\Delta\mathbf{u} - \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + a\nabla\varrho^\gamma]\,\mathrm{d}t + \mathbb{G}\,\mathrm{d}W & \text{in } Q, \\ \mathrm{d}\varrho = -\operatorname{div}(\varrho\mathbf{u})\,\mathrm{d}t & \text{in } Q, \\ \varrho(0,\cdot) = \varrho_0 & \text{in } \mathbb{T}^3, \\ \varrho(0,\cdot)\mathbf{u}(0,\cdot) = \mathbf{q}_0 & \text{in } \mathbb{T}^3. \end{cases}$$

- Momentum equation in the weak sense: $\int_{\mathbb{T}^3} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \dots$ for all $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3)$;
- Mass equation in the renormalizes sense: $\mathrm{d}b(\varrho) = \dots$ for all $b \in C^1(\mathbb{R})$ with $b'(z) = 0$ for all $z \geq M_b$.

Known Results

- If we have no uniqueness for the deterministic problem we expect martingale solutions, i.e. there is a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}})$ and a Brownian motion $\tilde{W}$ on it such that a weak solutions $(\tilde{\varrho}, \tilde{\mathbf{u}})$...
- Many literature for stochastic Navier Stokes equations starting with Bensoussan-Temam in 1973; Martingale solutions by Flandoli-Gątarek 1995.
- Knowledge about compressible fluids very limited: only the case $\mathbb{G} = \mathbb{G}(\varrho) = \varrho$ is considered (Tornatore for $d = 2$ in 2000; Feireisl, Maslowski, Novotný for $d = 3$ in 2013).

Main result

Theorem (Breit & Hofmanová, IUMJ, 2016)

Under appropriate assumptions on the initial law Λ there is a finite energy weak martingale solution to the stochastic compressible Navier-Stokes equations provided $\gamma > \frac{3}{2}$ substract to periodic boundary conditions.

- Adaption of theory by Feireisl et all to the stochastic setting;
- Wiener process $W = \sum_k \beta_k \mathbf{e}_k$, $(\mathbf{e}_k)$ ONB of HS $\mathfrak{U}$;
- General noise $\mathbb{G} = \mathbb{G}(\varrho, \varrho\mathbf{u}) \in L^2(0, T; L_2(\mathfrak{U}; W^{-l,2}(\mathbb{T}^3)))$ with $\mathbb{G}(\varrho, \varrho\mathbf{u})\mathbf{e}_k = \mathbf{g}_k(\cdot, \varrho, \varrho\mathbf{u})$, e.g.

$$\mathbb{G}(\varrho, \varrho\mathbf{u}) = \varrho\, dW, \quad \mathbb{G}(\varrho, \varrho\mathbf{u}) = \varrho\mathbf{u}\, dW.$$

Concept of solution (1)

A finite energy weak martingale solution is a quantity

$$\big((\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P}),\varrho,\mathbf{u},W\big).$$

- $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$ is a stochastic basis,
- W is an $(\mathscr{F}_t)$-cylindrical Wiener process,
- $\varrho\geq 0$ is $(\mathscr{F}_t)$-adapted and $\varrho\in C_w([0,T];L^\gamma(\mathbb{T}^3))$ $\mathbb{P}$-a.s.,
- $\mathbf{u}$ is a random variable and $\mathbf{u}\in L^2(0,T;W^{1,2}(\mathbb{T}^3))$ $\mathbb{P}$-a.s.,
- $\varrho\mathbf{u}$ is $(\mathscr{F}_t)$-adapted $\varrho\mathbf{u}\in C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))$ $\mathbb{P}$-a.s.,
- $\Lambda=\mathbb{P}\circ\big(\varrho(0),\varrho\mathbf{u}(0)\big)^{-1}$.

Concept of solution (2)

- $(\varrho, \mathbf{u})$ solves the momentum equation in the weak sense $\mathbb{P}$-a.s.
- $(\varrho, \mathbf{u})$ solves the continuity equation in the renormalized sense $\mathbb{P}$-a.s.
- We have the energy inequality

$$\mathrm{d}\int_{\mathbb{T}^3}\left(\frac{1}{2}\varrho(t)\big|\mathbf{u}(t)\big|^2+\frac{a}{\gamma-1}\varrho^{\gamma}(t)\right)\mathrm{d}x\;\mathrm{d}t+\mu\int_{\mathbb{T}^3}|\nabla\mathbf{u}|^2\,\mathrm{d}x$$
$$\leq\int_{\mathbb{T}^3}\mathbf{u}\cdot\mathbb{G}(\varrho,\varrho\mathbf{u})\,\mathrm{d}W+\frac{1}{2}\sum_{k\geq1}\int_{\mathbb{T}^3}\frac{|\mathbf{g}_k(\varrho,\varrho\mathbf{u})|^2}{\varrho}\;\mathrm{d}x\;\mathrm{d}t$$

$\mathbb{P}$-a.s. for all $t\in[0,T]$.

Strategy of the proof

- Four layer approximation;
- Add artificial pressure $\delta\varrho^{\beta}$ for $\beta > 4$;
- Add artificial viscosity $\varepsilon\Delta\varrho$ in continuity equation;
- Solve problem with artificial viscosity and pressure via Faedo-Galerkin approximation;
- Faedo-Galerkin approximation via stopping time, then stopping time $\to T$.
- Use Jakubowski-Skorokhod representations theorem for compactness; applies to quasi-polish spaces (e.g. Banach spaces with weak topology).

Invariant measures (1)

- Statistical approach to fluid mechanics: is there an equilibrium state such that time-averages of the observable tend to this state as $t \to \infty$?
- Well-known for finite dimensional stochastic differential equations: invariant measure defined via transition semigroup

$$P_t(\varphi)(x) = \mathbb{E}\varphi(X_t(x)), \quad \varphi \in C_b(\mathbb{R}),$$

where (X_t) solves... with initial datum x.

- Define dual P_t^* on space of probability measures by

$$\int_{\mathbb{R}} \varphi \, \mathrm{d}P_t^*\nu = \int_{\mathbb{R}} P_t\varphi \, \mathrm{d}\nu \quad \forall \varphi \in C_b(\mathbb{R}), \, \nu \in \mathcal{M}(\mathbb{R}).$$

Invariant measures (2)

- A measure μ is called invariant if $P_t^*\nu = \nu$ or, equivalently

  $$\int_{\mathbb{R}} P_t \varphi \,\mathrm{d}\nu = \int_{\mathbb{R}} \varphi \,\mathrm{d}\nu$$

  $\Rightarrow$ probability distribution of X_t is independent of t.
- A stochastic process is stationary if its probability distribution is independent of time. Example: (W_t) Wiener process $\Rightarrow W(t+h) - W(t)$ is stationary for fixed h.
- If (X_t) is stationary solution to some SDE its probability law is an invariant measure.
- Semigroup property $P_{t+s} = P_t \circ P_s$ requires uniqueness!!

Stationarity

Definition (Stationary stochastic process)

Let $\mathbf{U} = \{\mathbf{U}(t); t \in [0, \infty)\}$ be an X-valued stochastic process. We say that $\mathbf{U}$ is *stationary* provided the joint laws

$$\mathcal{L}(\mathbf{U}(t_1 + \tau), \ldots, \mathbf{U}(t_n + \tau)), \quad \mathcal{L}(\mathbf{U}(t_1), \ldots, \mathbf{U}(t_n))$$

on X^n coincide for all $\tau \geq 0$, for all $t_1, \ldots, t_n \in [0, \infty)$.

- Here $\mathcal{L}$ denotes the law on X^n, i.e.

$$\mathcal{L}(Y_1, ..., Y_n)(B) = \mathbb{P}((Y_1, ..., Y_n) \in B) \quad B \subset X^n$$

for X-valued random variable $Y_1, ..., Y_n$.

Incompressible Navier–Stokes equations

- Existence of stationary martingale solutions by Flandoli-Gątarek 1995;
- $L^2(0, T; W^{1,2}(\mathcal{O}))$ becomes (almost) $L^\infty(0, T; W^{1,2}(\mathcal{O}))$ $\Rightarrow$ Stationary solutions are smooth (but depend on time)!!!
- Existence of unique invariant measure by Da Prato-Debusche 2003 using Kolmogorov equation (equation for $\mathbb{E}(\varphi(\mathbf{u}(t, x)))$, where $\varphi \in C_b(L^2)$).
- Note that $\mathbf{u} \in C_w([0, T]; L^2(\mathcal{O}))$, so $\mathbf{u}(t) \in L^2(\mathcal{O})$ for ANY t.

Weak stationarity

Definition (Weakly stationary random variable)

Let $\mathbf{U} : \Omega \to \mathcal{D}'((0,\infty) \times \mathbb{T}^3)$ be weakly measurable. Let $\mathcal{S}_\tau$ be the time shift on the space of trajectories given by $\mathcal{S}_\tau \varphi(t) = \varphi(t+\tau)$. We say that $\mathbf{U}$ is *weakly stationary* provided the laws

$$\mathcal{L}\left(\langle \mathbf{U}, \mathcal{S}_{-\tau}\varphi_1\rangle, \dots, \langle \mathbf{U}, \mathcal{S}_{-\tau}\varphi_n\rangle\right), \quad \mathcal{L}\left(\langle \mathbf{U}, \varphi_1\rangle, \dots, \langle \mathbf{U}, \varphi_n\rangle\right)$$

on $\mathbb{R}^n$ coincide for all $\tau \geq 0$, $\varphi_1, \dots, \varphi_n \in C^\infty((0,\infty) \times \mathbb{T}^3)$.

- Here $\mathcal{L}$ denotes the law on $\mathbb{R}^n$, i.e.

$$\mathcal{L}(Y_1, ..., Y_n)(B) = \mathbb{P}((Y_1, ..., Y_n) \in B) \quad B \subset \mathbb{R}^n$$

for real valued random variable $Y_1, ..., Y_n$.

Properties

- weak stationarity stable under weak convergence
- weak stationarity of $\mathbf{u} \in L^2_{\rm loc}(0,\infty; W^{1,2}(\mathbb{T}^3))$ a.s.

$\Rightarrow$ $\mathcal{L}(\mathbf{u}) = \mathcal{L}(\mathcal{S}_\tau \mathbf{u})$ on $L^2_{\rm loc}(0,\infty; W^{1,2}(\mathbb{T}^3))$

$\Rightarrow$ $\mathcal{L}(\mathbf{u}(s)) = \mathcal{L}(\mathbf{u}(t))$ on $W^{1,2}(\mathbb{T}^3)$ for a.e. $s,t \in [0,\infty)$

- weak stationarity of $\varrho \in C_{\rm loc}\big([0,\infty); (L^\gamma(\mathbb{T}^3), w)\big)$ a.s.

$\Rightarrow$ ϱ is a stationary $L^\gamma(\mathbb{T}^3)$-valued stochastic process

- weak stationarity of $\varrho\mathbf{u} \in C_{\rm loc}\big([0,\infty); (L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3), w)\big)$ a.s.

$\Rightarrow$ $\varrho\mathbf{u}$ is a stationary $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$-valued stochastic process

Stationary solutions

Definition

A finite energy weak martingale solution $[\varrho, \mathbf{u}, W]$ is called *stationary* provided the joint law of the time shift $[\mathcal{S}_\tau \varrho, \mathcal{S}_\tau \mathbf{u}, \mathcal{S}_\tau W - W]$ on

$$L^p_{\rm loc}(0,\infty; L^\gamma(\mathbb{T}^3)) \times L^2_{\rm loc}(0,\infty; W^{1,2}(\mathbb{T}^3)) \times C_{\rm loc}([0,\infty); \mathfrak{U}_0)$$

is independent of $\tau \geq 0$, for all $p \in [1, \infty)$.

Theorem (Breit, Feireisl, Hofmanová, Maslowski '17)

Let the total mass be given by $M_0 \in (0, \infty)$, *that is,*

$$M_0 = \int_{\mathbb{T}^3} \varrho(t, x) \, \mathrm{d}x \quad \textit{for all} \quad t \in (0, \infty).$$

Then there exists a stationary finite energy weak martingale solution $[\varrho, \mathbf{u}, W]$ *satisfying complete slip boundary conditions.*

- More restrictive assumptions on noise:

$$\mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W = \varrho\mathbb{F}(\varrho, \varrho\mathbf{u})\mathrm{d}W$$

  with $\mathbb{F}$ bounded.
- Extension to no-slip b.c. possible (Korn-Poincaré inequality needed).

Four layer approximation scheme

- χ smooth, nonincreasing, $\chi \equiv 1$ on $(-\infty, 0]$, $\chi \equiv 0$ on $[1, \infty)$
- artificial viscosity - ε
- artificial pressure in the momentum equation - δ

$$\mathrm{d}\varrho + \operatorname{div}(\varrho \mathbf{u})\mathrm{d}t = \varepsilon \Delta \varrho \, \mathrm{d}t - 2\varepsilon\varrho \, \mathrm{d}t + \chi \left(\frac{1}{M_0} \int_{\mathbb{T}^3} \varrho \, \mathrm{d}x \right) \mathrm{d}t$$

$$\mathrm{d}(\varrho \mathbf{u}) + \left[\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho^\gamma + \delta \nabla \varrho^\beta - \varepsilon \Delta(\varrho \mathbf{u}) \right] \mathrm{d}t = \mu \Delta \mathbf{u} \mathrm{d}t + \mathbb{G}(\varrho, \varrho \mathbf{u}) \, \mathrm{d}W$$

- Faedo-Galerkin finite-dimensional approximation - N
- stopping time argument - R

Aim: $R \to \infty$, $N \to \infty$, $\varepsilon \to 0$, $\delta \to 0$

Four layer approximation scheme

- existence of an invariant measure on the basic level by Krylov-Bogoliubov method:
  1. strong Feller property (continuous dependence on initial data)
  2. solution $(\varrho, \mathbf{u})$ is a Markov process
  3. tightness of

$$\left\{ \frac{1}{T} \int_0^T \mathcal{L}[\mathbf{u}(t)] \, \mathrm{d}t; \; T > 0 \right\}, \; \left\{ \frac{1}{T} \int_0^T \mathcal{L}[\varrho(t)] \, \mathrm{d}t; \; T > 0 \right\}.$$

- new global-in-time estimates needed
- stationarity preserved under limit procedures

Additional difficulties in comparison to existence

- global-in-time estimates not controlled by the initial data
- new estimates established at every approximation step
- generalized energy inequality needed
- modified method of effective viscous flux
- if $\mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W \rightsquigarrow \varrho\mathbf{f}(x)\,\mathrm{d}t$, global bounds only for $\gamma > \frac{5}{3}$

Navier–Stokes–Fourier equations

- Heat-conducting fluids depending on temperature ϑ;
- Add entropy balance for specific entropy s

$$\mathrm{d}(\varrho s) + \left[\operatorname{div}(\varrho s \mathbf{u}) - \operatorname{div}\left(\frac{\kappa \nabla \vartheta}{\vartheta}\right)\right] \mathrm{d}t = \frac{1}{\vartheta}\left(\mu |\nabla \mathbf{u}|^2 + \kappa \frac{|\nabla \vartheta|^2}{\vartheta}\right) \mathrm{d}t$$

or equivalent equation for internal energy e, where

$$p(\varrho, \vartheta) = \varrho^{\frac{5}{3}} + \tfrac{a}{3}\vartheta^4,\ e(\varrho, \vartheta) = \tfrac{3}{2}\varrho^{\frac{2}{3}} + a\tfrac{\vartheta^4}{\varrho},\ s(\varrho, \vartheta) = \tfrac{4a}{3}\tfrac{\vartheta^3}{\varrho}.$$

- Existence of weak martingale solutions;
- Non-existence of stationary solutions (deterministic case: long-time behaviour only for r.h.s. $\varrho \nabla \mathbf{G}(x)$).

- D. Breit & M. Hofmanová: *Stochastic Navier–Stokes equations for compressible fluids.* **Indiana Univ. Math. J.** 65, 1183–1250. (2016)
- D. Breit, E. Feireisl & M. Hofmanová: *Incompressible limit for compressible fluids with stochastic forcing.* **Arch. Rational Mech. Anal.** 222, 895–926. (2016)
- D. Breit, E. Feireisl & M. Hofmanová: *Compressible fluids driven by stochastic forcing: The relative energy inequality and applications.* To appear in **Commun. Math. Phys.**
- D. Breit, E. Feireisl, M. Hofmanová & B. Maslowski: *Stationary solutions to the compressible Navier–Stokes system driven by stochastic forces.* Preprint at arXiv:1703.03177v1