A Brief Introduction to Optimal Transport Theory: Exercises

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Exercise 2.1. Let $T : \mathbb{R} \to \mathbb{R}$ be the translation $T(x) = x + 1$. Let $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$ be probability densities on $\mathbb{R}$. Show that $T\#f = g$. Define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = 2x$. Show that $S\#f \neq g$.

Exercise 2.5 (Strict convexity of h does not imply $h'' > 0$)**.** Find an example of a strictly convex function $h : \mathbb{R} \to \mathbb{R}$ such that $h''(x) = 0$ for some $x \in \mathbb{R}$.

Exercise 2.6. Show that $h_6(x) = x \log x$, $x \in (0, \infty)$, is strictly convex. Show that $h_7(x) = x^{1/2}$, $x \in (0, \infty)$, is strictly concave.

Exercise 3.5. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 2 - x$. Let $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$. Use Lemma 3.4 to show that $T\#f = g$.

Exercise 3.7. Check the values in the table in Example 3.6. Use Jensen's inequality to prove that T_1 is the *worst* transport map for the concave cost $h(s) = |s|^{1/2}$.

Exercise 3.8. Let $X = [0, 1]$, $Y = [1, 2]$, $f = \chi_{[0,1]}$, $g = \chi_{[1,2]}$, $c(x, y) = h(|y - x|)$ with $h(s) = (s + 1) \log(s + 1)$, $s \geq 0$. Find an optimal transport map.

Exercise 3.9 (Non-uniqueness for linear costs)**.** Let $X, Y \subset \mathbb{R}$ be bounded and $c(x, y) = h(y - x)$ where $h : X \to Y$ is a linear function. Show that *every* admissible transport map is optimal, i.e., show that if $T : X \to Y$, $T\#f = g$, then

$$M(T) = \mathcal{T}_c(f, g).$$

Hint: Compute $M(T)$ and show that it is independent of T.

Exercise 3.10 (Non-uniqueness for non-strictly convex costs: Book shifting)**.** Let $X = [0, 2]$, $Y = [1, 3]$, $f = \frac{1}{2}\chi_{[0,2]}$, $g = \frac{1}{2}\chi_{[1,3]}$, $c(x, y) = h(y - x)$ with $h(s) = |s|$. Let $T_1(x) = x + 1$ and

$$T_2(x) = \begin{cases} x + 2 & \text{if } x \in [0, 1], \\ x & \text{if } x \in (1, 2]. \end{cases}$$

Observe that f and g have mass in common in the interval $[1, 2]$. The map T_2 leaves the common mass fixed and only transports mass from $[0, 1]$ to $[2, 3]$. Show that T_1 and T_2 are both optimal transport maps:

$$M(T_1) = M(T_2) = \mathcal{T}_c(f, g).$$

Exercise 3.12 (A challenging exercise: Behaviour of quadratic transport under translations)**.** Let $X = Y = \mathbb{R}$ and c be the quadratic cost $c(x, y) = (x-y)^2$. For $a \in \mathbb{R}$, define the translation $\tau_a : \mathbb{R} \to \mathbb{R}$ by $\tau_a(x) = x-a$. Let $f \circ \tau_a$ denote the composition $(f \circ \tau_a)(x) = f(\tau_a(x)) = f(x-a)$. In this exercise we show that

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) = \mathcal{T}_c(f, g) + (b-a)^2 + 2(b-a)(m_g - m_f) \tag{0.1}$$

where $a, b \in \mathbb{R}$ and

$$m_f = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x, \quad m_g = \int_{-\infty}^{\infty} y g(y)\,\mathrm{d}y$$

and the centres of mass of f and g.

(i) Let $T\#f = g$. Define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = T(x-a) + b$. Show that $S\#(f \circ \tau_a) = g \circ \tau_b$.

(ii) Show that

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) \leq \mathcal{T}_c(f, g) + (b-a)^2 + 2(b-a)(m_g - m_f).$$

Hint: Let T be an optimal transport map transporting f to g, which means that $\mathcal{T}_c(f, g) = \int_{-\infty}^{\infty} |T(x) - x|^2 f(x)\,\mathrm{d}x$. By part (i),

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) \leq \int_{-\infty}^{\infty} |S(x) - x|^2 f(\tau_a(x))\,\mathrm{d}x.$$

(iii) Use a similar argument to part (ii) to show that

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) \geq \mathcal{T}_c(f, g) + (b-a)^2 + 2(b-a)(m_g - m_f).$$

Combining (ii) and (iii) proves (0.1). Hint: Start with an optimal map T transporting $f \circ \tau_a$ to $g \circ \tau_b$. Use it to construct an admissible map S transporting f to g.

(iv) Use (0.1) to give an alternative proof that $\mathcal{T}_c(\chi_{[0,1]}, \chi_{[1,2]}) = 1$.

Exercise 4.3 (Non-uniqueness of optimal Kantorovich potential pairs)**.** Show that if (ϕ, ψ) is an optimal Kantorovich potential pair, then so is $(\phi + a, \psi - a)$ for any $a \in \mathbb{R}$.

Exercise 4.8. Fill in the missing details for Example 4.6.

Exercise 4.9. Derive an optimal Kantorovich potential pair for the book shifting problem from Exercise 3.10.

Exercise 4.10. Prove that T_2 is the *worst* transport map for the convex cost $h(s) = s^2$ from Example 3.6. Hint: This is equivalent to proving that T_2 is the best transport map for the concave cost $\tilde{h}(s) = -s^2$. Verify this by constructing an optimal Kantorovich potential pair (ϕ, ψ) such that $D(\phi, \psi) = M(T_2)$ for the cost $\tilde{h}(s) = -s^2$.