A Brief Introduction to Optimal Transport Theory: Solutions to Exercises

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Exercise 2.1. Let $T : \mathbb{R} \to \mathbb{R}$ be the translation $T(x) = x + 1$. Let $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$ be probability densities on $\mathbb{R}$. Show that $T\#f = g$. Define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = 2x$. Show that $S\#f \neq g$.

Solution. Let $B \subseteq \mathbb{R}$. Then

$$\begin{aligned}
\int_{T^{-1}(B)} f(x)\,\mathrm{d}x &= \int_{T^{-1}(B)} \chi_{[0,1]}(x)\,\mathrm{d}x \\
&= \int_{T^{-1}(B)\cap[0,1]} 1\,\mathrm{d}x \\
&= \int_{B\cap[1,2]} 1\,\mathrm{d}y \qquad\qquad \text{(change of variables } y = T(x) = x+1\text{)} \\
&= \int_B \chi_{[1,2]}(y)\,\mathrm{d}y \\
&= \int_B g(y)\,\mathrm{d}y
\end{aligned}$$

and so $T\#f = g$. If we choose $B = [0,1]$ then

$$\int_B g(y)\,\mathrm{d}y = 0$$

but

$$\int_{S^{-1}(B)} f(x)\,\mathrm{d}x = \int_0^{1/2} f(x)\,dx = \frac{1}{2}.$$

Therefore $g \neq S\#f$.

Exercise 2.5 (Strict convexity of h does not imply $h'' > 0$)**.** Find an example of a strictly convex function $h : \mathbb{R} \to \mathbb{R}$ such that $h''(x) = 0$ for some $x \in \mathbb{R}$.

Solution. Take, for example, $h(x) = x^4$, $x \in \mathbb{R}$. Then h is strictly convex but $h''(0) = 0$.

Exercise 2.6. Show that $h_6(x) = x \log x$, $x \in (0,\infty)$, is strictly convex. Show that $h_7(x) = x^{1/2}$, $x \in (0,\infty)$, is strictly concave.

Solution. Just check that $h_6'' > 0$ and $h_7'' < 0$.

Exercise 3.5. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 2 - x$. Let $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$. Use Lemma 3.4 to show that $T\#f = g$.

Solution. Let $X = Y = \mathbb{R}$ and let $\varphi : Y \to \mathbb{R}$ be bounded. Then

$$\begin{aligned}\int_X \varphi(T(x))f(x)\,\mathrm{d}x &= \int_0^1 \varphi(2-x)\,\mathrm{d}x \\ &= \int_1^2 \varphi(y)\,\mathrm{d}y && \text{(change of variables } y = T(x) = 2-x) \\ &= \int_Y \varphi(y)g(y)\,\mathrm{d}y.\end{aligned}$$

Since this holds for all bounded functions $\varphi : Y \to \mathbb{R}$, then $T\#f = g$ by Lemma 3.4.

Exercise 3.7. Check the values in the table in Example 3.6. Use Jensen's inequality to prove that T_1 is the *worst* transport map for the concave cost $h(s) = |s|^{1/2}$.

Solution. It is an easy calculus exercise to check the values in the table and we just give the solution for the second part of the exercise. Let T be any admissible transport map, $T\#f = g$. Since the map $s \mapsto s^{1/2}$, $s \geq 0$, is concave, then the map $s \mapsto -s^{1/2}$, $s \geq 0$, is convex and so by Jensen's inequality

$$\begin{aligned}-M(T) &= -\int_0^1 |T(x)-x|^{1/2} f(x)\,\mathrm{d}x \\ &\geq -\left(\int_0^1 |T(x)-x| f(x)\,\mathrm{d}x\right)^{1/2} \\ &= -\left(\int_0^1 T(x)f(x)\,\mathrm{d}x - \int_0^1 xf(x)\,\mathrm{d}x\right)^{1/2} \\ &= -\left(\int_1^2 yg(y)\,\mathrm{d}y - \int_0^1 xf(x)\,\mathrm{d}x\right)^{1/2} && \text{(by (3.1) with } \varphi(y) = y) \\ &= -\left(\frac{3}{2} - \frac{1}{2}\right)^{1/2} \\ &= -h(1) = -M(T_1).\end{aligned}$$

Multiplying by -1 gives

$$M(T) \leq M(T_1)$$

as required. The same argument shows that the translation T_1 is the worst transport map for any concave cost.

Exercise 3.8. Let $X = [0,1]$, $Y = [1,2]$, $f = \chi_{[0,1]}$, $g = \chi_{[1,2]}$, $c(x,y) = h(|y-x|)$ with $h(s) = (s+1)\log(s+1)$, $s \geq 0$. Find an optimal transport map.

Solution. The cost h is convex since $h''(s) = 1/(s+1) > 0$. Therefore the same arguments as in Example 3.6 show that the translation $T(x) = x + 1$ is an optimal transport map.

Exercise 3.9 (Non-uniqueness for linear costs)**.** Let $X, Y \subset \mathbb{R}$ be bounded and $c(x,y) =$

$h(y-x)$ where $h : X \to Y$ is a linear function. Show that *every* admissible transport map is optimal, i.e., show that if $T : X \to Y$, $T\#f = g$, then

$$M(T) = \mathcal{T}_c(f,g).$$

Hint: Compute $M(T)$ and show that it is independent of T.

Solution. We have

$$\begin{aligned} M(T) &= \int_X c(x,T(x))f(x)\,\mathrm{d}x \\ &= \int_X h(T(x)-x)f(x)\,\mathrm{d}x \\ &= \int_X h(T(x))f(x)\,\mathrm{d}x - \int_X h(x)f(x)\,\mathrm{d}x \qquad \text{(since } h \text{ is linear)} \\ &= \int_Y h(y)g(y)\,\mathrm{d}y - \int_X h(x)f(x)\,\mathrm{d}x \end{aligned}$$

by equation (3.1) with $\varphi = h$. Therefore $M(T)$ is independent of T and every admissible transport map is optimal.

Exercise 3.10 (Non-uniqueness for non-strictly convex costs: Book shifting)**.** Let $X = [0,2]$, $Y = [1,3]$, $f = \frac{1}{2}\chi_{[0,2]}$, $g = \frac{1}{2}\chi_{[1,3]}$, $c(x,y) = h(y-x)$ with $h(s) = |s|$. Let $T_1(x) = x+1$ and

$$T_2(x) = \begin{cases} x+2 & \text{if } x \in [0,1], \\ x & \text{if } x \in (1,2]. \end{cases}$$

Observe that f and g have mass in common in the interval $[1,2]$. The map T_2 leaves the common mass fixed and only transports mass from $[0,1]$ to $[2,3]$. Show that T_1 and T_2 are both optimal transport maps:

$$M(T_1) = M(T_2) = \mathcal{T}_c(f,g).$$

Solution. We have

$$M(T_1) = \int_0^2 c(x,T_1(x))\,f(x)\,\mathrm{d}x = \int_0^2 |T_1(x)-x|\,\tfrac{1}{2}\,\mathrm{d}x = \int_0^2 1\cdot\tfrac{1}{2}\,\mathrm{d}x = 1$$

and

$$M(T_2) = \int_0^2 c(x,T_2(x))\,f(x)\,\mathrm{d}x = \int_0^2 |T_2(x)-x|\,\tfrac{1}{2}\,\mathrm{d}x = \int_0^1 2\cdot\tfrac{1}{2}\,\mathrm{d}x = 1.$$

These maps are optimal since, for any admissible map T such that $T\#f = g$,

$$\begin{aligned} \int_0^2 |T(x)-x|\,f(x)\,\mathrm{d}x &\geq \left|\int_0^2 (T(x)-x)\,f(x)\,\mathrm{d}x\right| \\ &= \left|\int_0^2 T(x)\,f(x)\,\mathrm{d}x - \int_0^2 x\,f(x)\,\mathrm{d}x\right| \\ &= \left|\int_1^3 y\,g(y)\,\mathrm{d}y - \int_0^2 x\,f(x)\,\mathrm{d}x\right| \qquad \text{(by (3.1) with } \varphi(y)=y\text{)} \\ &= \left|\frac{1}{2}\int_1^3 y\,\mathrm{d}y - \frac{1}{2}\int_0^2 x\,\mathrm{d}x\right| \\ &= 1. \end{aligned}$$

Exercise 3.12 (A challenging exercise: Behaviour of quadratic transport under translations)**.** Let $X = Y = \mathbb{R}$ and c be the quadratic cost $c(x, y) = (x-y)^2$. For $a \in \mathbb{R}$, define the translation $\tau_a : \mathbb{R} \to \mathbb{R}$ by $\tau_a(x) = x-a$. Let $f \circ \tau_a$ denote the composition $(f \circ \tau_a)(x) = f(\tau_a(x)) = f(x-a)$. In this exercise we show that

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) = \mathcal{T}_c(f, g) + (b-a)^2 + 2(b-a)(m_g - m_f) \tag{0.1}$$

where $a, b \in \mathbb{R}$ and

$$m_f = \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x, \quad m_g = \int_{-\infty}^{\infty} y g(y)\, \mathrm{d}y$$

and the centres of mass of f and g.

(i) Let $T \# f = g$. Define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = T(x-a) + b$. Show that $S\#(f \circ \tau_a) = g \circ \tau_b$.

(ii) Show that

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) \le \mathcal{T}_c(f, g) + (b-a)^2 + 2(b-a)(m_g - m_f).$$

Hint: Let T be an optimal transport map transporting f to g, which means that $\mathcal{T}_c(f, g) = \int_{-\infty}^{\infty} |T(x) - x|^2 f(x)\, \mathrm{d}x$. By part (i),

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) \le \int_{-\infty}^{\infty} |S(x) - x|^2 f(\tau_a(x))\, \mathrm{d}x.$$

(iii) Use a similar argument to part (ii) to show that

$$\mathcal{T}_c(f \circ \tau_a, g \circ \tau_b) \ge \mathcal{T}_c(f, g) + (b-a)^2 + 2(b-a)(m_g - m_f).$$

Combining (ii) and (iii) proves (0.1). Hint: Start with an optimal map T transporting $f \circ \tau_a$ to $g \circ \tau_b$. Use it to construct an admissible map S transporting f to g.

(iv) Use (0.1) to give an alternative proof that $\mathcal{T}_c(\chi_{[0,1]}, \chi_{[1,2]}) = 1$.

Solution.

(i) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be bounded. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi(S(x))(f \circ \tau_a)(x)\, \mathrm{d}x &= \int_{-\infty}^{\infty} \varphi(T(x-a) + b) f(x-a)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varphi(T(\tilde{x}) + b) f(\tilde{x})\, \mathrm{d}\tilde{x} && (\tilde{x} = x - a) \\
&= \int_{-\infty}^{\infty} \varphi(y + b) g(y)\, \mathrm{d}y && (\text{since } T\#f = g) \\
&= \int_{-\infty}^{\infty} \varphi(\tilde{y}) g(\tilde{y} - b)\, \mathrm{d}\tilde{y} && (\tilde{y} = y + b) \\
&= \int_{-\infty}^{\infty} \varphi(\tilde{y})(g \circ \tau_b)(\tilde{y})\, \mathrm{d}\tilde{y}.
\end{aligned}$$

Therefore $S\#(f \circ \tau_a) = g \circ \tau_b$, as required.

(ii) Let T be an optimal transport map transporting f to g, which means that $\mathcal{T}_c(f,g) = \int_{-\infty}^{\infty} |T(x)-x|^2 f(x)\,\mathrm{d}x$. Let $S(x) = T(x-a)+b$. Then $S\#(f\circ\tau_a) = (g\circ\tau_b)$ by part (i) and so

$$\begin{aligned}
&\mathcal{T}_c(f\circ\tau_a, g\circ\tau_b) \le M(S) \\
&= \int_{-\infty}^{\infty} (S(x)-x)^2 (f\circ\tau_a)(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} (T(x-a)+b-x)^2 f(x-a)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} (T(\tilde{x})+b-\tilde{x}-a)^2 f(\tilde{x})\,\mathrm{d}\tilde{x} && (\tilde{x} = x-a) \\
&= \int_{-\infty}^{\infty} (T(\tilde{x})-\tilde{x})^2 f(\tilde{x})\,\mathrm{d}\tilde{x} + (b-a)^2 \int_{-\infty}^{\infty} f(\tilde{x})\,\mathrm{d}\tilde{x} \\
&\qquad + 2(b-a)\int_{-\infty}^{\infty} (T(\tilde{x})-\tilde{x}) f(\tilde{x})\,d\tilde{x} \\
&= \mathcal{T}_c(f,g) + (b-a)^2 + 2(b-a)\left(\int_{-\infty}^{\infty} T(\tilde{x})f(\tilde{x})\,d\tilde{x} - \int_{-\infty}^{\infty} \tilde{x} f(\tilde{x})\,d\tilde{x}\right) \\
&= \mathcal{T}_c(f,g) + (b-a)^2 + 2(b-a)\left(\int_{-\infty}^{\infty} y g(y)\,dy - \int_{-\infty}^{\infty} \tilde{x} f(\tilde{x})\,d\tilde{x}\right) && (T\#f = g) \\
&= \mathcal{T}_c(f,g) + (b-a)^2 + 2(b-a)(m_g - m_f)
\end{aligned}$$

as required.

(iii) This is similar to part (ii). Let T be an optimal transport map transporting $f\circ\tau_a$ to $g\circ\tau_b$, which means that $\mathcal{T}_c(f\circ\tau_a, g\circ\tau_b) = \int_{-\infty}^{\infty} |T(x)-x|^2 (f\circ\tau_a)(x)\,\mathrm{d}x$. Let $S(x) = T(x+a)-b$. It can be shown that $S\#f = g$ (this is very similar to part (i)). Therefore

$$\mathcal{T}_c(f,g) \le M(S) = \int_{-\infty}^{\infty} (S(x)-x)^2 f(x)\,\mathrm{d}x.$$

The rest of the calculation is similar to part (ii).

(iv) Just take $f = g = \chi_{[0,1]}$, $a = 0$, $b = 1$. Then $f\circ\tau_a = \chi_{[0,1]}$, $g\circ\tau_b = \chi_{[1,2]}$, $m_f = m_g = \frac{1}{2}$ and so by equation (0.1)

$$\mathcal{T}_c(\chi_{[0,1]}, \chi_{[1,2]}) = 0 + (1-0)^2 + 2(1-0)(\tfrac{1}{2} - \tfrac{1}{2}) = 1$$

as we found in Example 3.6.

Exercise 4.3 (Non-uniqueness of optimal Kantorovich potential pairs)**.** Show that if (ϕ,ψ) is an optimal Kantorovich potential pair, then so is $(\phi+a, \psi-a)$ for any $a\in\mathbb{R}$.

Solution. Let $\tilde{\phi} = \phi + a$ and $\tilde{\psi} = \psi - a$. First we check that $(\tilde{\phi},\tilde{\psi})$ is an admissible pair:

$$\tilde{\phi}(x) + \tilde{\psi}(y) = \phi(x) + a + \psi(y) - a = \phi(x) + \psi(y) \le c(x,y)$$

and so $\tilde{\phi} \oplus \tilde{\psi} \le c$, as required. Now we check that $(\tilde{\phi}, \tilde{\psi})$ is optimal:

$$\begin{aligned}
D(\tilde{\phi}, \tilde{\psi}) &= \int_X \tilde{\phi}(x) f(x)\,\mathrm{d}x + \int_Y \tilde{\psi}(y) g(y)\,\mathrm{d}y \\
&= \int_X (\phi(x) + a) f(x)\,\mathrm{d}x + \int_Y (\psi(y) - a) g(y)\,\mathrm{d}y \\
&= \int_X \phi(x) f(x)\,\mathrm{d}x + \int_Y \psi(y) g(y)\,\mathrm{d}y + a\left(\int_X f(x)\,\mathrm{d}x - \int_Y g(y)\,\mathrm{d}y\right) \\
&= D(\phi, \psi) + a(1 - 1) \\
&= D(\phi, \psi) \\
&= \mathcal{T}_c(f, g)
\end{aligned}$$

as required.

Exercise 4.8. Fill in the missing details for Example 4.6.

Solution. It is an easy calculus exercise to check the values in the table. We show how to derive an optimal potential pair (ϕ, ψ) for the cost $h(s) = |s|$.

Let $h(s) = |s|$, $c(x, y) = h(x - y) = |x - y|$. By Corollary 4.4, if $T_1(x) = x + 1$ and (ϕ, ψ) are optimal, then

$$\phi'(x) = c_x(x, T_1(x)) = \mathrm{sgn}(x - T_1(x)) = \mathrm{sgn}(-1) = -1 \quad \text{for } x \in [0, 1].$$

Integrating gives $\phi(x) = -x + a$, $x \in [0, 1]$. We can choose $a = 0$ by Exercise 4.3. Using Corollary 4.4 again (and again assuming that T_1 is optimal) gives

$$\psi(T_1(x)) = c(x, T_1(x)) - \phi(x) = |x - T_1(x)| - (-x) = 1 + x \quad \text{for } x \in [0, 1].$$

By setting $y = T_1(x) = x + 1$ we find that

$$\psi(y) = 1 + (y - 1) = y \quad \text{for } y \in [1, 2].$$

Therefore $\phi(x) = -x$ and $\psi(y) = y$, as desired.

The calculation is similar for the cost $h(s) = |s|^{1/2}$ (this time use the map $T_2(x) = 2 - x$).

Exercise 4.9. Derive an optimal Kantorovich potential pair for the book shifting problem from Exercise 3.10.

Solution. One possible choice is $\phi(x) = -x$, $\psi(y) = y$. Another choice is $\phi(x) = -|x - 1|$, $\psi(y) = |y - 1|$.

Exercise 4.10. Prove that T_2 is the *worst* transport map for the convex cost $h(s) = s^2$ from Example 3.6. Hint: This is equivalent to proving that T_2 is the best transport map for the concave cost $\tilde{h}(s) = -s^2$. Verify this by constructing an optimal Kantorovich potential pair (ϕ, ψ) such that $D(\phi, \psi) = M(T_2)$ for the cost $\tilde{h}(s) = -s^2$.

Solution. Use the same method as in Example 4.6 to derive the optimal Kantorovich potential pair

$$(\phi(x), \psi(y)) = (2(x - 1)^2, -2(y - 1)^2),$$

which satisfies

$$D(\phi, \psi) = M(T_2) = -\frac{4}{3}$$

for the cost $\tilde{h}(s) = -s^2$.