SCALE-FREE PERCOLATION

Maria Deijfen
Stockholm University

Joint work with Remco van der Hofstad and Gerard Hooghiemstra.

EMPIRICAL NETWORKS

- Social relations
- Communication nets
- Internet/www etc.

Properties:

- Heavy-tailed degree distribution

  Often: Fraction of vertices with degree $> k$ decays like $k^{-\gamma}$ (power-law).

- Short distances between vertices.

INHOMOGENEOUS RANDOM GRAPHS

n vertices

Assign weight $W \sim F$ indep. to each vertex.

$P(\text{edge}(i,j)) = f(W_i, W_j)$ $\left[\text{for instance } f(W_i,W_j) = \frac{W_i W_j}{n} \wedge 1\right]$

Edges indep. given the weights.

- F power-law $\Rightarrow$ Limiting degree distribution power-law with same exponent
- Phase-transition for distances:

  $\text{Var}(\text{degree}) < \infty \Rightarrow \log n$

  $\text{Var}(\text{degree}) = \infty \Rightarrow \log\log n$

LONG-RANGE PERCOLATION

$x, y \in \mathbb{Z}^d$

$P(\text{edge } (x,y)) \sim \lambda |x-y|^{-\alpha}$ as $|x-y| \to \infty$.

Edges independent.

$d=1$ $\alpha \leq 2 \Rightarrow$ Percolation possible.

$\alpha > 2 \Rightarrow$ No percolation

$d \geq 2$ Focus on how long-range edges affect properties of ∞ component (e.g. distances).

INHOMOGENEOUS L-R PERCOLATION

Assign weight $W_x \geq 0$ to $x \in \mathbb{Z}^d$.

$\{W_x\}$ iid

$$P(\text{edge } (x,y)) = 1 - e^{-\frac{\lambda W_x W_y}{|x-y|^\alpha}} \sim \frac{\lambda W_x W_y}{|x-y|^\alpha}$$

Edges independent given the weights.

α: Long-range parameter.

λ: Percolation parameter.

THE DEGREES

slowly varying

Assume $P(W > w) = w^{-\beta} \cdot L(w)$

D_x = degree of x

$\gamma = \frac{\alpha}{d} \cdot \beta$

$$\boxed{\text{THEOREM. If } \alpha > d \text{ and } \gamma > 1, \text{ then } P(D_0 > s) = s^{-\gamma} \cdot \ell(s).}$$

slowly varying

$\boxed{\begin{array}{l}\textbf{THEOREM. If:} \\ \quad a)\ \alpha \le d, \text{ or} \\ \quad b)\ \alpha > d \text{ and } \gamma \le 1, \\ \text{then } P(D_0 = \infty \mid W_0 > 0) = 1.\end{array}}$

$\therefore$ Infinite degrees for $\alpha \le d$ or $\gamma \le 1$.

PERCOLATION

$C(x)$ = component of x

$\theta(\lambda) = P(|C(0)| = \infty)$ percolation probab.

$\lambda_c = \inf\{\lambda : \theta(\lambda) > 0\}$

1) When is $\underbrace{\lambda_c < \infty}_{\text{percolation for large } \lambda}$?

2) When is $\underbrace{\lambda_c > 0}_{\text{no percolation for small } \lambda}$?

Assume: $\alpha > d$ and $\gamma > 1$.

1. When is $\lambda_c < \infty$?

- In $d \geq 2$: As soon as $P(W=0) \neq 1$.
- In $d = 1$.

  $\alpha \in (1,2]$: As soon as $P(W=0) \neq 1$.

  $\alpha > 2$: If $\gamma < 2$ (and $P(W=0) \neq 1$), but not if $\gamma > 2$.

  $\operatorname{Var} D < \infty$.

(2) When is $\lambda_c > 0$?

For all d: $\lambda_c \begin{cases} = 0 & \text{if } \gamma < 2 \\ > 0 & \text{if } \gamma > 2. \end{cases}$

$\therefore \boxed{\lambda_c > 0 \iff \operatorname{Var} D < \infty} \quad (\gamma \neq 2)$

Assume $P(W=0) \neq 1$.

	$\gamma=1$ $\mathrm{Var}\, D = \infty$	$\gamma=2$ $\mathrm{Var}\, D < \infty$
$d \geq 2$	Always perc.	Non-trivial critical value
$d=1$ $\alpha \in (1,2]$	Always perc.	Non-trivial critical value
$d=1$ $\alpha > 2$	Always perc	Never perc.

DISTANCES

$d(0,x)$ = graph distance between 0 and x

How does $d(0,x)$ grow with $|x|$?

$\gamma = 1$ $\mathrm{Var}\, D = \infty$	$\gamma = 2$ $\mathrm{Var}\, D < \infty$
$d(0,x) \sim \log\log\lvert x\rvert$	$d(0,x) \gtrsim \log\lvert x\rvert$
	(If $\alpha > 2d$: $d(0,x) \gtrsim \lvert x\rvert^{\varepsilon}$)

OPEN PROBLEMS

- Is the log sharp for $\gamma > 2$, $\alpha \in (d, 2d]$?
  When $W \equiv w$: no $\left((\log|x|)^{\Delta},\ \Delta > 1 \right)$

- Determine exponent μ s.th.
  $d(0,x) \sim |x|^{\mu}$ for $\gamma > 2$, $\alpha > 2d$.

- Critical behavior.

SUMMARY

1. Power-law degrees
   Tail-exponent : γ

2. In $d \geq 2$: non-trivial critical value
   if $\gamma > 2$, always percolation
   if $\gamma < 2$.

3. Phase-transition for distances
   at $\gamma = 2$.