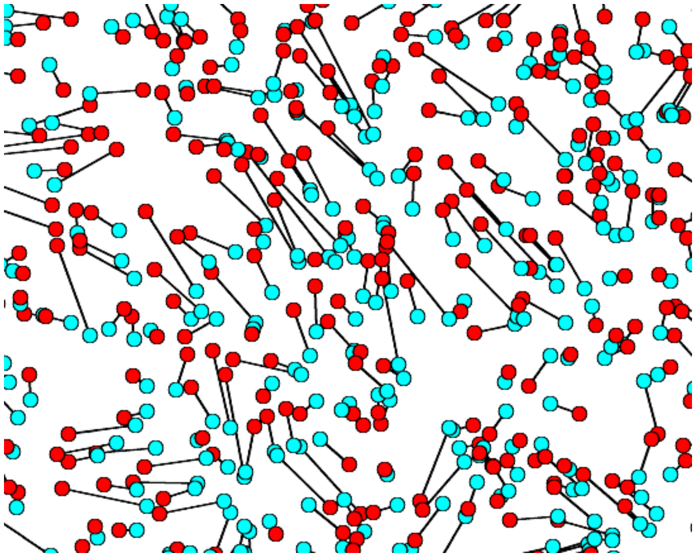
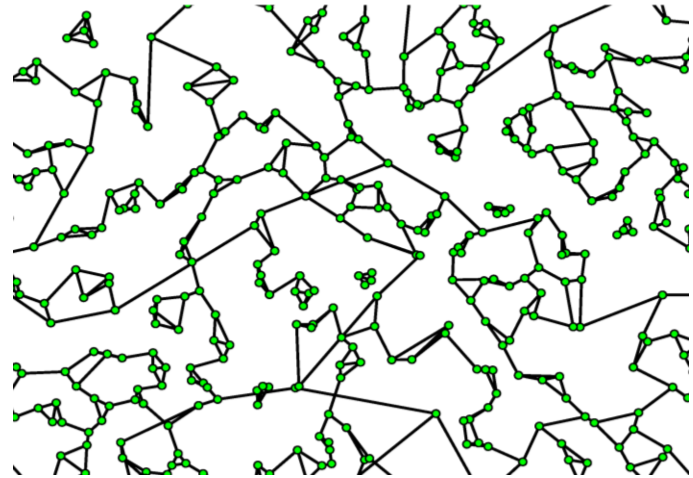
Invariant Matching

Alexander E. Holroyd, Microsoft Research

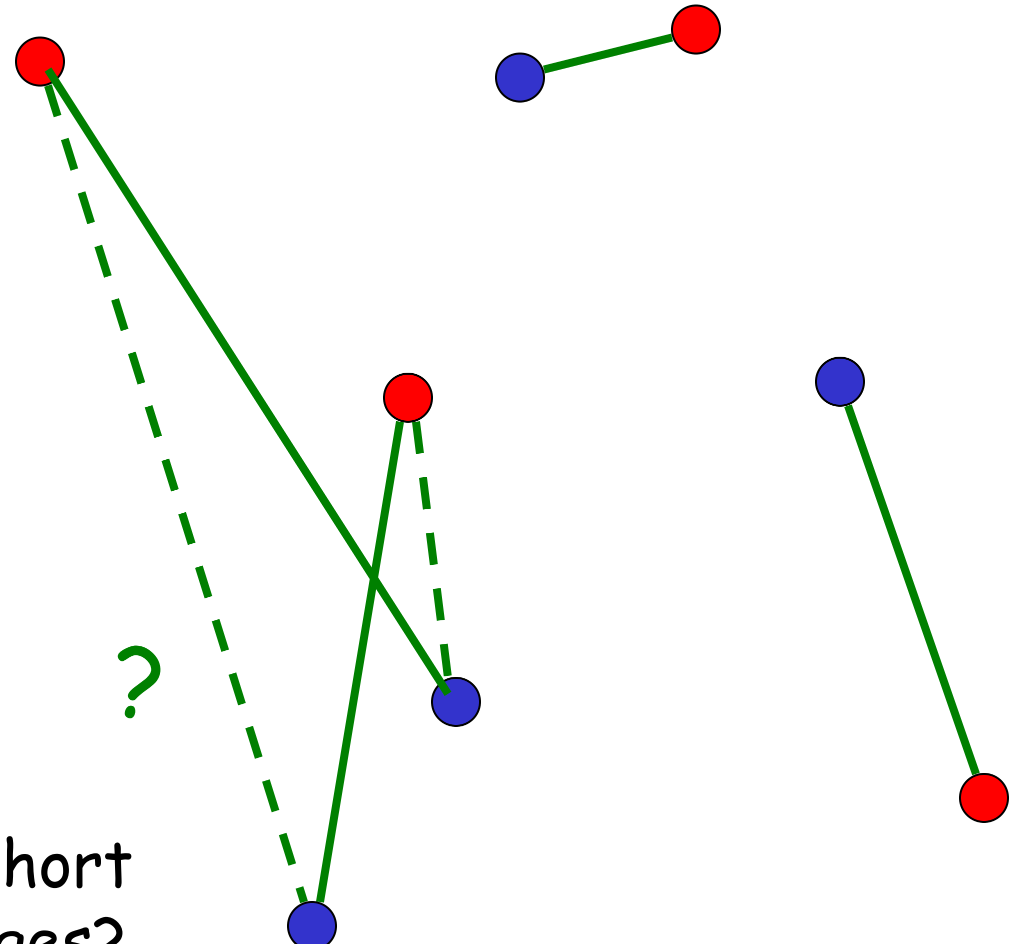




Red points

Blue points

Perfect matching

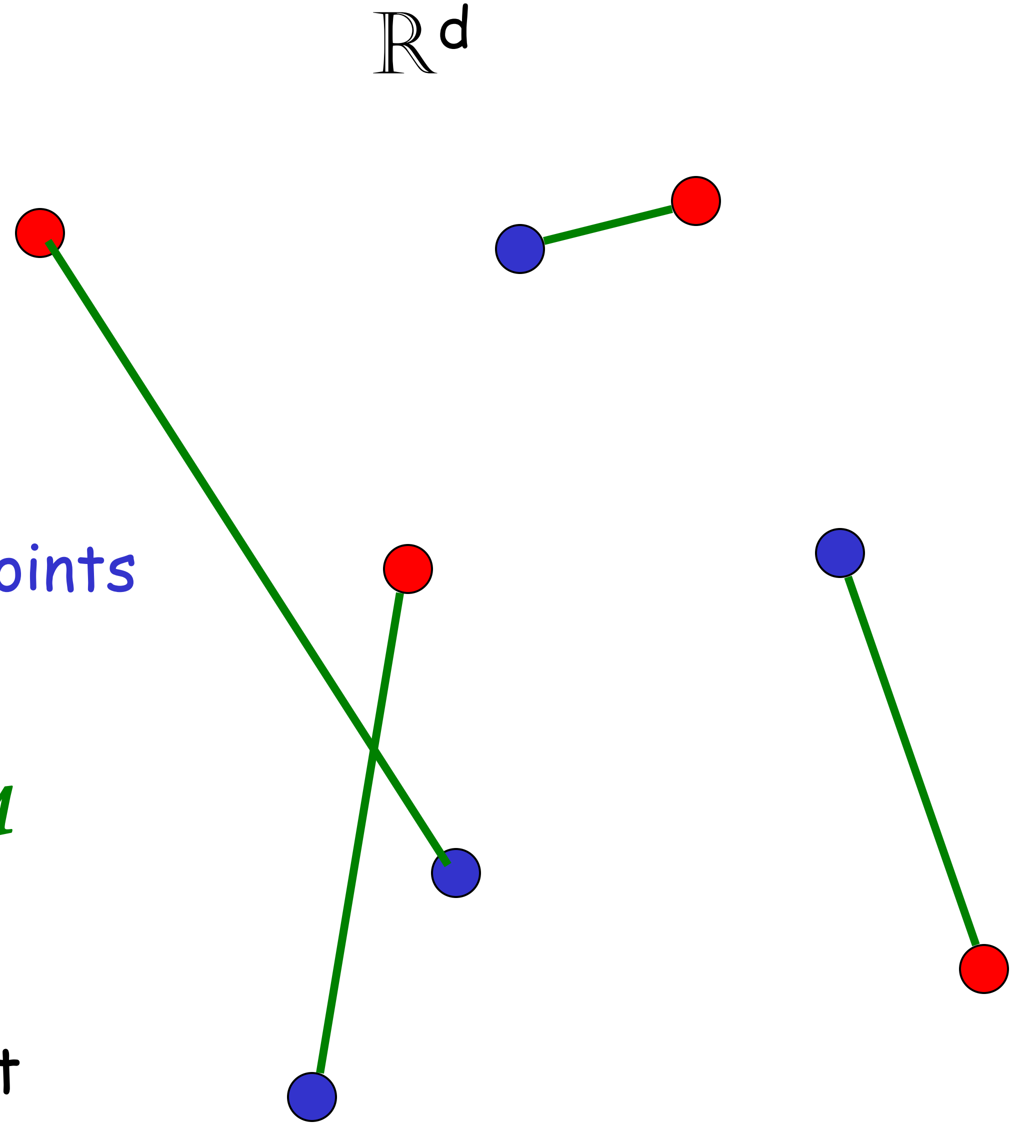


Questions:

Quantitative- how short can we make the edges?

Geometric...

Local/greedy/non-random matching rules?



Intensity-1 Poisson process $\mathcal{R}$ of red points

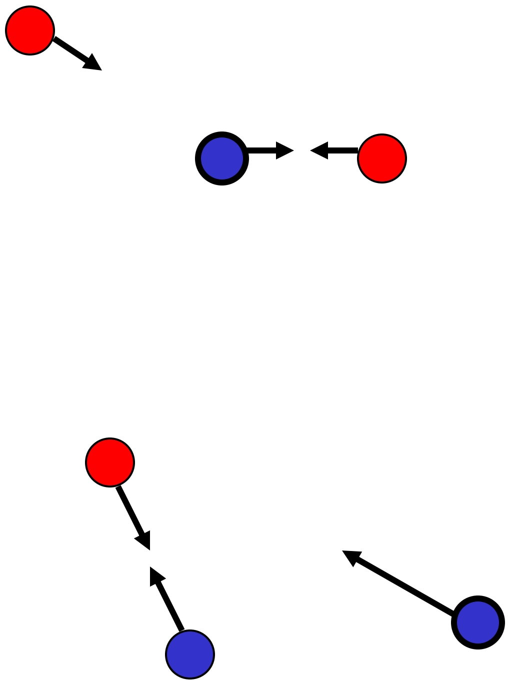
Independent intensity-1 Poisson process $\mathcal{B}$ of blue points

Random perfect matching scheme $\mathcal{M}$

Assume $(\mathcal{R}, \mathcal{B}, \mathcal{M})$ translation-invariant in law

Example: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.



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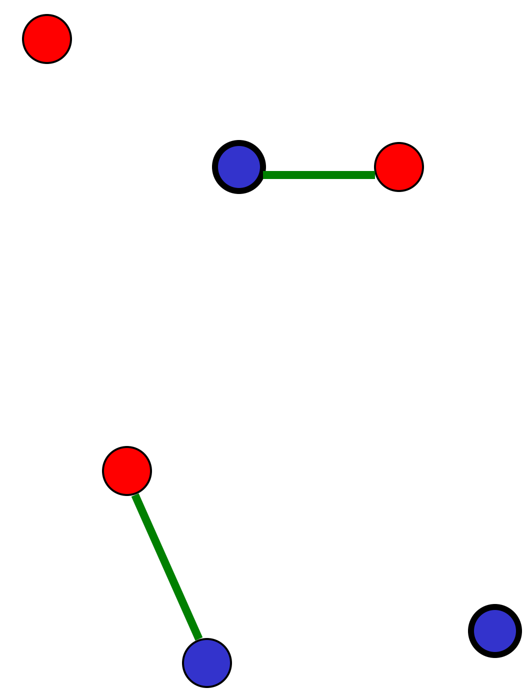
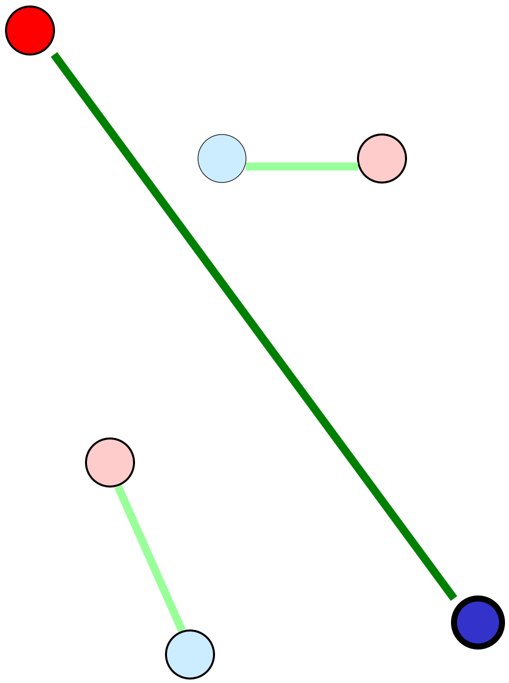


Example: Gale-Shapley stable matching.

- Match all *mutually closest* red/blue pairs.

- Remove them

- Repeat indefinitely



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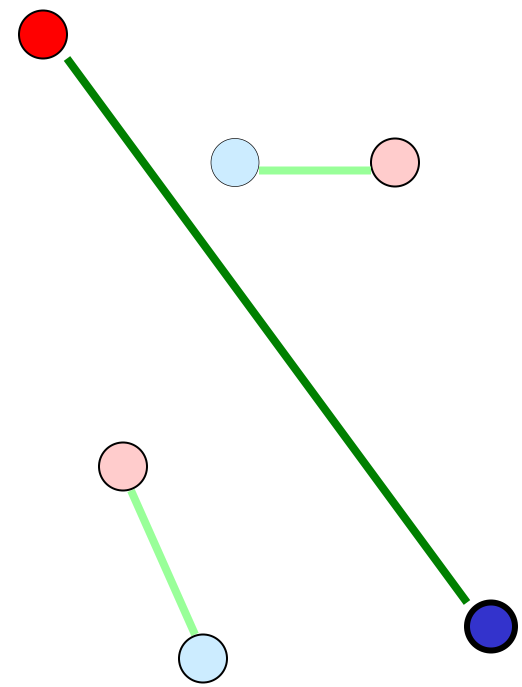

Why does every point get matched?

R={∃ unmatched red point}
B={∃ unmatched blue point}

- Ergodicity ⇒ P(R),P(B) ∈ {0,1}

- Algorithhm ⇒ cannot have P(R)=P(B)=1

- Averaging/Mass-transport/Fubini: unmatched red & blue points have equal intensity, so cannot have one=1, other=0

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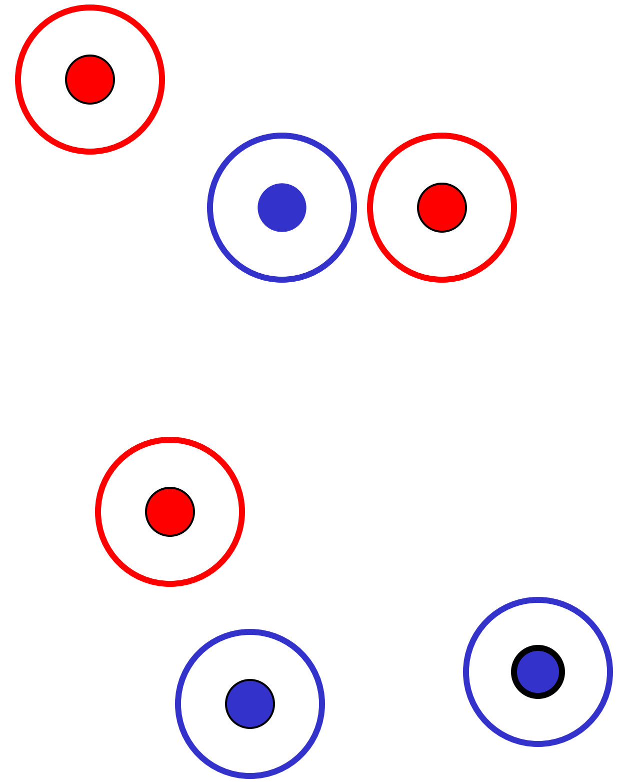

Alternative description:
ball-growing

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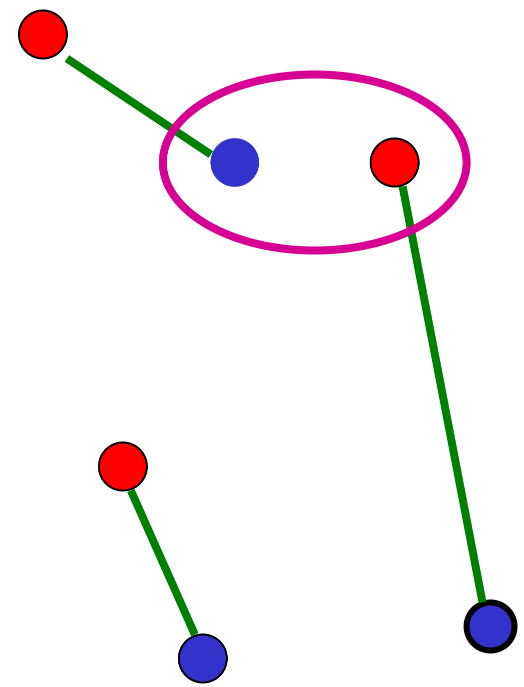
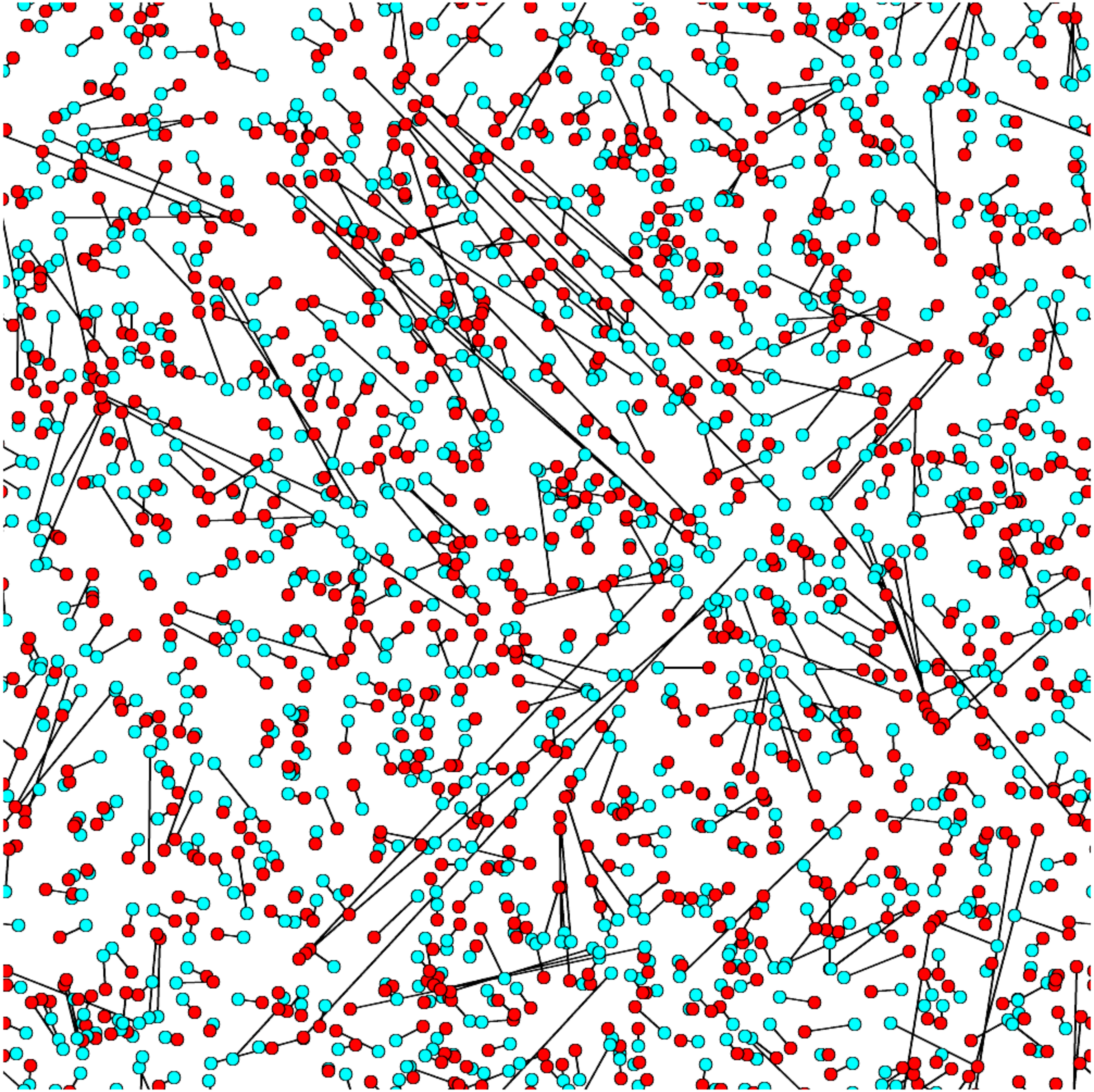


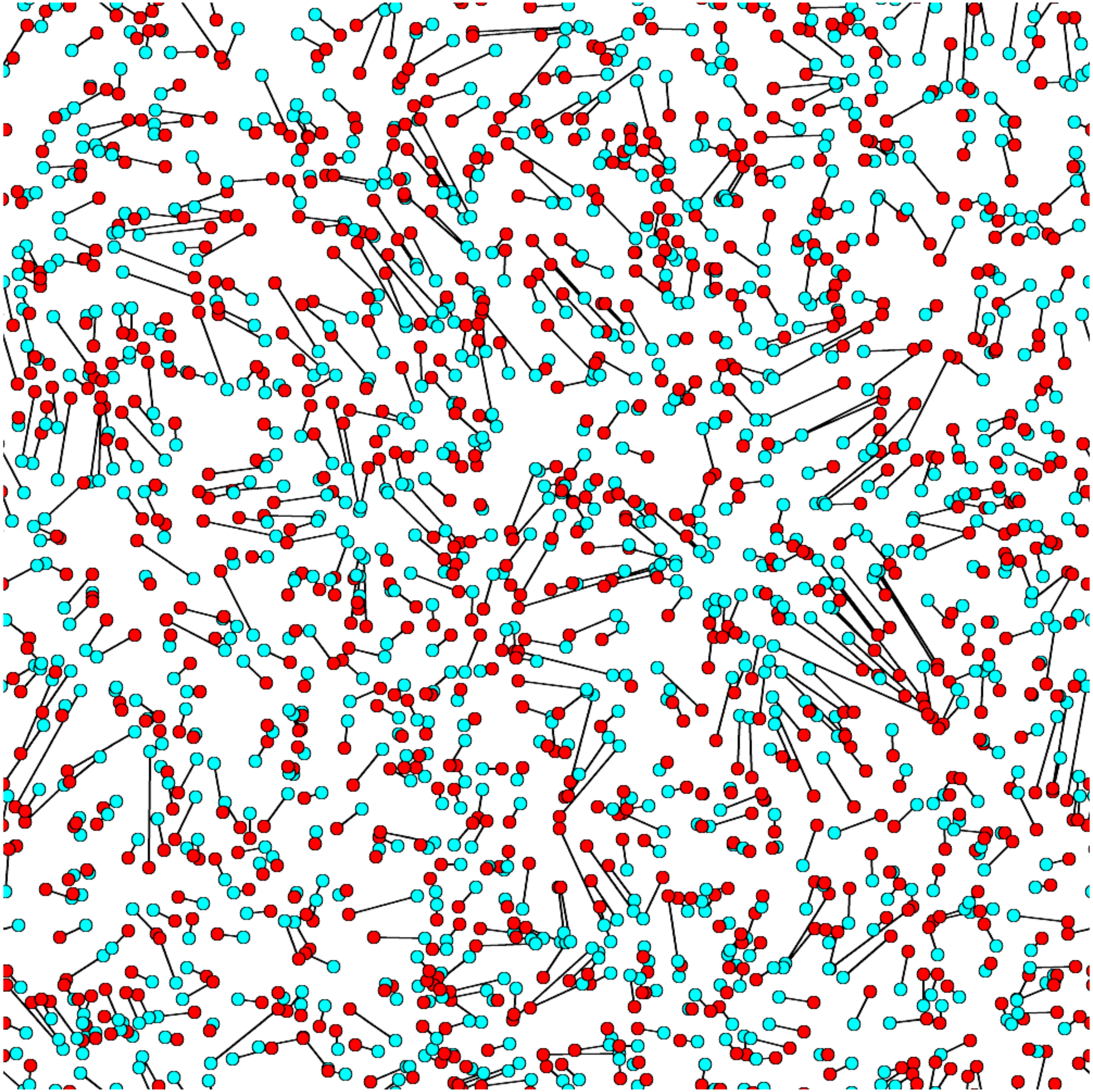
Alternative description:
ball-growing

Alternative description:
unique matching with
no *unstable* pairs



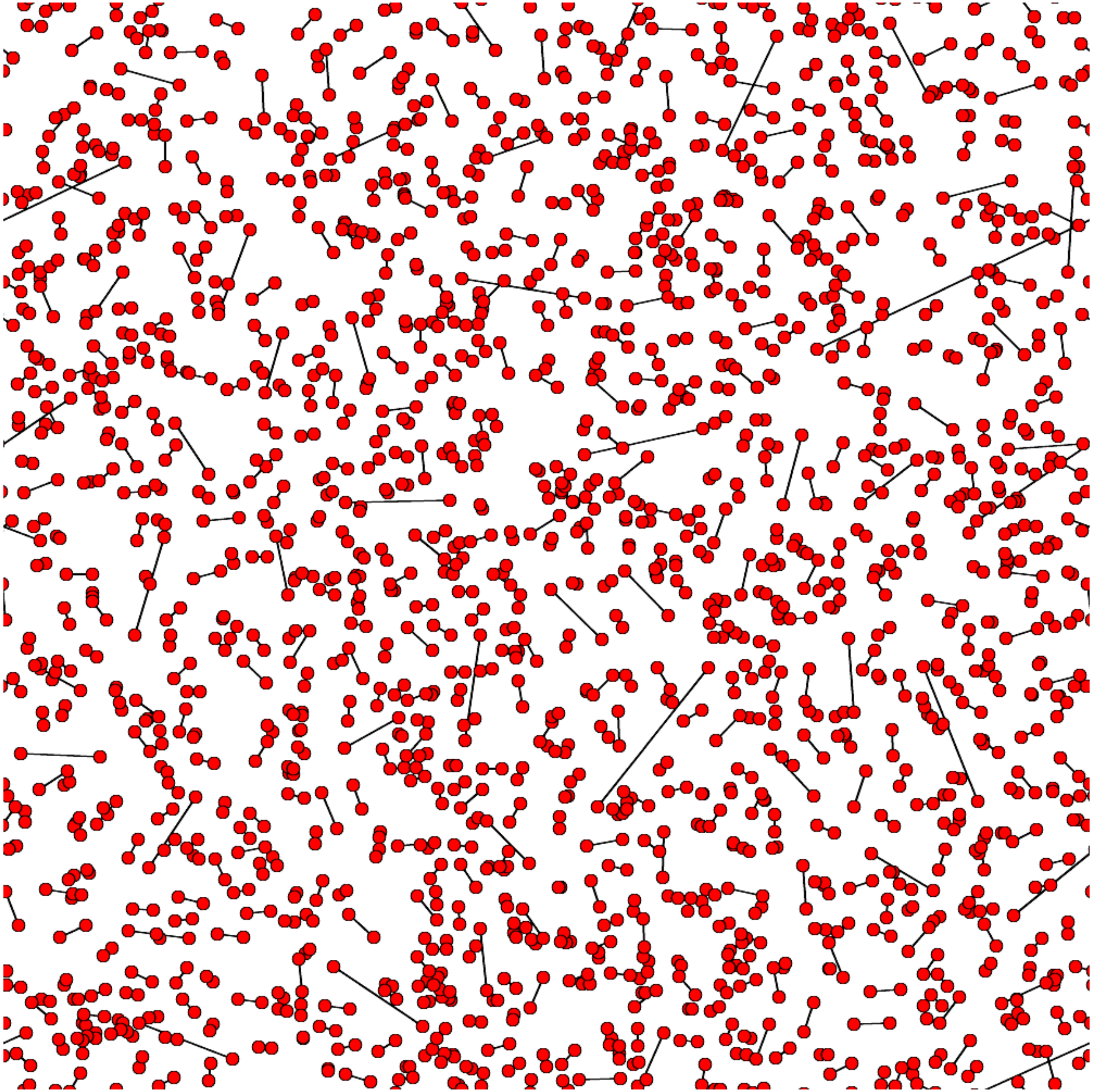
Two-colour stable matching

(on torus)



Two-colour minimum-length matching

(on torus)



One-colour stable matching

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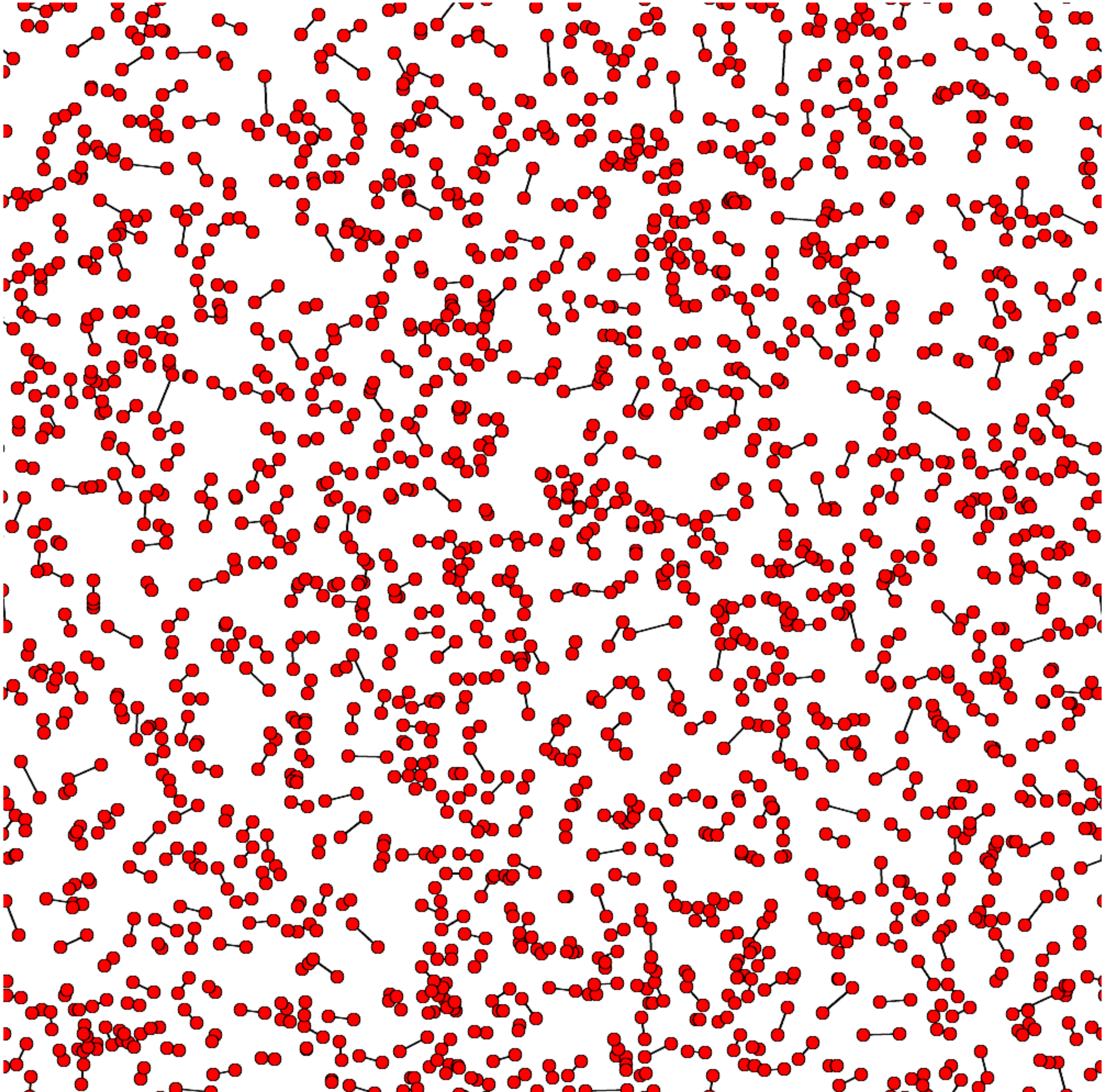


One-colour
minimum-
length
matching

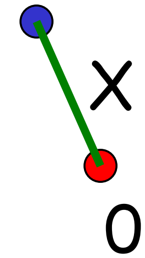
(on torus)

Call a matching scheme

- a *factor* if $\mathcal{M} = f(\mathcal{R}, \mathcal{B})$
  (e.g. stable matching)

- *randomized* if not

Given a matching scheme $\mathcal{M}$,



denote X = length of "typical edge"

$= |0-\mathcal{M}(0)|$ "conditioned" on {0 is red}

(i.e. under Palm measure P*)

i.e. $P^*(X \le r) :=$

$E \# \{\text{red points } z \in [0,1)^d \text{ with } |z-\mathcal{M}(z)| \le r\}$

Question: how small can we make X

(in terms of tail behaviour)?

A trivial lower bound: for any matching,

$P^*(X > r) \ge P^*(\exists \text{ no other point in } B(0,r)) \ge e^{-cr^d}$

i.e. $E^* e^{cX^d} = \infty$

More results (H., Pemantle, Peres, Schramm 2008):

One color		Lower bound	Upper bound
Randomized	$d=1$ $d\geq 2$		
Factor	$d=1$ $d\geq 2$		
Stable	All d		

Two color		Lower bound	Upper bound
Randomized	$d=1$ $d=2$ $d\geq 3$		
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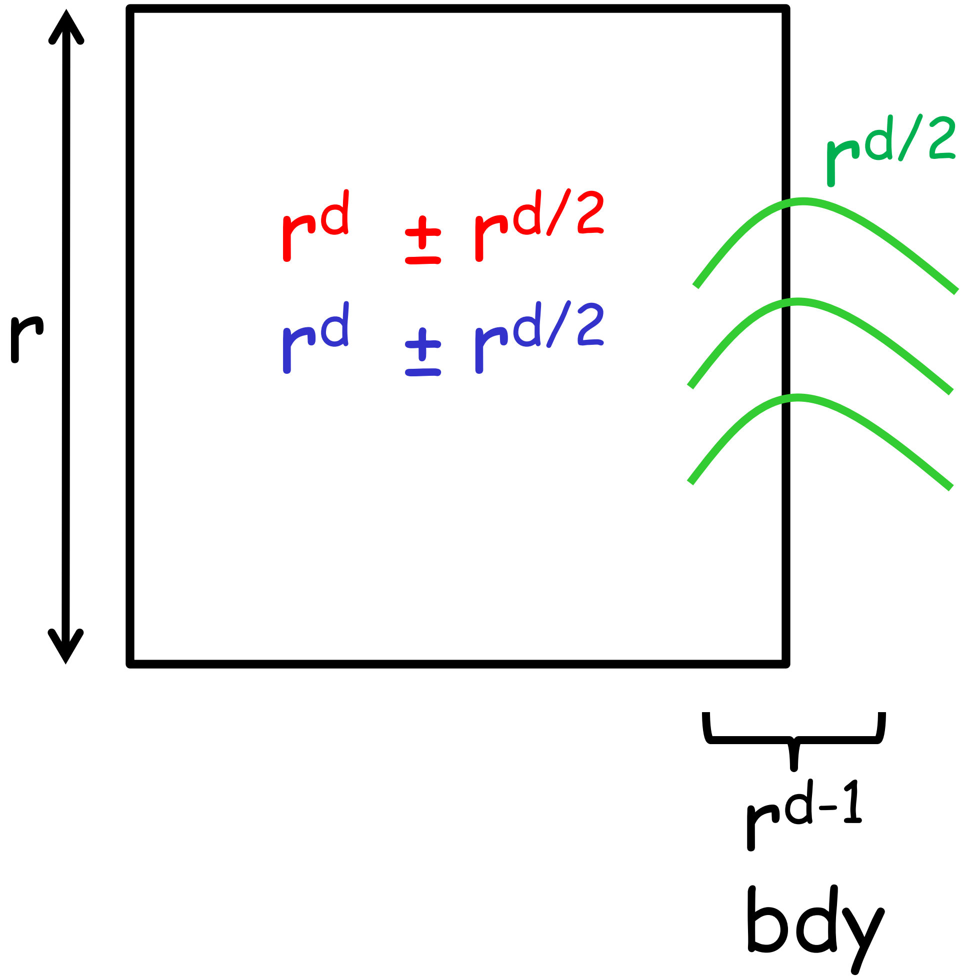
Heuristic reason:

$r^{d/2}$ excess



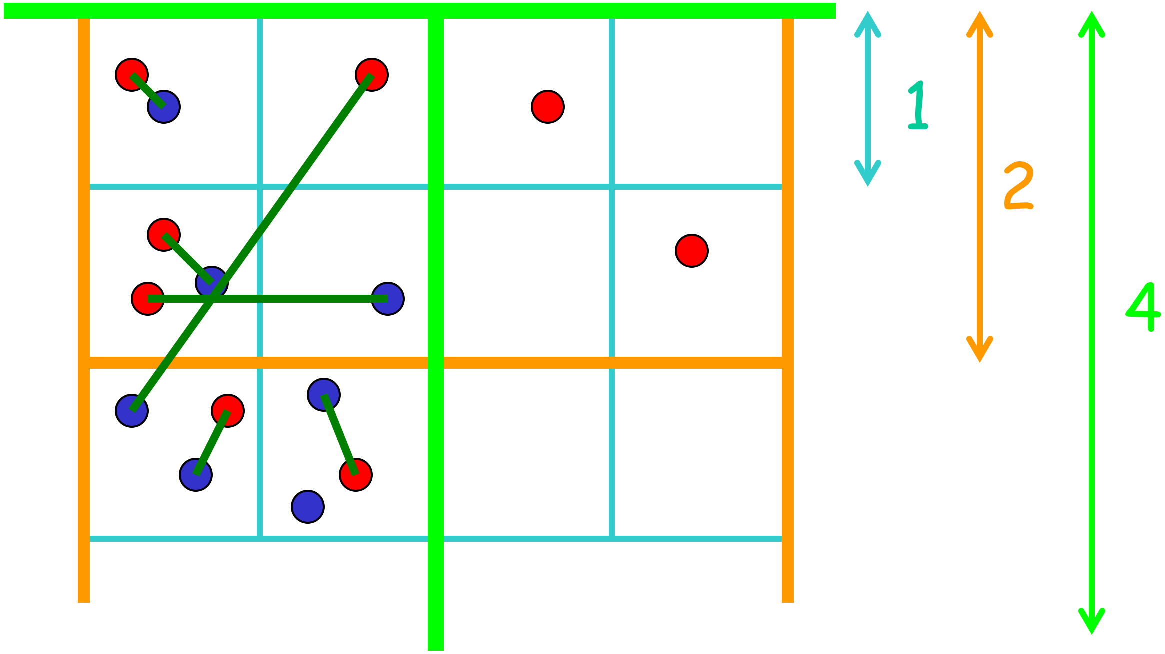
$d \le 2$: $r^{d/2} \ge r^{d-1}$

$P^*(X>r) \approx r^{d/2}/r^d$

$d \ge 3$: $r^{d/2} \ll r^{d-1}$

match "locally"

Two-color randomized matching



Invariant random dyadic partitioning

....

Match as much as possible within level-1 cubes

Match as much as possible within level-2 cubes

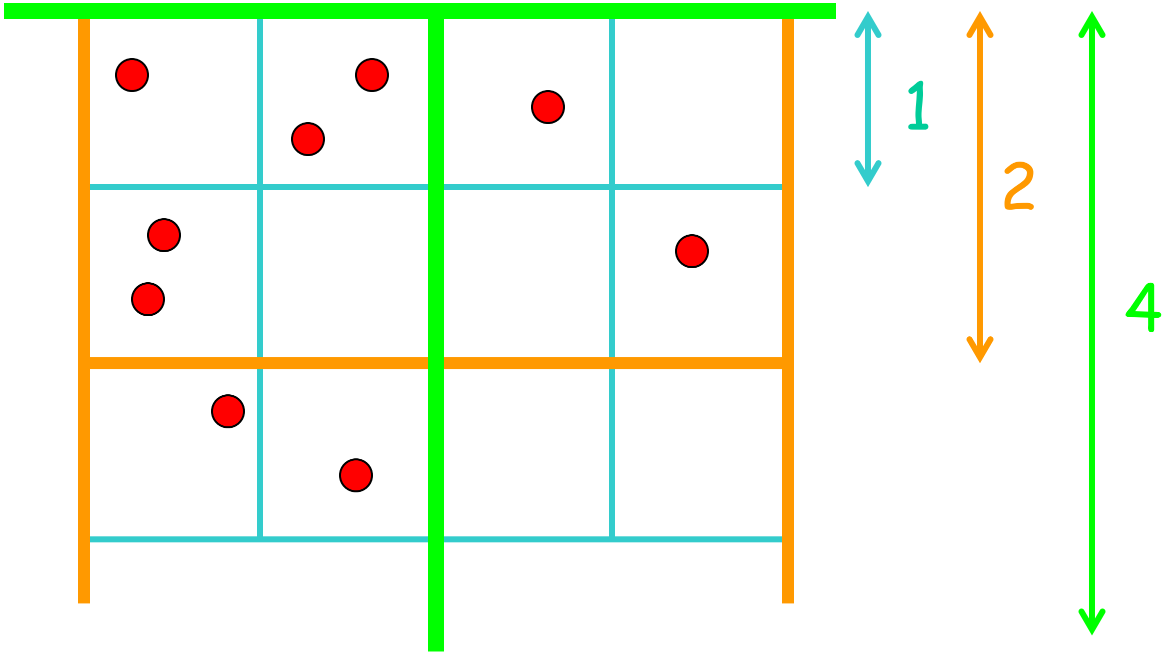
Etc.

$P^*(X > c2^k) \leq P^*(0 \text{ not matched by stage } k)$

$\leq P^*(0 \text{ in "excess" in its level-k cube}) \leq C\sqrt{[2^{kd}]}/2^{kd} = C(2^k)^{d/2}$

Two-color randomized matching: better method for $d \geq 3$

(based on Ajtai-Komlos-Tusnady)



Invariant random dyadic partitioning

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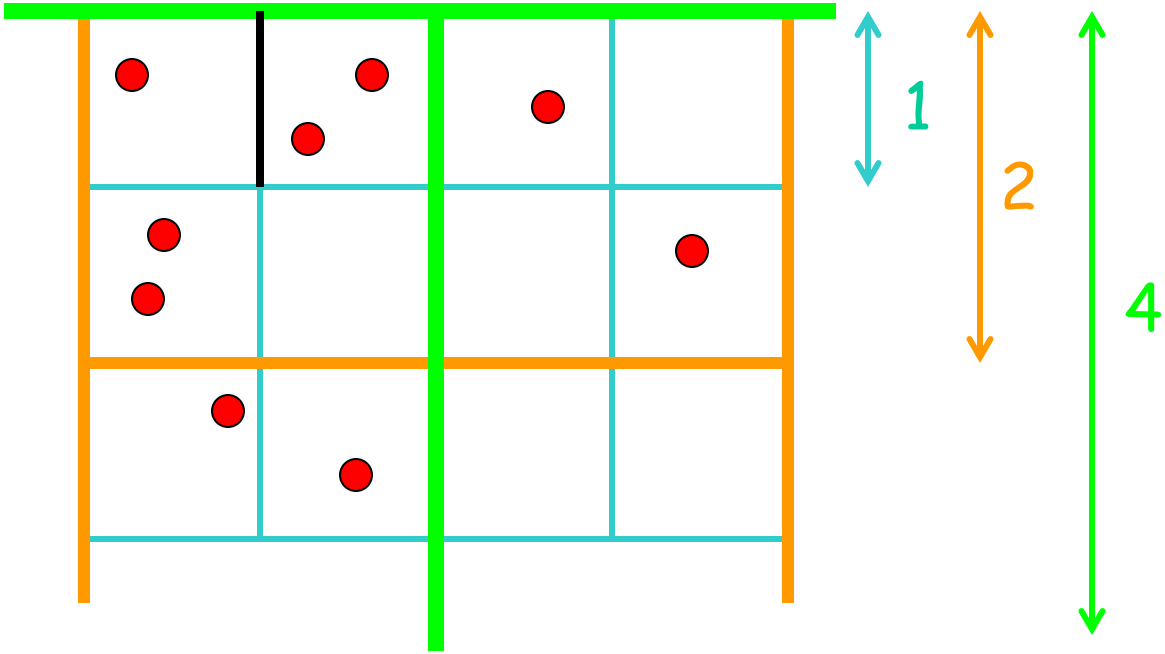

Invariant random dyadic partitioning

....

Repartition to equalize points per unit volume, affinely shift points

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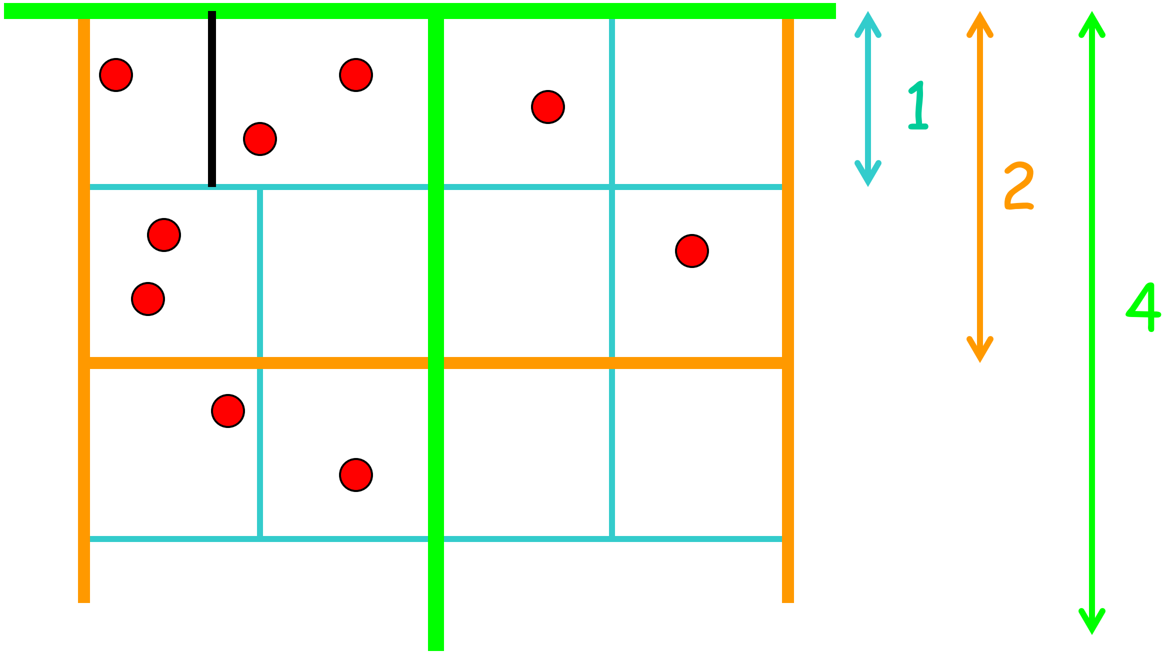

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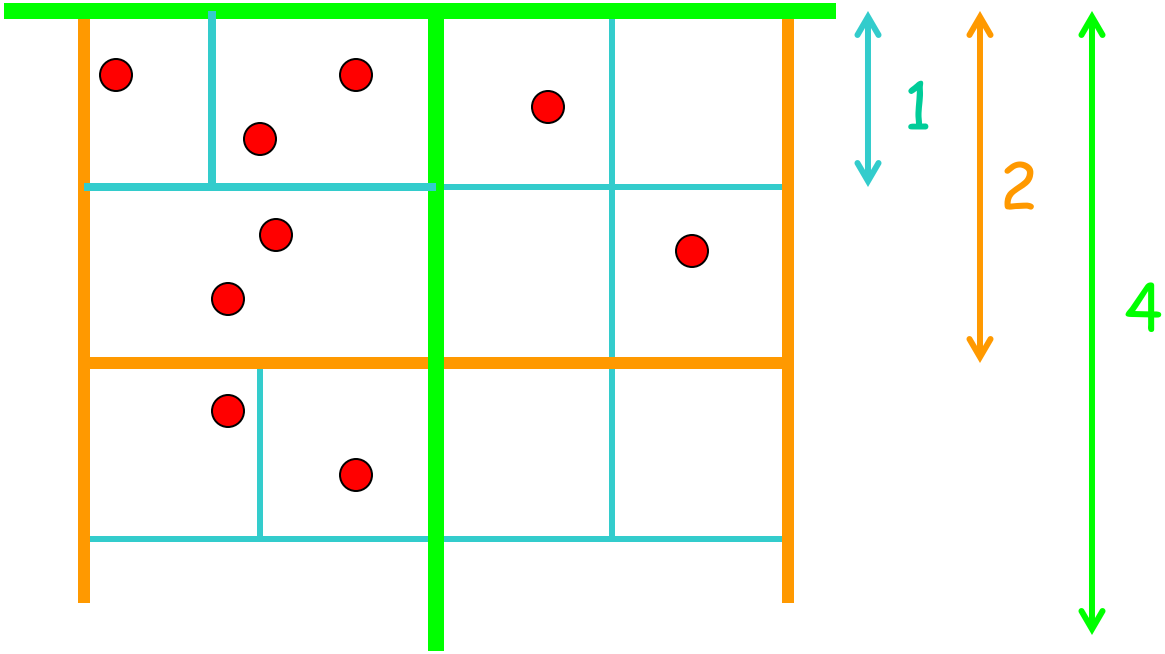

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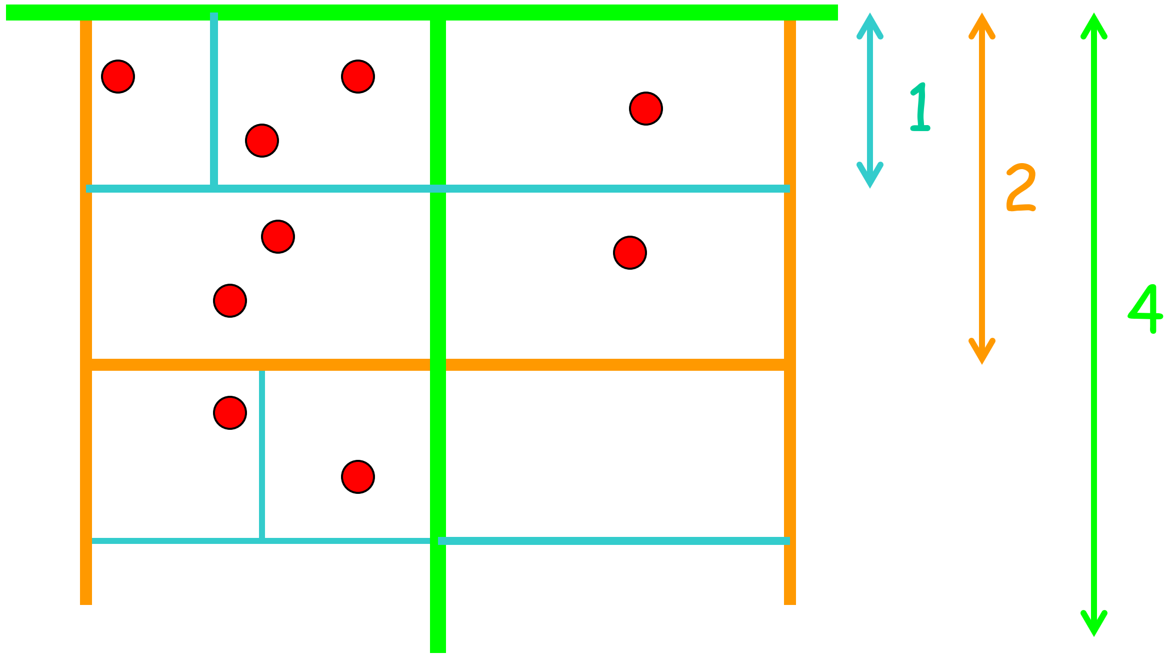

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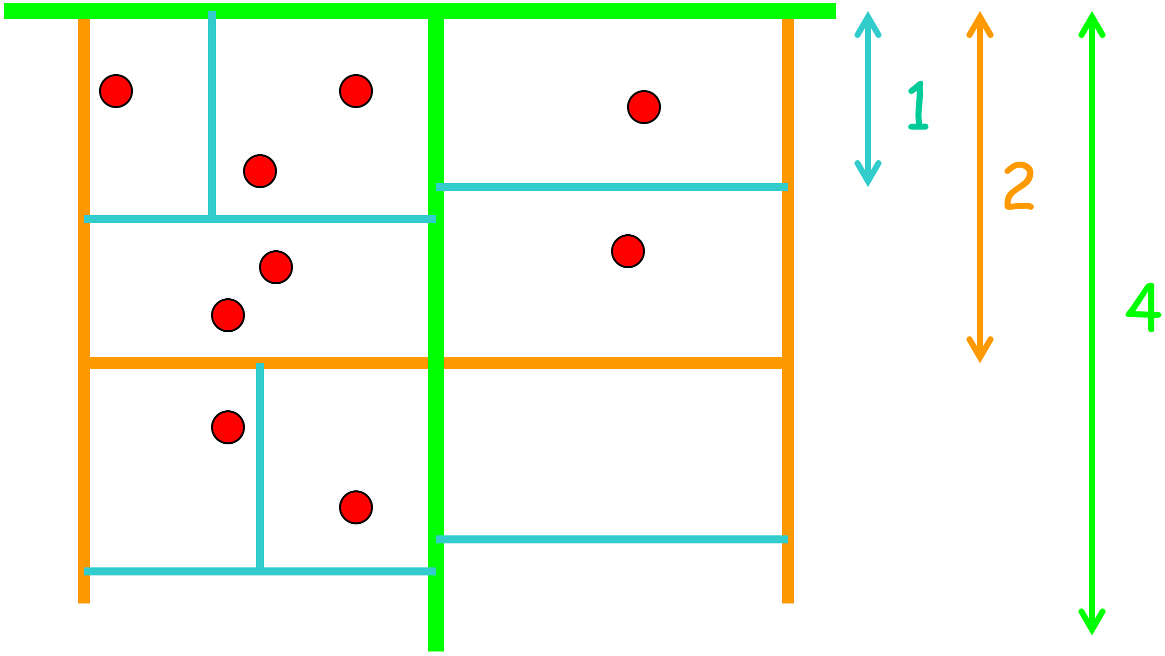

Invariant random dyadic partitioning

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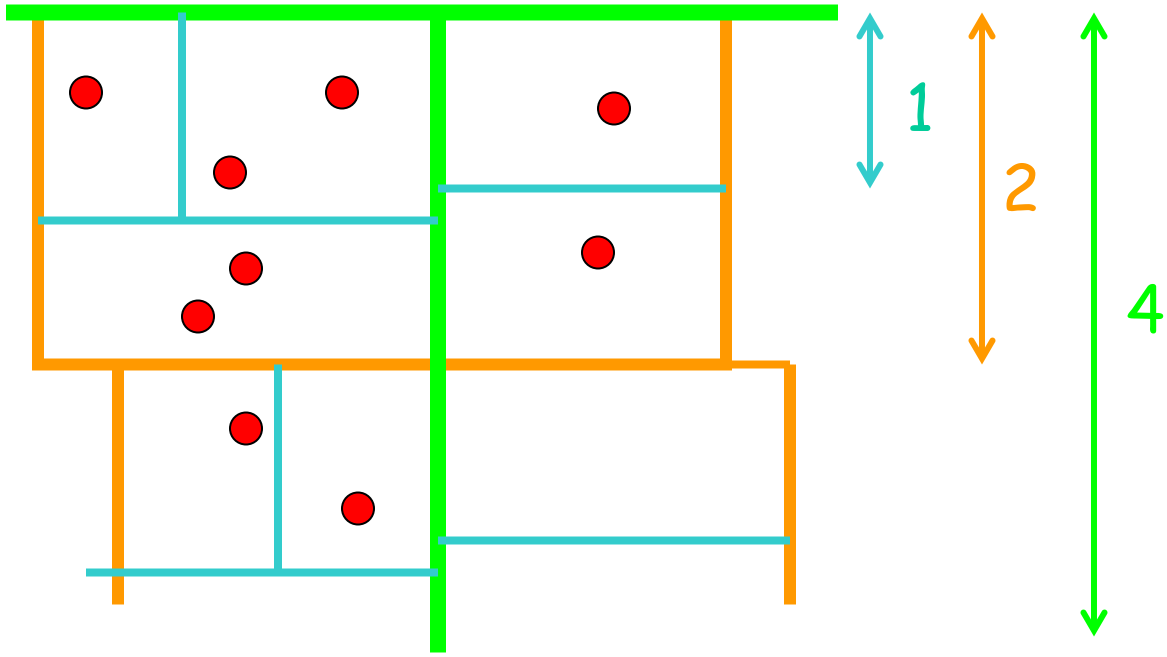
Repartition to equalize points per unit volume, affinely shift points

Iterate

Two-color randomized matching: better method for d≥3
(based on Ajtai-Komlos-Tusnady)



Invariant random dyadic partitioning

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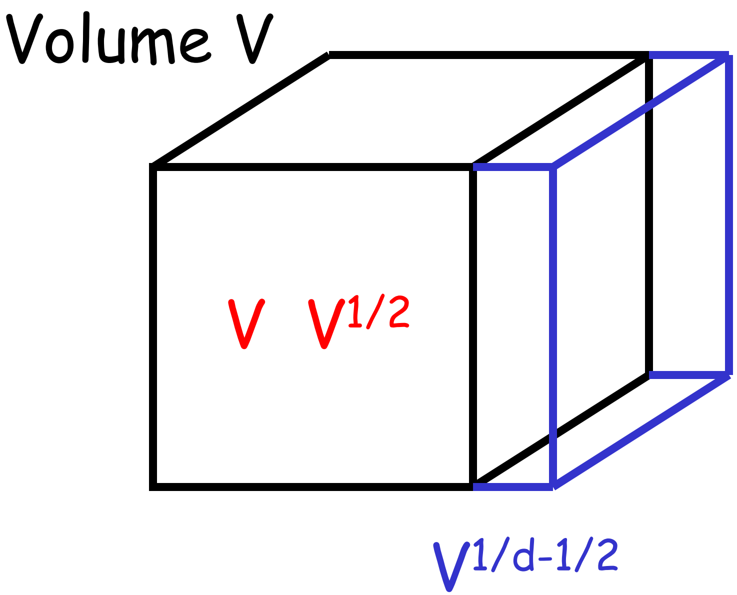
Repartition to equalize points per unit volume, affinely shift points

Iterate...

Get allocation of 1 unit volume to each point
... abstract arguments $\Rightarrow$ matching



Total distance moved by a typical point $\sum_{k=1}^{\infty} (2^k)^{1/d-1/2}$

$< \infty$ for d $\geq$ 3

Biggest deviation: empty cubes...

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Factor	$d=1$	$E^* e^{cX^d} = \infty$	
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Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C\, r^{-2/3+\varepsilon}$ [Soo]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$P^*(X > r) < C\, r^{-2d/(d+4)+\varepsilon}$ [Soo]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
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	$d=2$	$E^* X = \infty$	$P^*(X > r) < C\, r^{-1}$ [Timar]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
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Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C\, r^{-1}$
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C\, r^{-1}$ [Timar]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C\, r^{-1/2}$
	$d=2$	$E^* X = \infty$	
	$d\geq 3$	$E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{CX} < \infty$
	$d\geq 2$	$E^* e^{cX^d} = \infty$	
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d\geq 2$	$E^* e^{cX^d} = \infty$	
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d\geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
	$d\geq 3$	$E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{cX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Stable	All d	$E^* e^{cX^d} = \infty$	

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	

One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{CX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$P^*(X > r) < C r^{-d}$

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{CX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	
	$d \geq 3$	$E^* e^{cX^d} = \infty$	

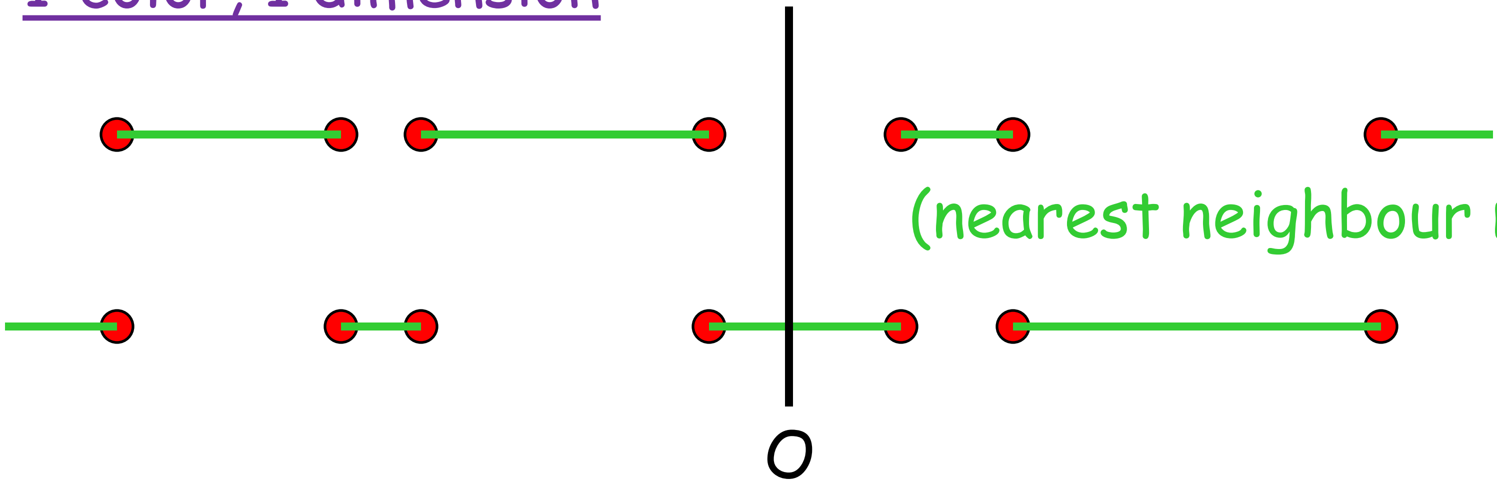
One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{cX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$P^*(X > r) < C r^{-d}$

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-0.496...}$
	$d \geq 3$	$E^* X^d = \infty$	$P^*(X > r) < C r^{-s(d)}$

One color		Lower bound	Upper bound
Randomized	$d=1$	$E^* e^{cX} = \infty$	$E^* e^{cX} < \infty$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 2$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Stable	All d	$E^* X^d = \infty$	$P^*(X > r) < C r^{-d}$

Two color		Lower bound	Upper bound
Randomized	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^d} < \infty$
Factor	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-1}$ [Timar]
	$d \geq 3$	$E^* e^{cX^d} = \infty$	$E^* e^{cX^{d-2}} < \infty$ [Timar]
Stable	$d=1$	$E^* X^{1/2} = \infty$	$P^*(X > r) < C r^{-1/2}$
	$d=2$	$E^* X = \infty$	$P^*(X > r) < C r^{-0.496...}$
	$d \geq 3$	$E^* X^d = \infty$	$P^*(X > r) < C r^{-s(d)}$

<u>1-color, 1 dimension</u>

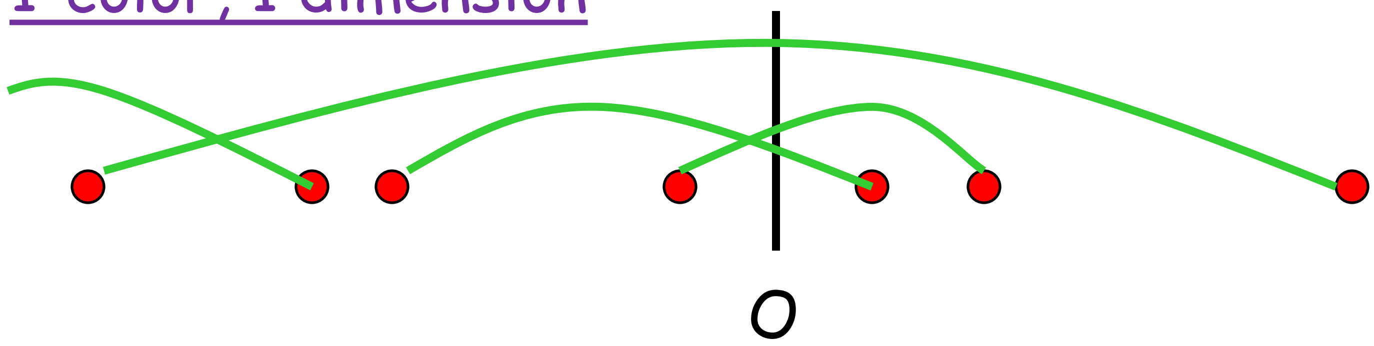


- a <u>randomized</u> matching with $P^*(X > r) = e^{-r}$

But ∄ a <u>factor</u> nearest neighbour matching

Any factor matching has $E^*X = \infty$. Proof:

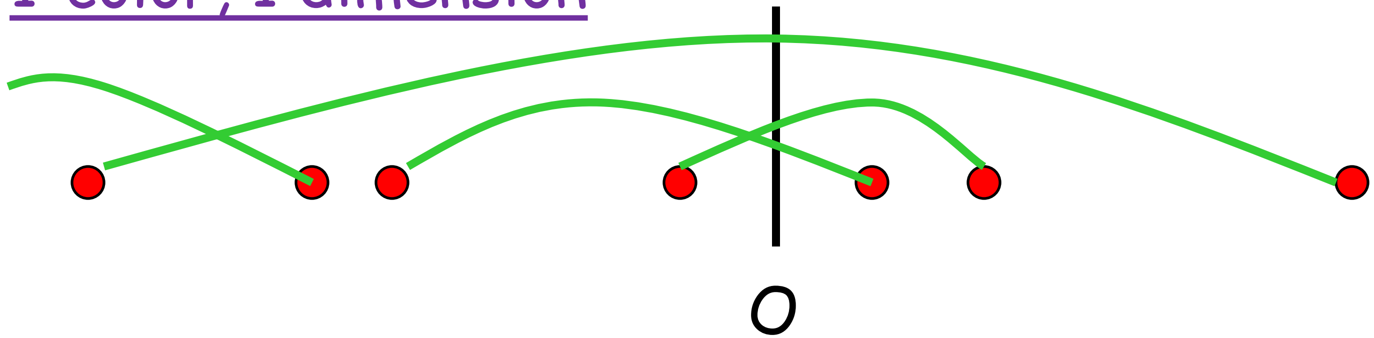
1-color, 1 dimension



Enough to show:

$$E(\#\text{ edges crossing } O) = \infty$$

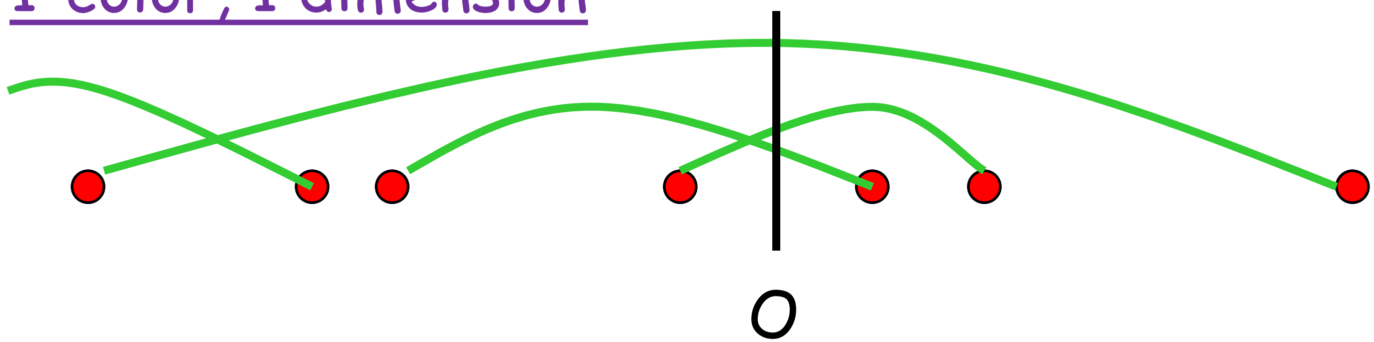
1-color, 1 dimension



Enough to show:

$$P(\#\text{ edges crossing } O = \infty) = 1$$

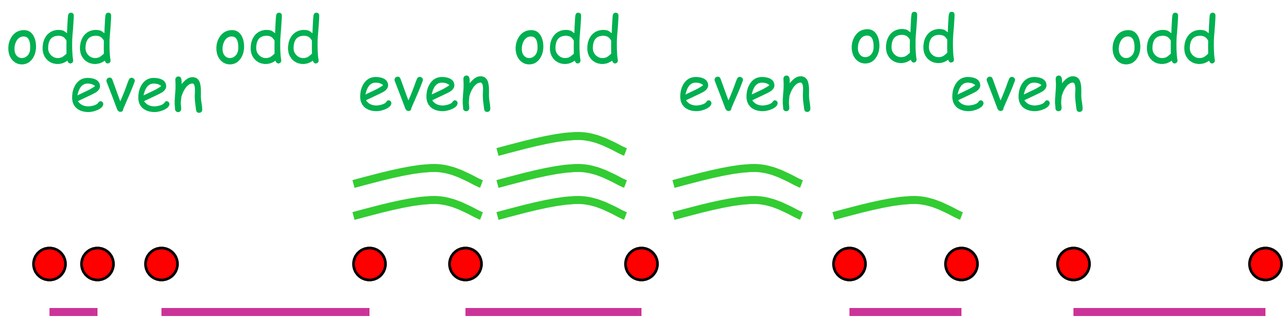
1-color, 1 dimension



Suppose:

$P(\#\text{ edges crossing } O < \infty) > 0$

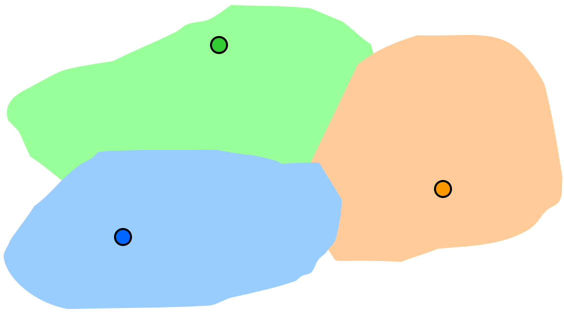
$\Longrightarrow$ $P(< \infty \text{ edges crossing every site}) = 1$



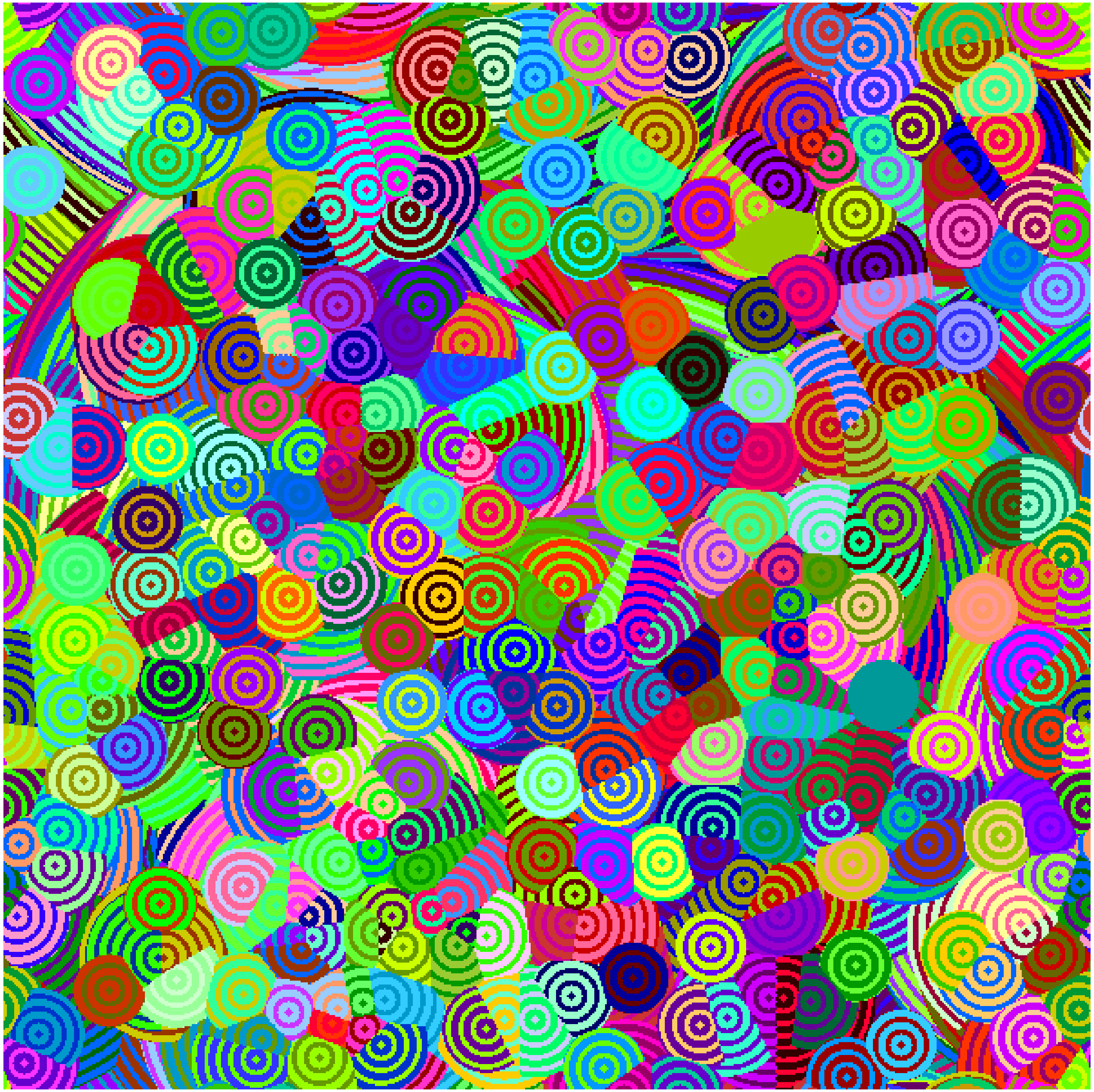
Rematch $\Rightarrow$ factor nearest neighbour matching! #

Variant problem: **allocation**

Given a point process of intensity 1 in R^d, partition space into cells of volume 1, with each cell allocated to a point, in a translation-invariant way.



Applications to Palm processes, shift coupling:
H., Peres; Thorisson, Last

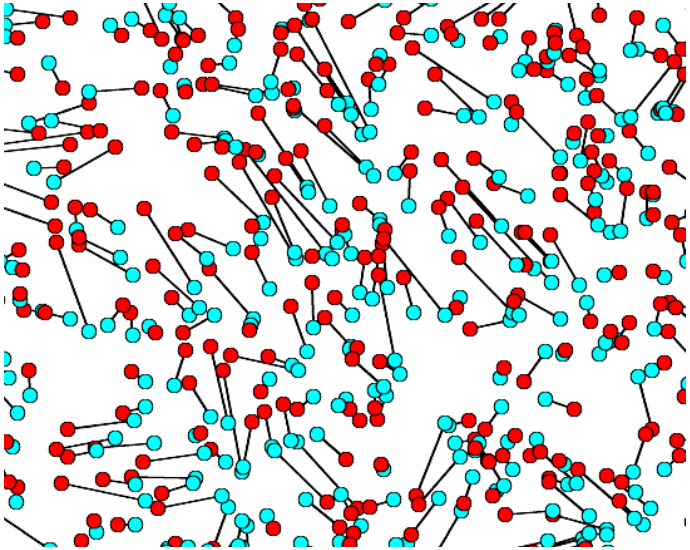


Stable allocation (Hoffman, H., Peres, 2005, 2009)

Gravtiational allocation, for Poisson process in d ≥ 3.
(Chaterjee, Peled, Peres, Romik, 2010).

Geometric questions for matchings:

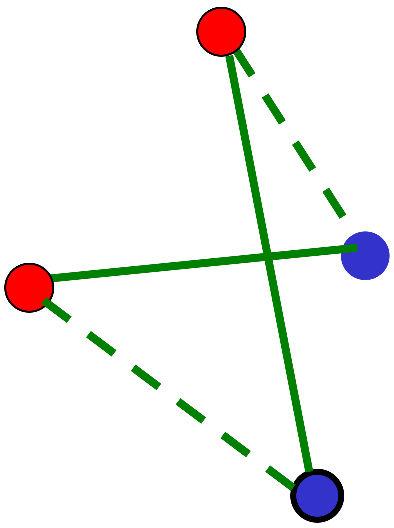
Q: For independent red and blue intensity-1 Poisson processes in R^2, does there exist a translation-invariant matching in which line segments joining matched pairs **do not cross**?



Proposition (H. 2009) Yes if we drop invariance, or for one color, or allow partial matching, or curved edges!

Q: For independent red and blue intensity-1 Poisson processes in R^2, does there exist a **minimal** translation-invariant matching, i.e. s.t. every finite set of edges minimizes the total length?

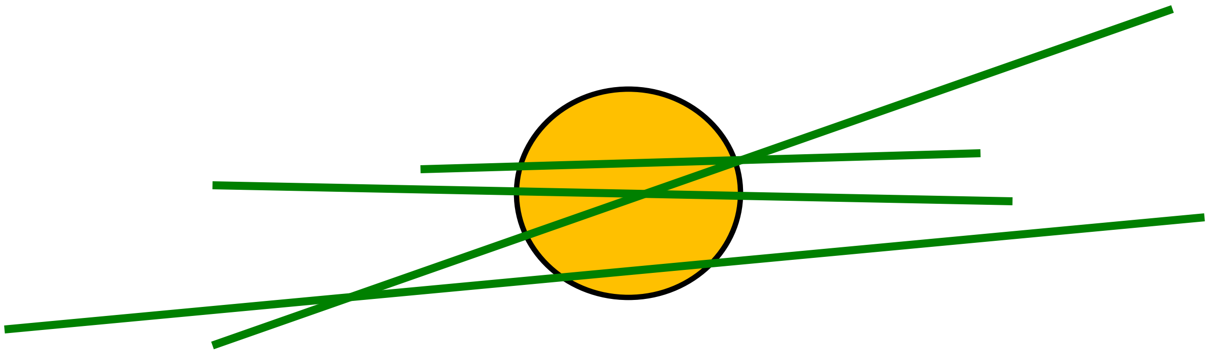
(If yes, then it would have no crossings)



Theorem (H. 2009) Yes in R^d, $d=1$ and $d\geq 3$

No in strip $R \times [0,1]$

For independent red and blue intensity-1 Poisson processes, does there exist a **locally finite** translation-invariant matching, i.e. s.t. any bounded set meets only finitely many edges?

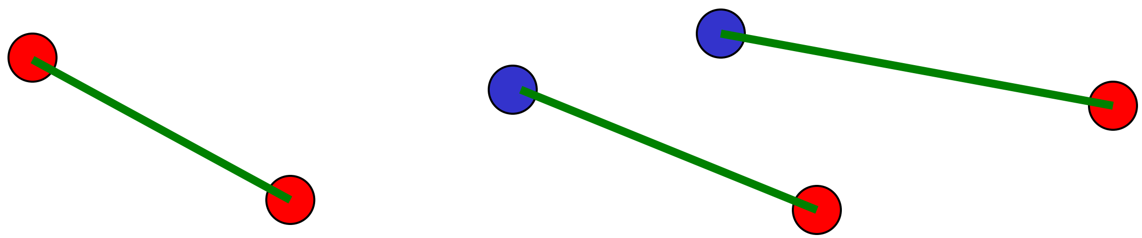


Theorem (H. 2009) Yes in R^d, $d \geq 2$
No in $d=1$, and strip

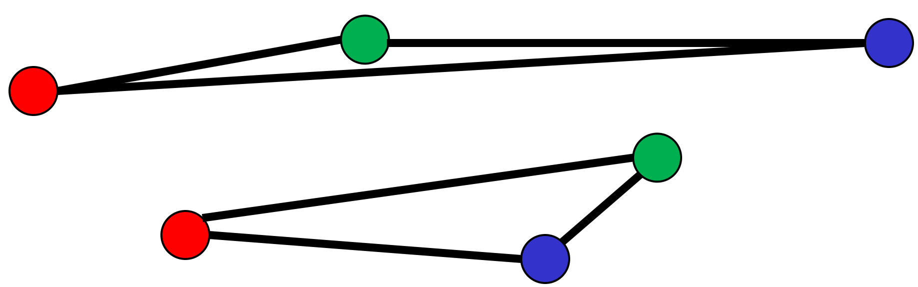
Multi-colour matching (Amir, Angel, H., in preparation)

Given independent Poisson processes of several colours. E.g.

may match red-blue or red-red



must match in red-green-blue families



families of RBB or RRRGB or GGY ....

X := diameter of "typical" family

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$ intensities of points each colour

Theorem

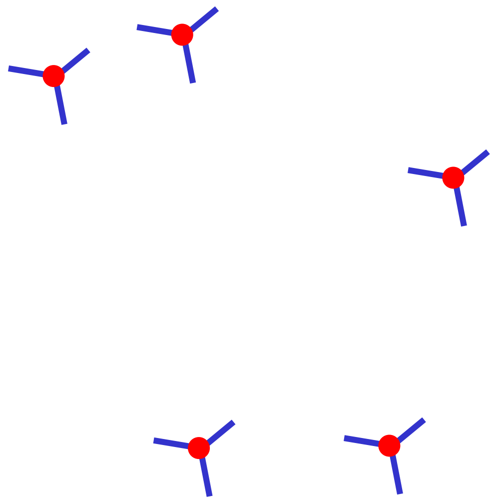
$\lambda \notin S \quad \Rightarrow$ no translation-invariant matching possible

$\lambda \in \partial S \quad \Rightarrow$ upper/lower bounds like 2-colour matching
(d/2 power in d≤2, exponential in volume in d≥3)

$\lambda \in S^o \quad \Rightarrow$ upper/lower bounds like 1-colour matching
(exponential in volume)

where $S \subset R^q$
is the cone generated by the allowed family vectors

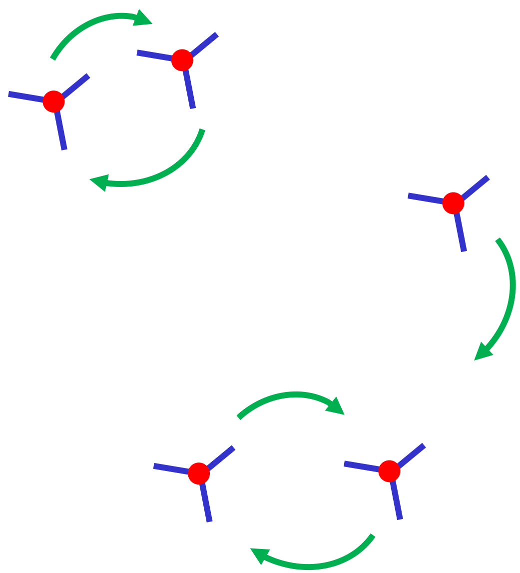
Stable Simple Graphs



$D = 3$

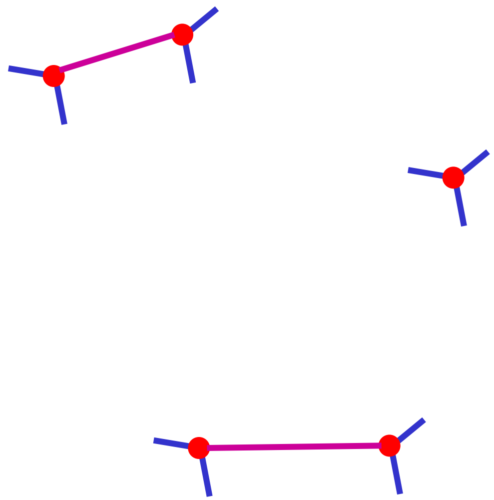
Assign each Poisson point x
a (deterministic or iid) degree D_x

Want a translation-invariant *simple* graph on the point process s.t. x has degree D_x



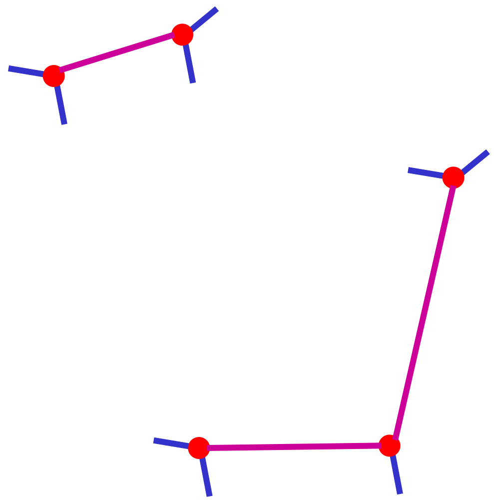
<u>Stable matching :</u>

- start with D_x stubs at each x
- each point x *looks at* closest other point with unused stubs and no edge to x already



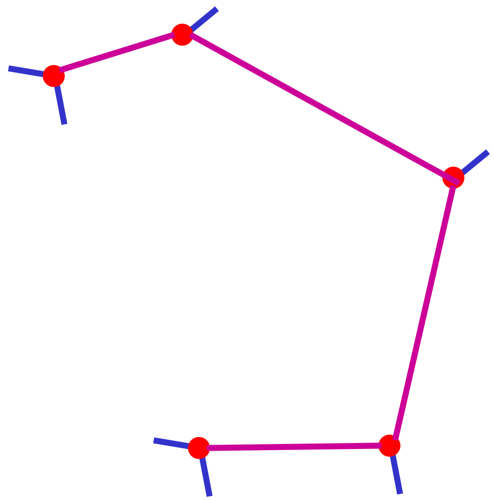
Stable matching:

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove one stub from each



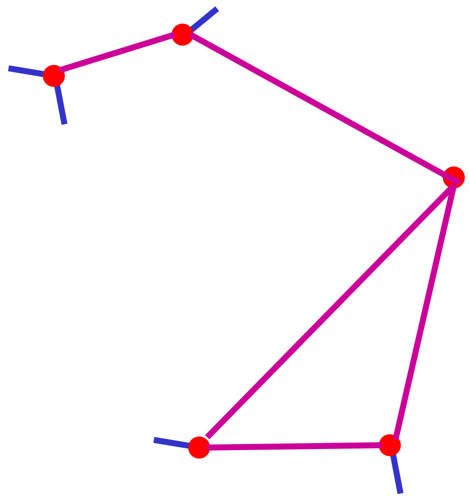
<u>Stable matching:</u>

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove stubs
- iterate



<u>Stable matching:</u>

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove stubs
- iterate



<u>Stable matching:</u>

- start with D stubs at every point
- each point x *looks at* closest other point with unused stubs and no edge to x already
- if x,y are looking at each other, match them, remove stubs
- iterate

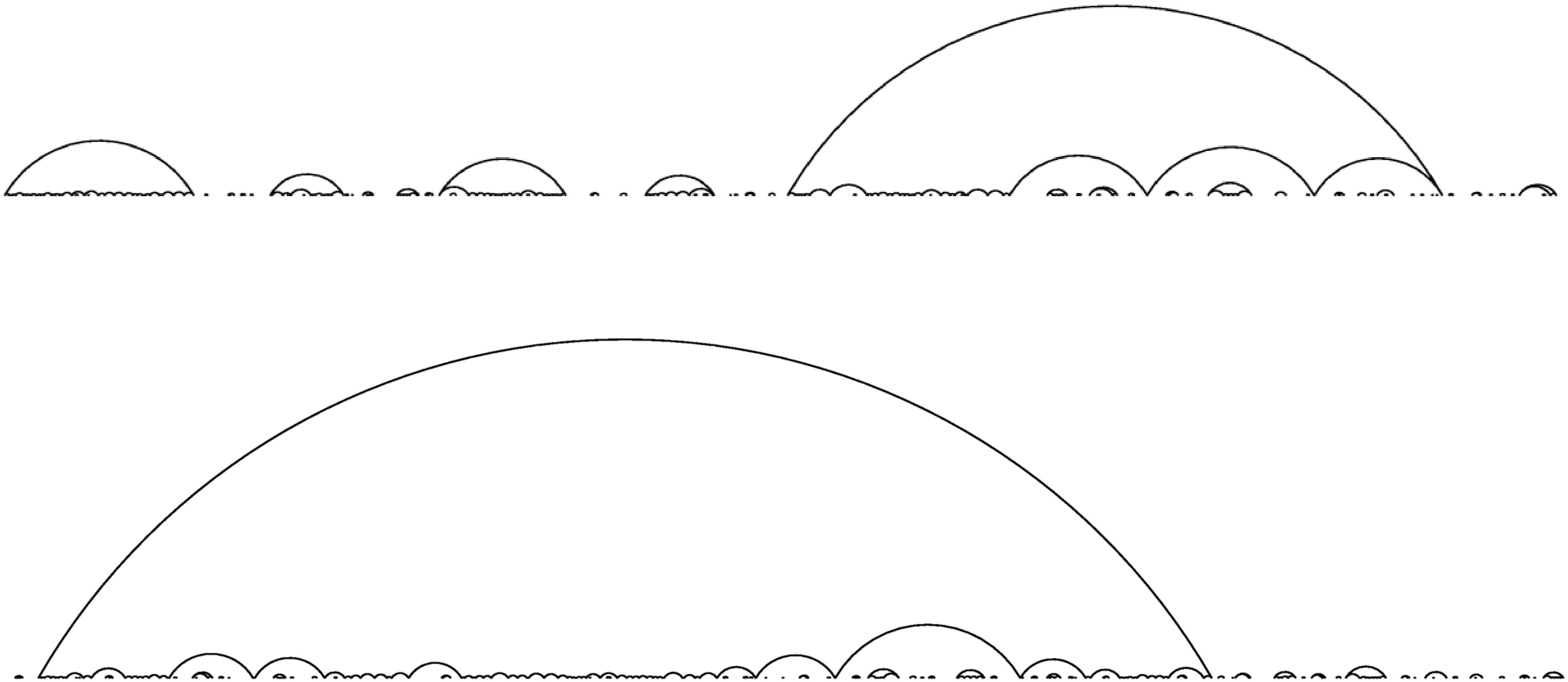
E.g.:
R^2, D=3

Central question: is there an infinite component

Theorem (Deijfen, Häggstrom, H., 2010)

- Yes if $d \geq 2$ and $P(D > k(d)) = 1$
- No if $P(D \in \{1,2\}) = 1$ with $P(D=1) > 0$.

Basic case: R^1, D=2:



Is there an infinite path?

not rigorously known in the case, but...

Theorem (Deijfen, H., Peres, 2011)
R^1 , D=2. For a certain event A_L, defined in terms of Poisson process on the *finite* interval [0,L],

if $P(A_L) > 0.97$ for some L, then
P(there is an infinite path) = 1.

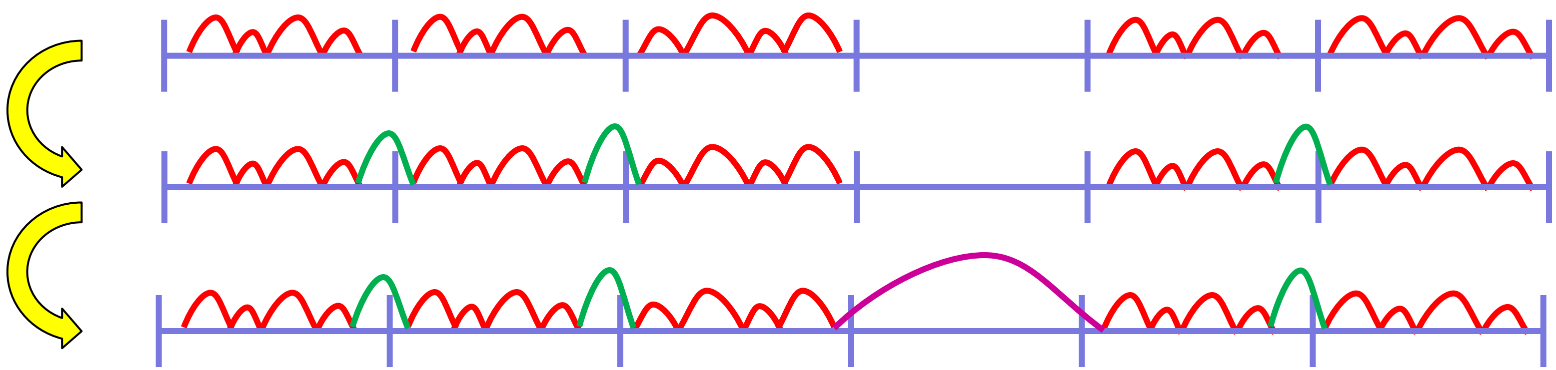
Simulations support $P(A_{13000}) > 0.97$
at the 99.9999% confidence level

(subject to trusting the software and the pseudo-random number generator)

In fact:

A_L = {there is a connection from $[0,\frac{L}{4}]$ to $[\frac{3L}{4},L]$
in subgraph constructed in $[0,L]$
assuming nothing about $[0,L]^c$ }

Proof proceeds by "hooking up" such intervals
on successively larger scales…



Open problems

Non-crossing 2-colour matching in R^2?

Better bounds for 2-colour stable matching?
e.g. d=2: $E^* X = \infty$ $P^*(X > r) < C\, r^{-0.496\ldots}$

Infinite component for stable Poisson graph
in R with degree D=2?

Stable multi-colour matching....?