A genealogy in Feller's branching diffusion with quadratic competition

Etienne Pardoux

with A. Wakolbinger

Ray–Knight's theorem as a genealogy in Feller's diffusion

- Consider the "standard" Feller branching diffusion

$$Z_t^x = x + 2\int_0^t \sqrt{Z_r^x}\,dW_r.$$

The branching property is the fact that Z_t^{x+y} has the same law as $Z_t^x + Z'^y_t$, where here we add two independent processes.

- Consider now the local time $\{L_s(y),\ s, y \geq 0\}$ of a reflected Brownian motion. Let $S_x = \inf\{s > 0,\ L_s(0) > x\}$. Then the second Ray–Knight theorem says that $\{L_{S_x}(t),\ t \geq 0\}$ has the same law as $\{Z_t^x,\ t \geq 0\}$.
- This may be understood as a description of the genealogy in Feller's branching diffusion, meaning that reflected Brownian motion codes (in the sense of Aldous) the "real tree" which describes the genealogy of the population which evolves according to Feller's diffusion.
- The description of the genealogy is easier and more clear for approximate models with finite population. Ray–Knight's theorem can be proved by taking the limit in finite population models, which clarifies the above claim. One might also discretize time.

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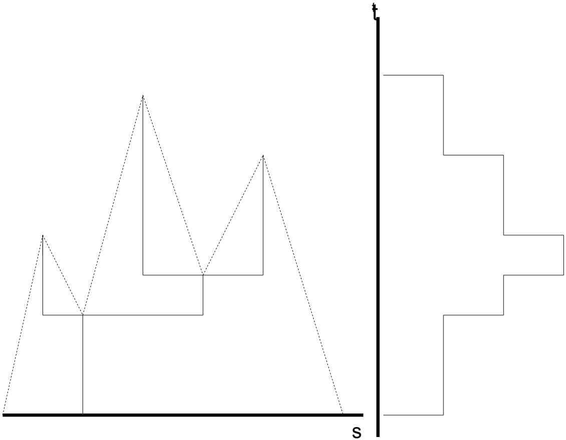
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An extension of Ray'–Knight's theorem to cope with Feller's diffusion with logistic growth

- Consider Feller's diffusion with logistic growth

$$Z_t^x = x + \int_0^t [\theta Z_r^x - \gamma (Z_r^x)^2] dr + 2 \int_0^t \sqrt{Z_r^x} dW_r \qquad (1)$$

  This is no longer a branching process. The quadratic term introduces "interaction between the branches".

- However, this is still a bona fide model for the evolution of a population, whose genealogy one might want to describe. This is the aim of this work.
- But before doing so, let us describe the law of the random field $\{Z_t^x,\ t, x \geq 0\}$. Contrary to the case of Feller's diffusion, $x \to Z_t^x$ is no longer a process with independent increments. It event not Markov for fixed t.
- But the process $\{Z_t^x,\ t \geq 0\}_{\{x \geq 0\}}$ is a $E = C^c(\mathbb{R}_+, \mathbb{R}_+)$ –valued Markov process indexed by $x \geq 0$.

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The transition probabilities

- For each $x > 0$ and $z \in E$, let $\mathbf{P}_x(z, \cdot)$ be the distribution of $z + Z^{z,x}$, where $Z^{z,x}$ solves

$$Z_t^{x,z} = x + \int_0^t Z_u^{x,z}(\theta - \gamma[Z_u^{x,z} + 2z(u)])du + 2\int_0^t \sqrt{Z_u^{x,z}} dW_u,$$

  with W being a standard Brownian motion.

- Then

$$\mathbb{P}\left(Z^{y+x} \in \cdot \middle| Z^y = z\right) = \mathbf{P}_x(z, \cdot).$$

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- **Definition** Let $\{Z^x\}_{x\geq 0}$ be the $C^c(\mathbb{R}_+,\mathbb{R}_+)$-valued Markov process with transition semigroup $(\mathbf{P}_x)$.
- Let us describe the joint law of $\{(Z_t^x, Z_t^{x+y}),\ t\geq 0\}$, for some $x, y > 0$.

$$Z_t^x = x + \int_0^t \left[\theta Z_r^x - \gamma\,(Z_r^x)^2\right] dr + 2\sqrt{Z_r^x}dW_r,$$

$$V_t^{x,y} = y + \int_0^t \left[\theta V_r^{x,y} - \gamma\left\{(V_r^{x,y})^2 + 2Z_r^x V_r^{x,y}\right\}\right] dr + 2\sqrt{Z_r^x}dW'_r,$$

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where W and W' are two mutually independent standard Brownian motions.

- **Remark 1** For each $x > 0$, Z^x solves the SDE

$$dZ_t^x = \left[\theta Z_t^x - \gamma (Z_t^x)^2\right] dt + 2\sqrt{Z_t^x} dW_t^x, \; Z_0^x = x,$$

where $\{W_t^x, \; t \geq 0\}$ is a standard Brownian motion. We have

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- **Remark 2** $\{Z^x\}_{x\geq 0}$ is a jump–Markov process, whose infinitesimal generator can be described in terms of the Poisson process of excursions (in the sense of Pitman–Yor) of the above SDE.
- Moreover, we can couple the random field $\{Z_t^x, \; t \geq 0, x > 0\}$ with the random field $\{Y_t^x, \; t \geq 0, x > 0\}$ corresponding to the case $\gamma = 0$, in such a way that for all $0 < x < y$, $t > 0$, $Z_t^y - Z_t^x \leq Y_t^y - Y_t^x$. This entails in particular that $x \to Z_\cdot^x$ jumps only where $x \to Y_\cdot^x$ jumps. But the jumps of Y^x which reach time $t > 0$ can be described as the jump times of a Poisson process on $\mathbb{R}_+$.

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- Consider the following reflecting SDE driven by standard Brownian motion B

$$H_s = B_s + \frac{1}{2}L_s(0) + \frac{\theta}{2}s - \gamma\int_0^s L_r(H_r)dr,\ \ s \geq 0, \tag{2}$$

  Here and everywhere below, $\{L_s(t),\ s \geq 0,\ t \geq 0\}$ denotes the local time of the process $\{H_s,\ s \geq 0\}$ accumulated up to time s at level t.

- One can show with the help of Girsanov's theorem that equation (2) has a unique weak solution, which we assume to be defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Define for any $x > 0$ the stopping time

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  Theorem *The two random fields $\{L_{S_x}(t),\ t, x \geq 0\}$ and $\{Z_t^x,\ t, x \geq 0\}$ have the same law.*

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- In a paper with a third author Vi Le, we have proved that result via approximation by a sequence of finite population models.
- Z is approximated by the total mass Z^N of a population of individuals, each of which has mass $1/N$. The initial mass is $Z_0^N = \lfloor Nx \rfloor / N$, and Z^N follows a Markovian jump dynamics : from its current state k/N,

$$Z^N \text{ jumps to } \begin{cases} (k+1)/N \text{ at rate } kN\sigma^2/2 + k\theta \\ (k-1)/N \text{ at rate } kN\sigma^2/2 + k(k-1)\gamma/N. \end{cases} \quad (3)$$

- For $\gamma = 0$, this is (up to the mass factor $1/N$) as a Galton-Watson process in continuous time : each individual independently spawns a child at rate $N\sigma^2/2 + \theta$, and dies (childless) at rate $N\sigma^2/2$.

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- For $\gamma > 0$, the additional quadratic death rate destroys the independence, and hence also the branching property.
- Viewing the individuals alive at time t as being arranged "from left to right", and by decreeing that each of the pairwise fights (which happen at rate 2γ and always end lethal for one of the two involved individuals) is won by the individual to the left, we arrive at the additional death rate $2\gamma\mathcal{L}_i(t)/N$ for individual i, where $\mathcal{L}_i(t)$ denotes the number of individuals living at time t to the left of individual i.
- If we want only to show that for fixed $x > 0$, $Z^{x,N} \Rightarrow Z^x$, we could as well adopt a "symmetric killing" scenario. The "left to right" scenario is crucial for getting

$$(Z^{x_1,N}, Z^{x_2,N}, \dots, Z^{x_k,N}) \Rightarrow (Z^{x_1}, Z^{x_2}, \dots, Z^{x_k}).$$

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- Let us now exploit an approach to a similar result by Norris, Rogers, Williams (1987). The idea is to start from the case $\theta = \gamma = 0$ where we can apply the classical Ray–Knight theorem, and to apply the same Girsanov transformation jointly to the population process and to the exploration process.
- On the side of the exploration process, the Girsanov–Radon–Nikodym ratio reads

$$G_s := \exp\left(M_s - \frac{1}{2}\langle M\rangle_s\right), \quad \text{where}$$

$$M_s = \int_0^s \left\{\frac{\theta}{2} - \gamma L_r(H_r)\right\} dB_r.$$

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$$G_{S_x}, \quad \text{where } S_x = \inf\{s > 0,\ L_s(0) > x\}.$$

- For that sake, we would need to prove a priori that $S_x < \infty$ a. s. under the transformed measure (the one with $\theta, \gamma \neq 0$). This is true as a consequence of our theorem, but we have no a priori proof of that fact.
- We introduce an approximation. Let us consider our exploration process $\{H_s\}$ (with $\theta = \gamma = 0$) reflected below an arbitrary point K (which eventually will go to ∞).
- Let H^K denote Brownian motion reflected inside the interval $[0, K]$, i. e. the solution of the SDE

$$H_s^K = B_s + \frac{1}{2}L_s^K(0) - \frac{1}{2}L_s^K(K^-),\ s \geq 0,$$

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Lemma

For any $0 < K < K'$ the processes $\{L^K_{S_x^K}(t),\ 0 \le t \le K\}$ and $\{L^{K'}_{S_x^{K'}}(t),\ 0 \le t \le K\}$ have the same distribution.

- There exists a probability measure $\tilde{\mathbb{P}}^K$ such that for all $s > 0$,

$$\left.\frac{d\tilde{\mathbb{P}}^K}{d\mathbb{P}}\right|_{\mathcal{F}_s} = \exp\left(M_s^K - \frac{1}{2}\langle M^K\rangle_s\right), \text{ with } M_s^K = \int_0^s \left[\frac{\theta}{2} - \gamma L_r^K(H_r^K)\right] dB_r.$$

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Proposition

For any $K > 0$, the process $\{L^K_{S^K_x}(t),\ t \geq 0\}$ is under $\tilde{\mathbb{P}}^K$ a solution of equation (1), killed at time K.

- Tanaka's formula gives for $0 \leq t < K$ the identity

$$L^K_{S^K_x}(t) = L^K_{S^K_x}(0) + 2\int_0^{S^K_x} \mathbf{1}_{\{H^K_s \leq t\}} dB_s,$$

- On the other hand, from the second Ray–Knight theorem, $\{L^K_{S^K_x}(t),\ 0 \leq t < K\}$ is a $\mathbb{P}$–martingale with quadratic variation given by

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- which with respect to a suitable filtration (in the time t variable) turns out to be a $\mathbb{P}$-martingale. This is where the techniques from Norris, Rogers, Williams (1987) are essential.

- We define for all $0 \leq t \leq K$ (suppressing the superscript K in the defined quantities)

$$A(s,t) := \int_0^s \mathbf{1}_{\{H_r^K \leq t\}} dr, \quad \tau(s,t) := \inf\{r : \ A(r,t) > s\},$$
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$$\mathcal{F}(s,t) := \sigma(\{\xi(r,t) : \ r \leq s\}), \quad \mathcal{E}_t := \mathcal{F}(\infty,t).$$

- Walsh (1978) shows that $\{\mathcal{E}_t, \ 0 \leq t \leq K\}$ is a right–continuous filtration, and Norris, Rogers, Williams (1987) show that $\{N_t^K, \ 0 \leq t \leq K\}$ is an $(\mathcal{E}_t)$–martingale s. t.

$$\langle N^K, Y^K \rangle_t = \int_0^{S_x^K} \mathbf{1}_{\{H_s^K \leq t\}} \left(\theta - 2\gamma L_s^K(H_s^K)\right) ds,$$

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- Reexpressing the r.h.s. of the above identity via the occupation times formula yields

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- Since the quadratic variation remains unchanged under a Girsanov transformation, we infer that $\langle R^K \rangle_t = 4 \int_0^t L^K_{S^K_x}(u)du$, $0 \le t < K$. Consequently, there exists a Brownian motion $\{W_t,\ t \ge 0\}$ such that $L^K_{S^K_x}(t)$ solves for $0 \le t < K$ the whished SDE.

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Work in progress (with my student Mamadou Ba)

- Suppose we want to consider a more general model of the type

$$Z_t^x = x - \int_0^t f(Z_r^x)dr + 2\int_0^t \sqrt{Z_r^x}dW_r,$$

where say f is of class C^1 such that $f(0) = 0$ and $f(z) > 0$ for z large enough.

- We can still prove a Ray–Knight type theorem to describe the genealogy of a population whose size would evolve according to such an SDE.
- An intuition on which type of interaction in the population this SDE would model can be best explained by looking back at the approximate discrete model, where (in the case say $f > 0$), f adds a death rate equal to

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