Scaling limits of planar random growth models

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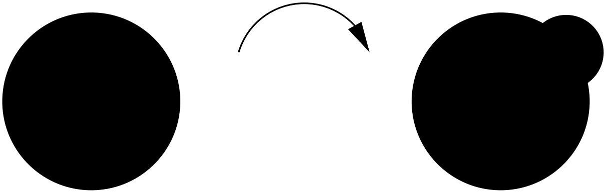
Motivation

- Model spatial configuration of populations of individuals that grow randomly but are constrained from moving.
- Describe growth that arises by reproduction (bacterial cells on a petri dish) or by immigration (trees in a large forest).
- Limitation - constrained to 2-dimensions, but still many potential applications.

Conformal mapping representation of single particle

Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$. Let $K_0 = \mathbb{C} \setminus D_0$ be the closed unit disk. Consider a simply connected set $D_1 \subset D_0$, such that $P = D_1^c \setminus K_0$ has diameter $d \in (0, 1]$ and $1 \in \overline{P}$. The set P models an incoming particle, which is attached to the unit disk at 1. We use the unique conformal mapping $f_P : D_0 \to D_1$ as a mathematical description of the particle.

Basic conformal mapping from the exterior disk



Conformal mapping representation of a cluster

Let $P_1, P_2, \ldots$ be a sequence of particles with $\mathrm{diam}(P_j) = d_j$. Let $\theta_1, \theta_2, \ldots$ be a sequence of angles. Define rotated copies $f_{P_j}^{\theta_j}(z)$ of the maps $\{f_{P_j}\}$ so that $f_{P_j}^{\theta_j}(D_0) = e^{i\theta_j} f_{P_j}(D_0)$. Take $\Phi_0(z) = z$, and recursively define

$$\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}^{\theta_n}(z), \quad n = 1, 2, \ldots.$$

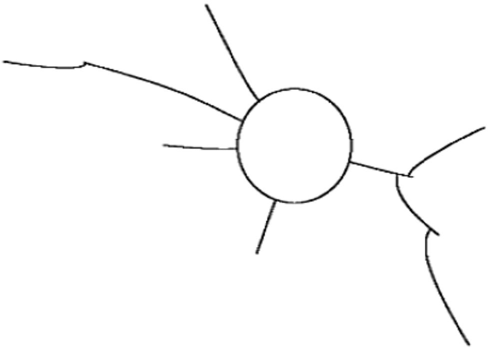
This generates a sequence of conformal maps $\Phi_n : D_0 \to D_n = \mathbb{C} \setminus K_n$, where $K_{n-1} \subset K_n$ are growing compact sets, or clusters.

Generalised Hastings-Levitov clusters

By choosing the sequences $\{\theta_j\}$ and $\{d_j\}$ in different ways, it is possible to describe a wide class of growth models.

In the Hastings-Levitov family of models $\mathrm{HL}(\alpha)$, $\alpha \in [0, 2]$, the θ_j are chosen to be independent uniform random variables on the unit circle which corresponds to the attachment point at the *n*th step being distributed according to harmonic measure at infinity for K_{n-1}. The particles are usually taken to be "slits" with diameters taken as $d_j = d/|\Phi'_{j-1}(e^{i\theta_j})|^{\alpha/2}$. Heuristically, the case $\alpha = 1$ corresponds to the Eden model (biological cell growth) and the case $\alpha = 2$ is a candidate for off-lattice DLA.

HL(0) cluster after a few arrivals with $d = 1$



Anisotropic Hastings-Levitov model

Anisotropic Hastings-Levitov, $\mathrm{AHL}(\nu)$, is a variant of the $\mathrm{HL}(0)$ model in which $\theta_1, \theta_2, \ldots$ are i.i.d. random variables on the unit circle with common law ν and $d_j = d$.

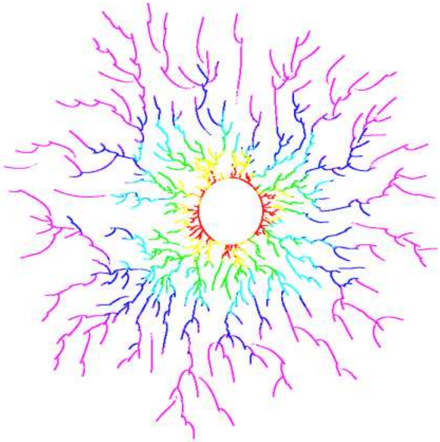
Models can be further generalised by allowing $P_1, P_2, \ldots$ to be chosen randomly from a class of suitable shapes, even with $d_1, d_2, \ldots$ i.i.d. random variables (independent of $\{\theta_j\}$) satisfying certain conditions, however our results are not sensitive to these changes.

The use of more general distributions for the angles is a way of introducing anisotropy or localization in the growth. It is suggested that such anisotropic Hastings-Levitov models may provide a description for the growth of bacterial colonies where the concentration of nutrients is directional.

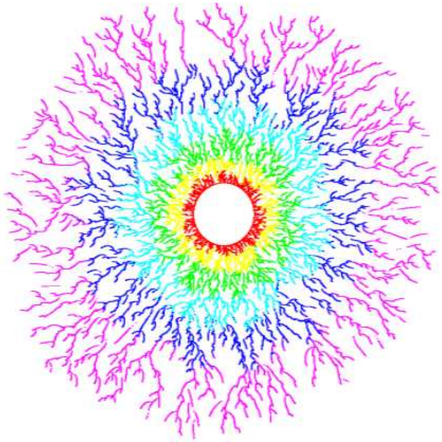
Natural scaling limits

From the physical point of view, it is natural to consider particle sizes that are very small compared to the overall size of the cluster. We consider scaling limits where we scale the particle sizes and let the number of particles grow at a rate depending on the particle diameters.

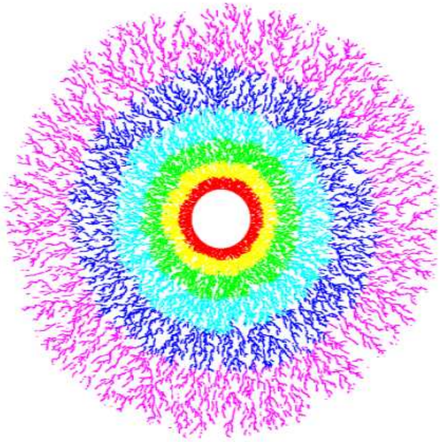
HL(0) cluster after 800 arrivals with $d = 0.1$



HL(0) cluster after 5 000 arrivals with $d = 0.04$



HL(0) cluster after 20 000 arrivals with $d = 0.02$



Loewner chains

If μ_t is a family of probability measures on the unit circle $\mathbb{T}$, the Loewner equation

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu_t(\zeta)$$

produces a family of conformal maps $f_t \colon D_0 \to \mathbb{C} \setminus K_t$, where K_t is a growing sequence of compact sets.

A shape theorem

Suppose that the particles $P_1, P_2, \ldots$ in AHL(ν) are chosen to be identical and symmetric, with diameter d (and a few technical conditions). If $O(d^{-2})$ particles are added, the macroscopic shape of the cluster converges almost surely as $d \to 0$ and the limit can be realized as the image of the solution to the Loewner equation driven by the angle measure ν evaluated at a suitable time.

The isotropic case

In the case $d\mu_t(\zeta) = |d\zeta|/2\pi$, the Loewner equation reduces to

$$\partial_t f_t(z) = z f_t'(z),$$

and we see that $f_t(z) = e^t z$, so that $K_t = e^t K_0$.

This shows that the macroscopic shape of the cluster grows like an expanding disc, as seen in the simulations above.

Angles chosen in an interval

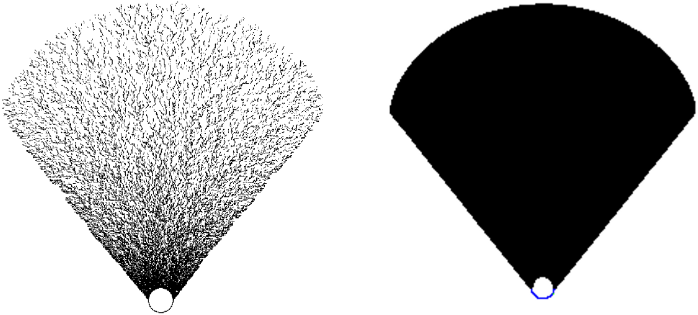
For $\eta \in (0,1]$, let θ_j be chosen uniformly in $[0,\eta]$. Then

$$d\nu(e^{2\pi ix}) = \frac{\chi_{[0,\eta]}(x)dx}{\eta}.$$

The clusters converge to the hulls of the Loewner chain described by the equation

$$\partial_t f_t(z) = zf_t'(z)\left(1+\frac{2}{\eta}\arctan\left[\frac{e^{i\pi\eta}\sin(\pi\eta)}{z-e^{i\pi\eta}\cos(\pi\eta)}\right]\right).$$

AHL on the half circle



Simulation of AHL(ν) and limiting Loewner hull, for $d = 0.02$ after 25000 arrivals, corresponding to $d\nu(e^{2\pi ix}) = 2\chi_{[0,1/2]}(x)dx$.

Angles chosen from a density with m-fold symmetry

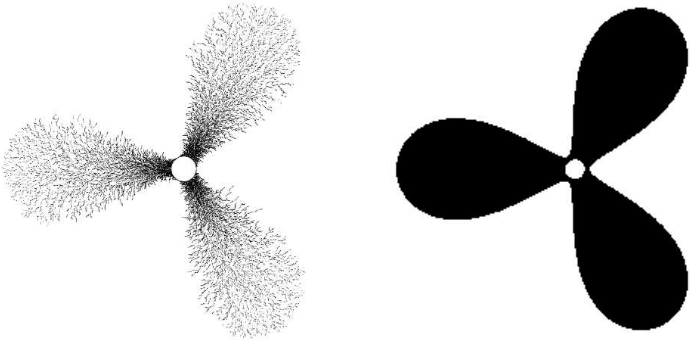
For fixed $m \in \mathbb{N}$, choose θ_j distributed according to the density

$$d\nu(e^{2\pi i x}) = 2\sin^2(m\pi x)dx.$$

The clusters converge to the hulls of the Loewner chain described by the equation

$$\partial_t f_t(z) = z f_t'(z)\left(1 - \frac{1}{z^m}\right).$$

AHL for a measure with 3-fold symmetry



Simulation of AHL(ν) and limiting Loewner hull, for $d = 0.02$ after 25000 arrivals, corresponding to $d\nu(e^{2\pi ix}) = 2\sin^2(3\pi x)dx$.

Location of particles

Let $c = \log f_P'(\infty)$ be the logarithmic capacity of the particle. Then for $\epsilon > 0$ and $m \in \mathbb{N}$ (can depend on d subject to constraints), with high probability as $d \to 0$, for all $n \leq m$ and all $n' \geq m + 1$,

$$|z - e^{cn + i\Theta_n}| \leq \epsilon e^{cn} \quad \text{for all } z \in P_n,$$

$$\mathrm{dist}(w, K_n) \leq \epsilon e^{cn} \quad \text{whenever } |w| \leq e^{cn},$$

$$|z| \geq (1 - \epsilon) e^{cm} \quad \text{for all } z \in P_{n'}.$$

Fingers and gap paths

Convenient to work in logarithmic space:

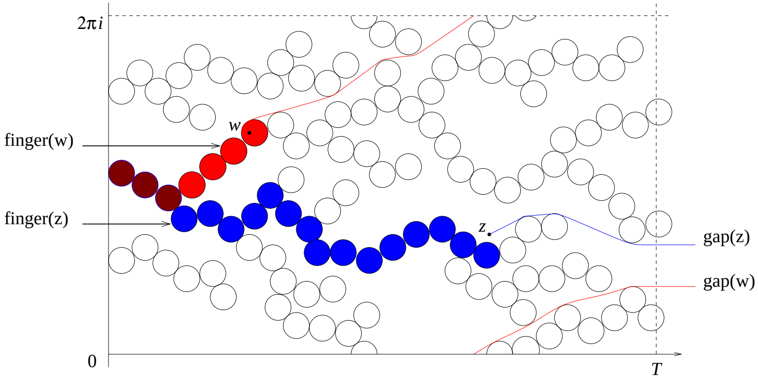
$$\widetilde{K}_n = \{z \in \mathbb{C} : e^z \in K_n\} \subseteq \mathbb{R}_+ \times \mathbb{R} \quad \text{(time-space)}.$$

Fix $N \in \mathbb{N}$.

For $\mathrm{Re}(z) \geq 0$, let finger(z) be the nearest particle to z in $\widetilde{K}_N$, together with all its "parent" particles.

Let gap(z) denote the unique minimal length path from the nearest point to z in $\overline{\widetilde{K}_N^c}$ to ∞ that does not leave $\overline{\widetilde{K}_N^c}$.

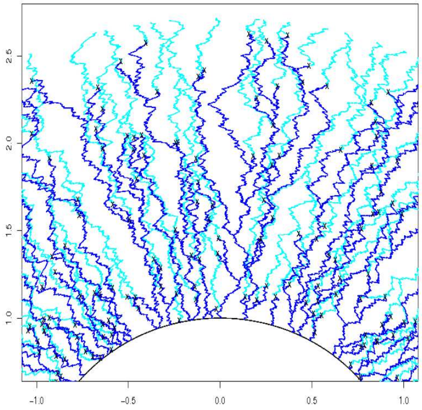
Diagram illustrating fingers and gap paths



Local limit result

For any fixed $T > 0$ and finite $E \subset [0, T] \times \mathbb{R}$, let $N = \lfloor c^{-1} T \rfloor$ so that K_N is approximately a disc of radius e^T. Under a rescaling of "space" by $d^{-1/2}$, the gap paths in K_N starting from points in E converge to coalescing Brownian motions starting from E and the fingers converge to coalescing backwards Brownian motions starting from E.

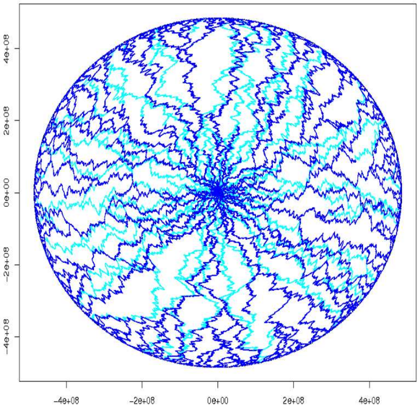
Local limit approximation of fingers and gap paths for $T = 1$ and $d = 0.01$



Global limit result

For any fixed $T > 0$ and finite $E \subset [0, T] \times \mathbb{R}$, let $N = \lfloor (cd)^{-1} T \rfloor$ so that K_N is approximately a disc of radius $e^{T/d}$. Under a rescaling of "time" by d, the gap paths in K_N starting from points in E converge to coalescing periodic Brownian motions starting from E and the fingers converge to coalescing periodic backwards Brownian motions starting from E.

Global limit approximation of fingers and gap paths for $T = 1$ and $d = 0.05$



Epidemics on Hastings-Levitov clusters?

- Epidemic spreads to particles at range $O(d)$ on K_∞?
- As above but immunity passed on to offspring?
- At time n epidemic spreads to particles at range $O(d)$ on K_n?
- As above, but children of infected particles have a different size to children of uninfected particles?
- As above, but distribution of Θ_{n+1} depends on arrangement of infected particles in K_n?
- Infected particles are removed from the cluster at certain rate?
- . . . ?

Problems likely to be mathematically very hard but perhaps simulations can reveal interesting results. Many natural potential applications.

References

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[3] J. Norris, A. Turner, *Hastings-Levitov aggregation in the small-particle limit,* arxiv:1106.3546, 2011.

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