# EPIDEMICS AND RUMOURS ON NETWORKS

**Valerie Isham**

Department of Statistical Science
University College London

# *Acknowledgements*


Maziar Nekovee

Complexity Group, BT Research & Centre for Computational Science, UCL

Simon Harden

Department of Statistical Science, UCL

## *Models for the spread of infection or information*

– homogeneous mixing, closed population (size $n$)

Epidemics: SIR model $\quad (X(t) + Y(t) + Z(t) = n)$

$$\text{susceptible} \quad \xrightarrow{\lambda XY/n} \quad \text{infective} \quad \xrightarrow{\delta Y} \quad \text{removed}$$

Rumours:

$$\text{ignorant} \quad \xrightarrow{\lambda XY/n} \quad \text{spreader} \quad \xrightarrow{??} \quad \text{stifler}$$

Spreaders in contact with spreaders or stiflers become stiflers

(Kendall, ‘57; Daley & Kendall ‘64,’65; Daley & Gani, ‘99)

## *Stifling options*

Undirected contacts:

spreader – spreader contacts $\Longrightarrow$ both stifled

Directed contacts: Maki-Thompson (1973) model

only initiating spreader is stifled spreader $\rightarrow$ stifler

$$\lambda(Y+Z)Y/n$$

or, spreaders become stifled with prob. $p$

$$\lambda p(Y+Z)Y/n$$

"forgetting"

$$\lambda p(Y+Z)Y/n+\delta Y$$

# *Nekovee et al (Physica A, 2007)*

Looked at properties of this rumour model superposed on a network structure.

Notation:

$n_k$ nodes of degree $k$ (*i.e. with* $k$ neighbours)

$X_k, Y_k, Z_k$ ignorants, spreaders, stiflers of degree $k$

and network degree-degree correlation function

$$p_{j\,k} = \mathrm{P}(\text{neighbour node has degree } k \mid \text{index node has degree } j)$$

*N et al* used approximation re dependence, so that the influence of the network is encapsulated in the matrix of $p_{j\,k}$s

Specifically, the total rate of "infection" of degree $k$ ignorants is

$$\lambda k X_k \sum_j p_{kj} \mathrm{P}(\text{node is spreader} \mid \text{degree } j \text{ and neighbour of ignorant node of degree } k).$$

$$\simeq \lambda k X_k \sum_j p_{k\,j} Y_j / n_j$$

Similarly, the total rate of stifling of degree $k$ spreaders is

$$\simeq Y_k \{ \delta + \lambda p k \sum_j p_{k\,j} [n_j - X_j]/n_j \}$$

$\Longrightarrow$ homogenous mixing structure (the "approximate" model)

no stifling ( $p = 0$ ) $\Longrightarrow$ SIR Model

*N et al*'s analysis is deterministic ( $n \to \infty$ )
(they also assume broadcast messages but deterministic analysis is unaffected)

## *Nekovee et al's results*

"Homogeneous" case (all nodes have same degree $k$ ) :

final size eqn $z = 1 - e^{-Rz}$ where

$$R = \lambda k(1+p)/(\delta + \lambda kp) = (1+p)/(\psi + p) \quad (\psi = \delta/(\lambda k))$$

For rumour to spread (non-zero $z$ ) need $R > 1$

If $\delta = 0$ (no forgetting) then $R > 1$ (no threshold)

If $\delta > 0$ then need $\delta < \lambda k$ $(\psi < 1)$ regardless of $p$ (*cf* SIR)

"Uncorrelated" case ($p_{j\,k} \propto kp_k$, where $p_k$ is marginal degree dn)

If $\delta > 0$ then, *to leading order in* $p$ , need

$$\delta < \lambda \left(\mu_K + \sigma_K^2/\mu_K\right)$$

Numerical results for simple random graph, scale-free network (random graph and initial spreader, deterministic rumour dynamics)

# *Stochastic rumour dynamics*

*– approximate (homogeneous mixing) model and fixed* $k$

*spreading rate* $\lambda k XY/n$ *stifling rate* $Y\{\delta + \lambda pk(1 - X/n)\}$

*Final size distribution - embedded Markov chain*

Time in state $(x, y)$: $T_{x\,y} \sim \exp\{u_{x\,y} = (\delta + \lambda pk)y + \lambda(1-p)kxy/n\}$

For $x \geq 0, y \geq 1$ and $x + y \leq n$

$$(x, y) \to (x-1, y+1) \text{ with prob } \phi_x = \frac{x}{(\psi+p)n+(1-p)x}$$

$$(x, y) \to (x, y-1) \text{ with prob } 1 - \phi_x = \overline{\phi}_x$$

$x(0) = n-1, y(0) = 1$ and states $(x, 0)$ are absorbing

Note that $\psi < 1, > 1$ according as $\phi_x >, < 0.5$

Absorption in state $(s, 0) \Longrightarrow$ final size is $n - 1 - s$

Let $\pi_{x\,y} = \mathrm{P}(\text{ever reach } (x,y))$ then

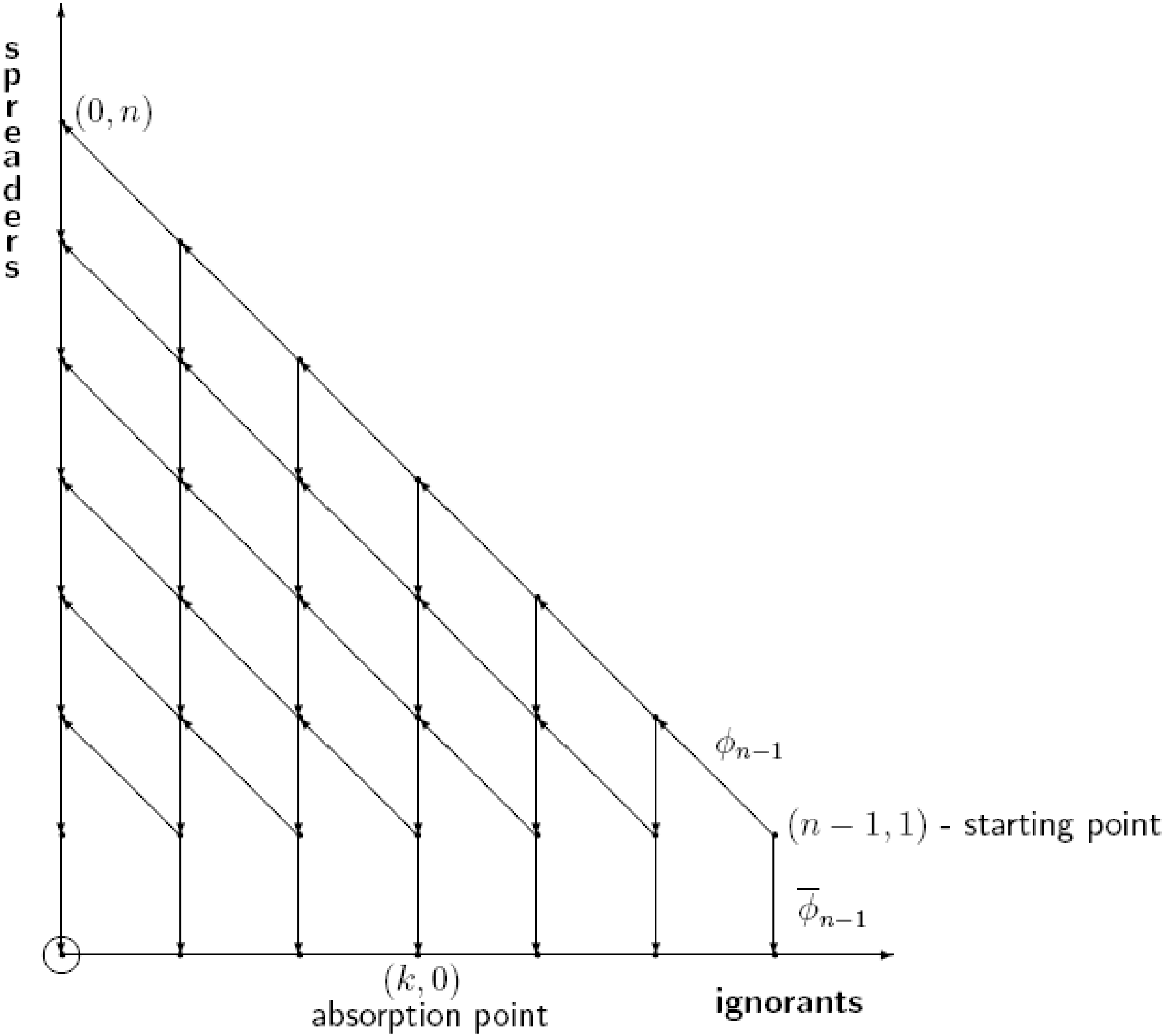
$$\begin{aligned}
\pi_{x\,y} &= \pi_{x+1\;y-1}\phi_{x+1} + \pi_{x\;y+1}\,(1-\phi_x) \text{ for } x \geq 0, y \geq 2, x+y \leq n-1. \\
\pi_{x\,y} &= \pi_{x+1\;y-1}\phi_{x+1} && \text{for } 0 \leq x \leq n-2, y = n-x \\
\pi_{x\,1} &= \pi_{x\,2}\,(1-\phi_x) && \text{for } 0 \leq x \leq n-2, \\
\pi_{x\,0} &= \pi_{x\,1}\,(1-\phi_x) && \text{for } 0 \leq x \leq n-1,
\end{aligned}$$

with initial condition $\pi_{n-1\;\;1} = 1$

the equations can be solved iteratively to give $\pi_{n-1-s\;\;0}$.

Similarly for the time to absorption *e.g.*

$$\mathrm{E}(T_{x\,y}) = 1/u_{x\,y} + \phi_x \mathrm{E}(T_{x-1\,y+1}) + (1-\phi_x)\mathrm{E}(T_{x\,y-1})$$


spreaders
$(0, n)$
$\phi_{n-1}$
$(n-1, 1)$ - starting point
$\overline{\phi}_{n-1}$
$(k, 0)$
absorption point
ignorants

or numerically, for final size distribution, using

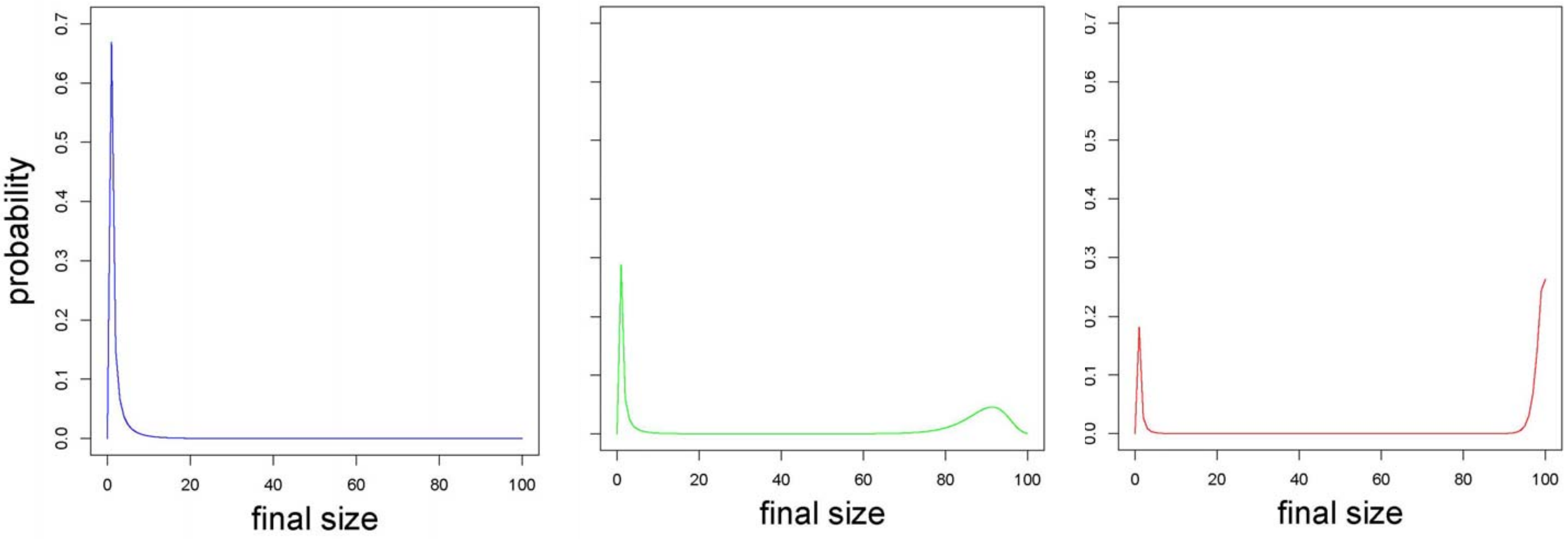
$$\pi_{n-j\ 0} = \left(\prod_{i=1}^{j-1} \phi_{n-i}\right) \overline{\phi}_{n-j}^{\,2} \sum_{i_1=2}^{j} \sum_{i_2=\max(i_1,3)}^{j} \cdots \sum_{i_{j-2}=\max(i_{j-3},j-1)}^{j} \overline{\phi}_{n-i_1} \cdots \overline{\phi}_{n-i_{j-2}}$$

for $k \geq 3$ together with $\pi_{n-1,0} = \overline{\phi}_{n-1}, \text{ and } \pi_{n-2,0} = \phi_{n-1}\overline{\phi}_{n-2}^{\,2}$

# *Thresholds for finite n*

Nåsell (1995): SIR model $(p = 0)$

threshold defined as value of $R_0 (= 1/\psi)$ for which the final size dn changes from J-shape to U-shape



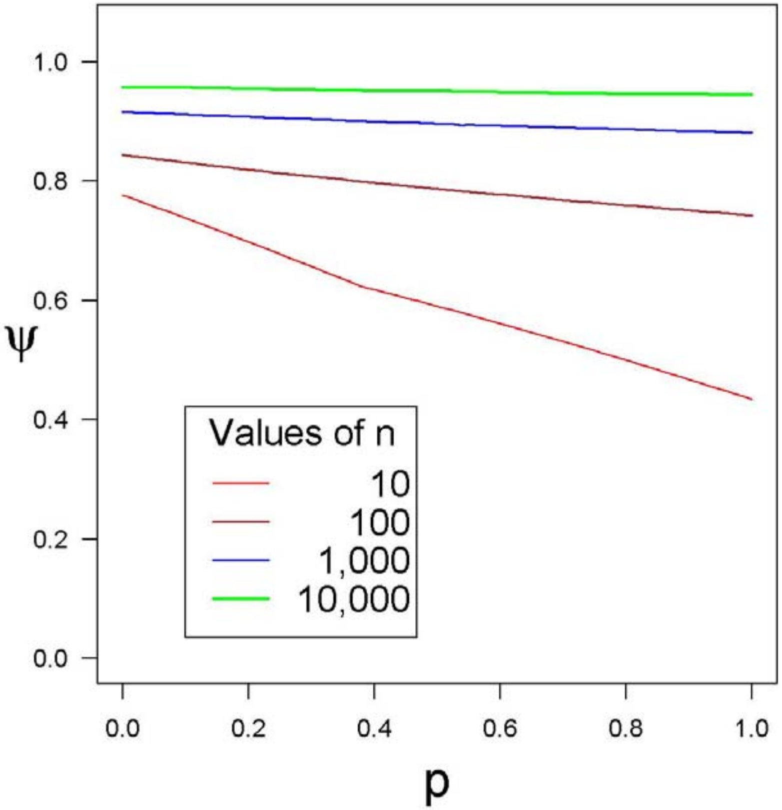
## Nåsell (1995):

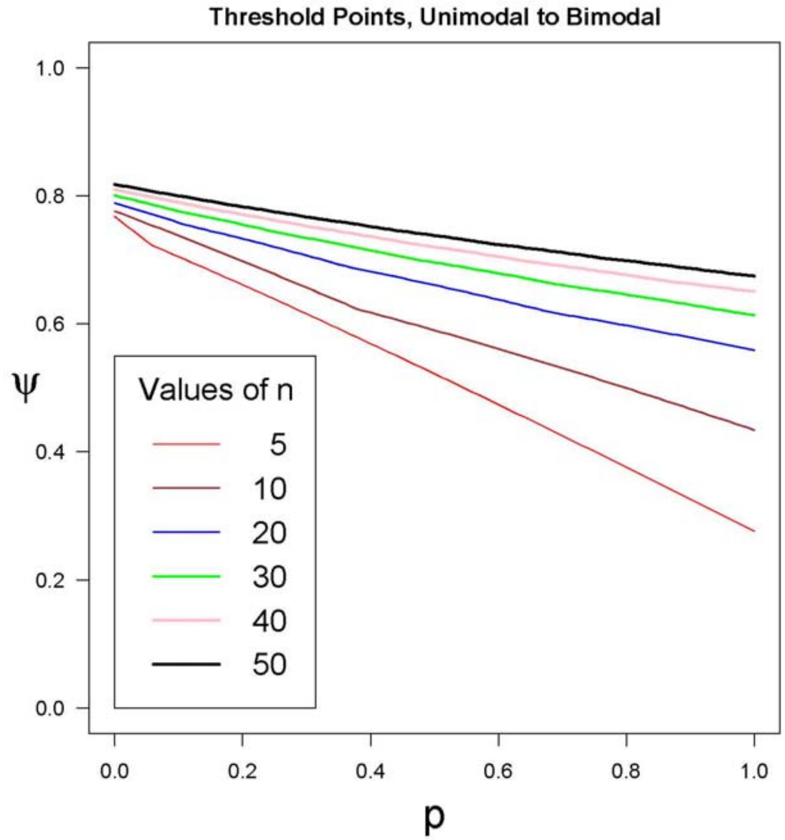
- conjecture based on numerical evaluation $R_0 \sim 1 + \rho/n^{1/3}$ (apparently for $n$ up to $10^2$)

## Ball & Nåsell (1994):

- theoretical support via approximation to the final size distribution as a mixture of the distribution for a small epidemic (branching process) and that for a large epidemic (normal distribution)
- numerical work up to $n = 10^4$

## Thresholds for rumour and SIR models

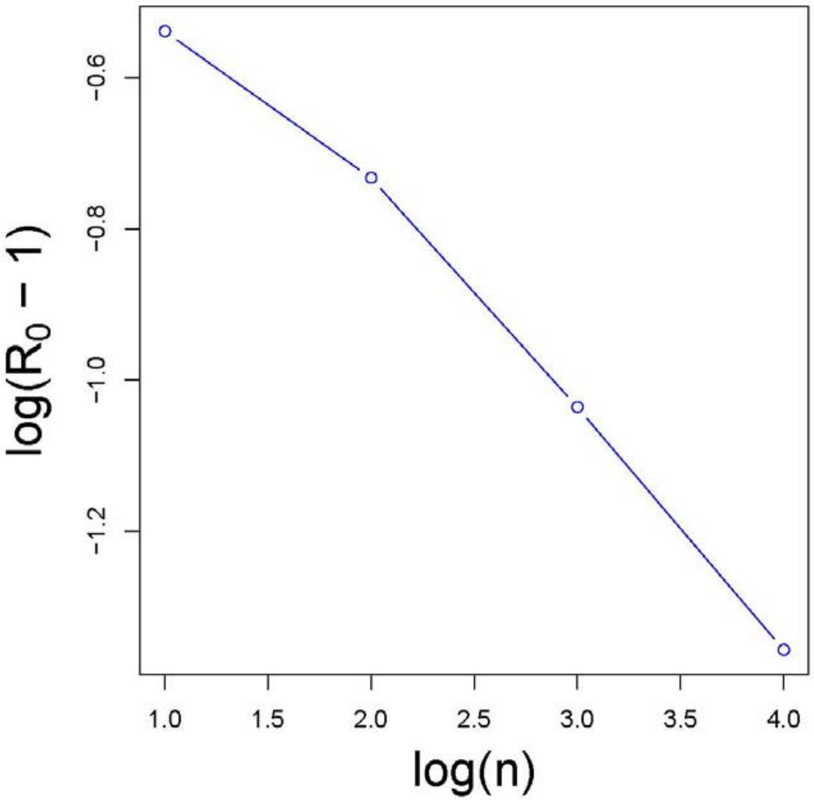


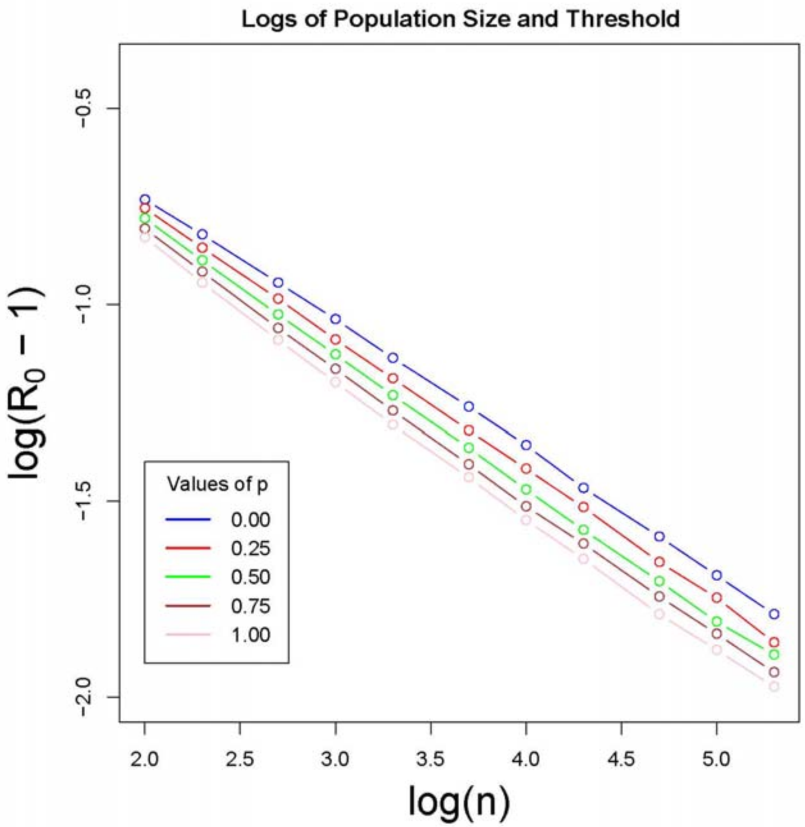




*rumours*: $\psi = \delta/(\lambda k)$ vs $p$

Threshold becomes independent of $p$ as $n$ increases

## *Thresholds for rumour and SIR models*

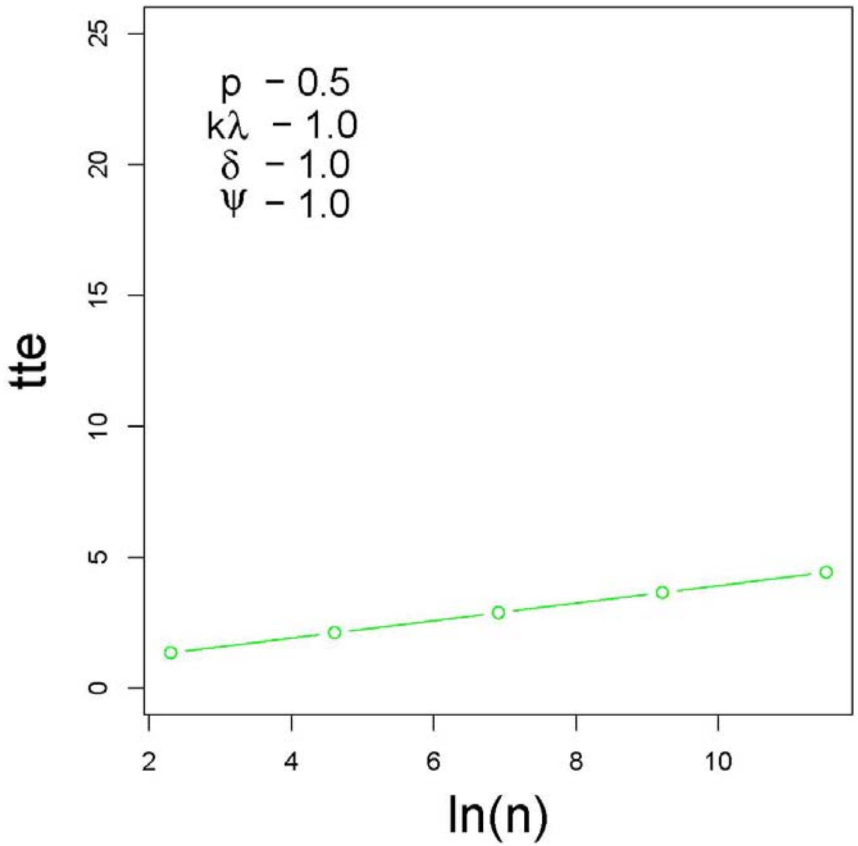


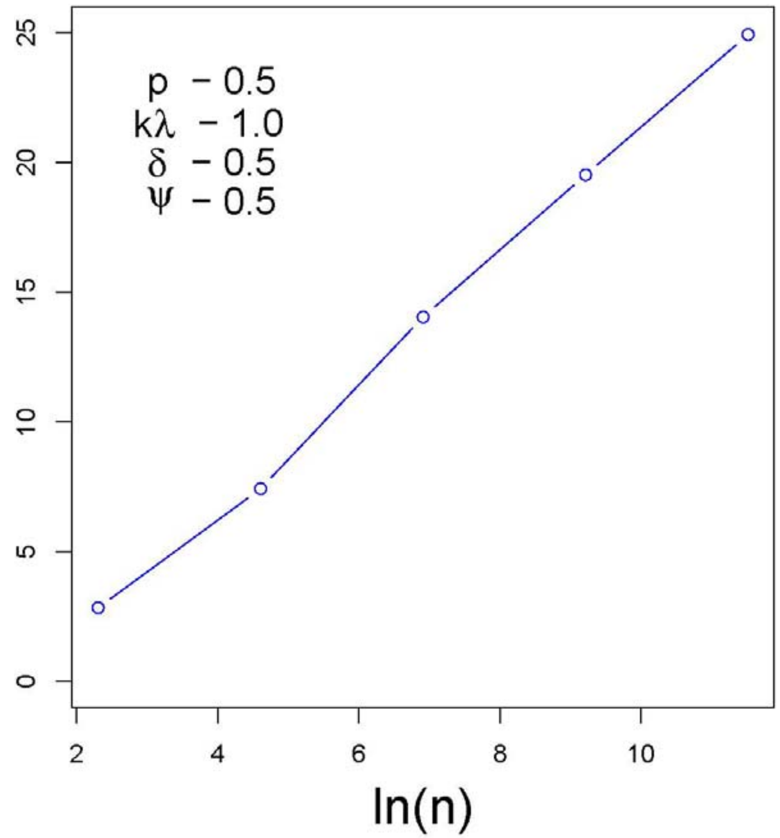

SIR: $p = 0 \ (R_0 = 1/\psi)$

$$R_0 = 1 + 0.879 n^{-0.3267} (\text{fitting to } 10^3 \leq n \leq 2 \times 10^5)$$

# *Mean duration of rumour spread*




$$\psi = 1, R = (1+p)/(\psi + p) = 1$$

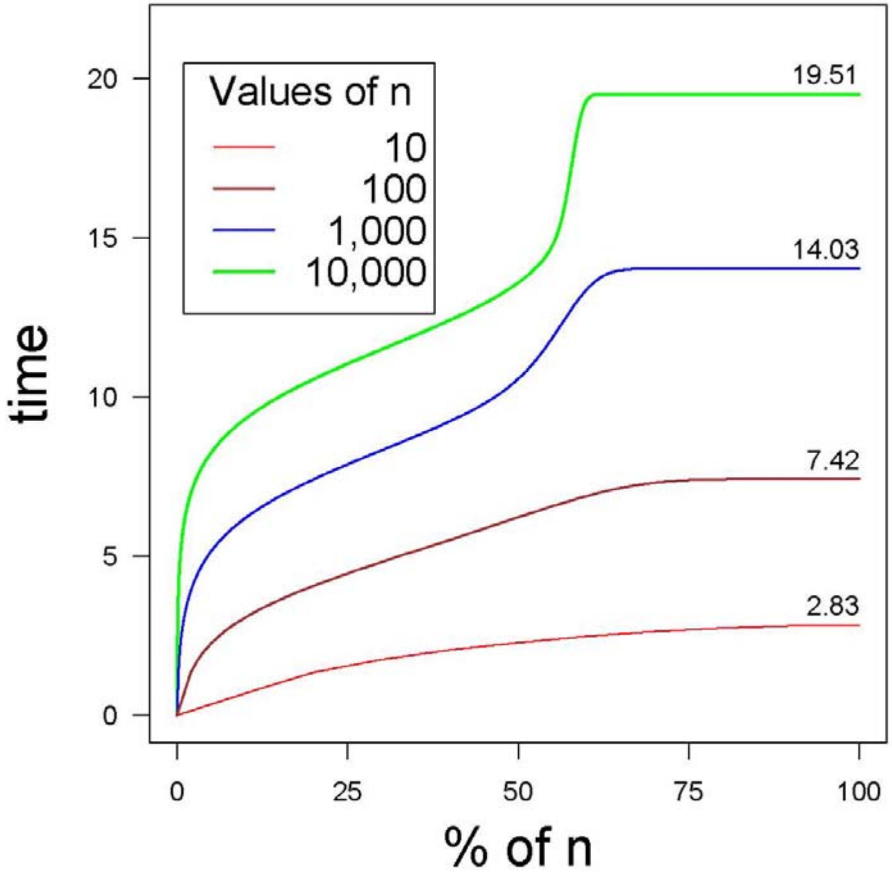



$$\psi = 0.5, R = 1.5$$

*cf Barbour (1975) for SIR*

# *Mean duration conditional on maximum number to whom rumour is spread*

$(\psi = 0.5, R = 1.5)$



## *Simulations of the full network model*

- *fixed k, comparison with "approximate" model*

  *the effect of ignoring dependence*

- *comparisons between network types*

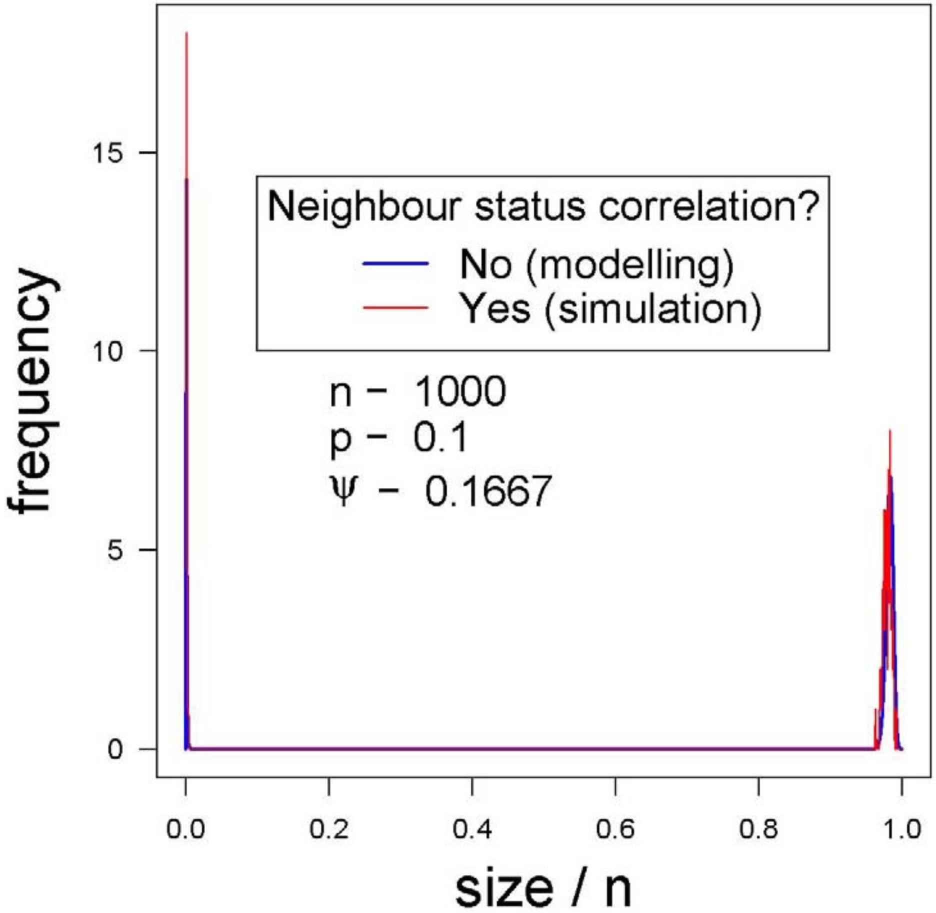
  *homogeneous (fixed $k$), uncorrelated*

  *simple random graph*

  *scale-free*

- *broadcast versus serial spreading*

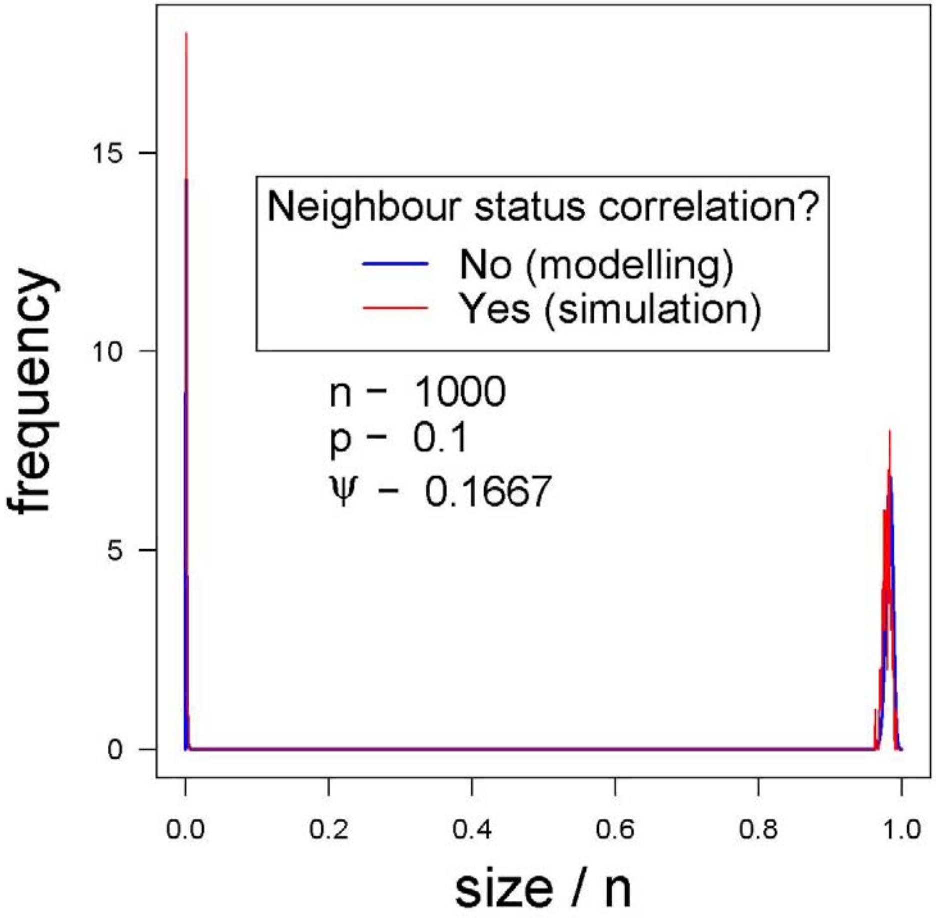
# *Effect of ignoring dependence on final size distribution and threshold – fixed k*




Expect threshold $\psi = \delta/\lambda k$ to be smaller for simulations than for approximation – magnitude of difference?

For simulation $\lambda = 0.012, k = 50, \delta = 0.1$

## *Effect of ignoring dependence on final size distribution and threshold – fixed k*




For simulation $\lambda = 0.012, k = 50, \delta = 0.1$

With $n = 1000, p = 0.1$ threshold for model is

$\psi = 0.921$

*eg*

$\delta = 0.547 \text{ if } \lambda = 0.012, k = 50$

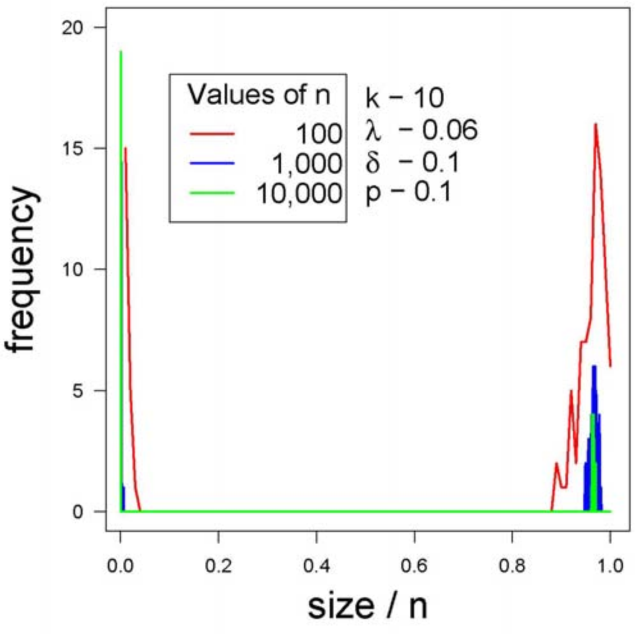
For simulation

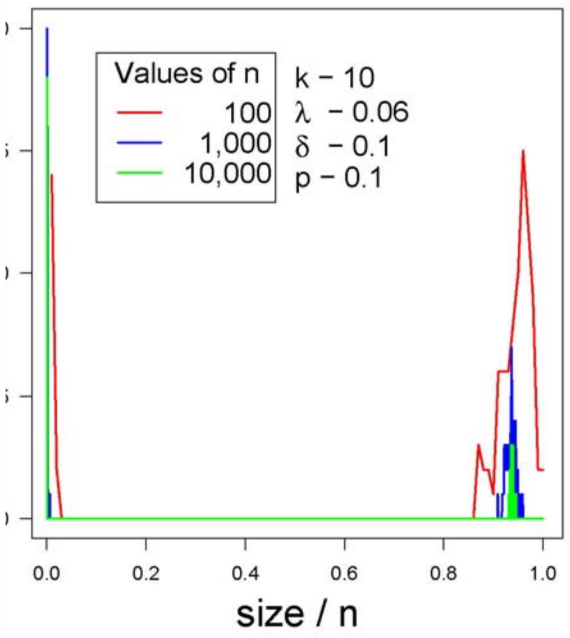
$\text{if } \lambda = 0.012, k = 50$

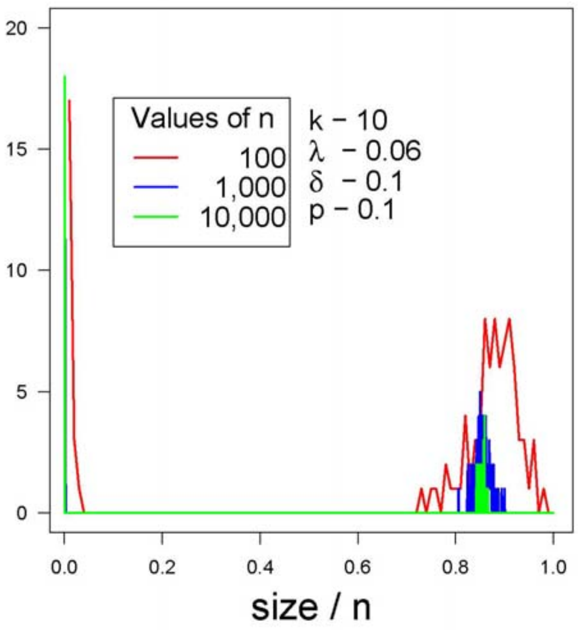
then threshold is smaller? $(0.4 < \delta < 0.5)$

# Effect of network structure

- effect of $n$ on shape of final size distribution




homogeneous

$k = 10$

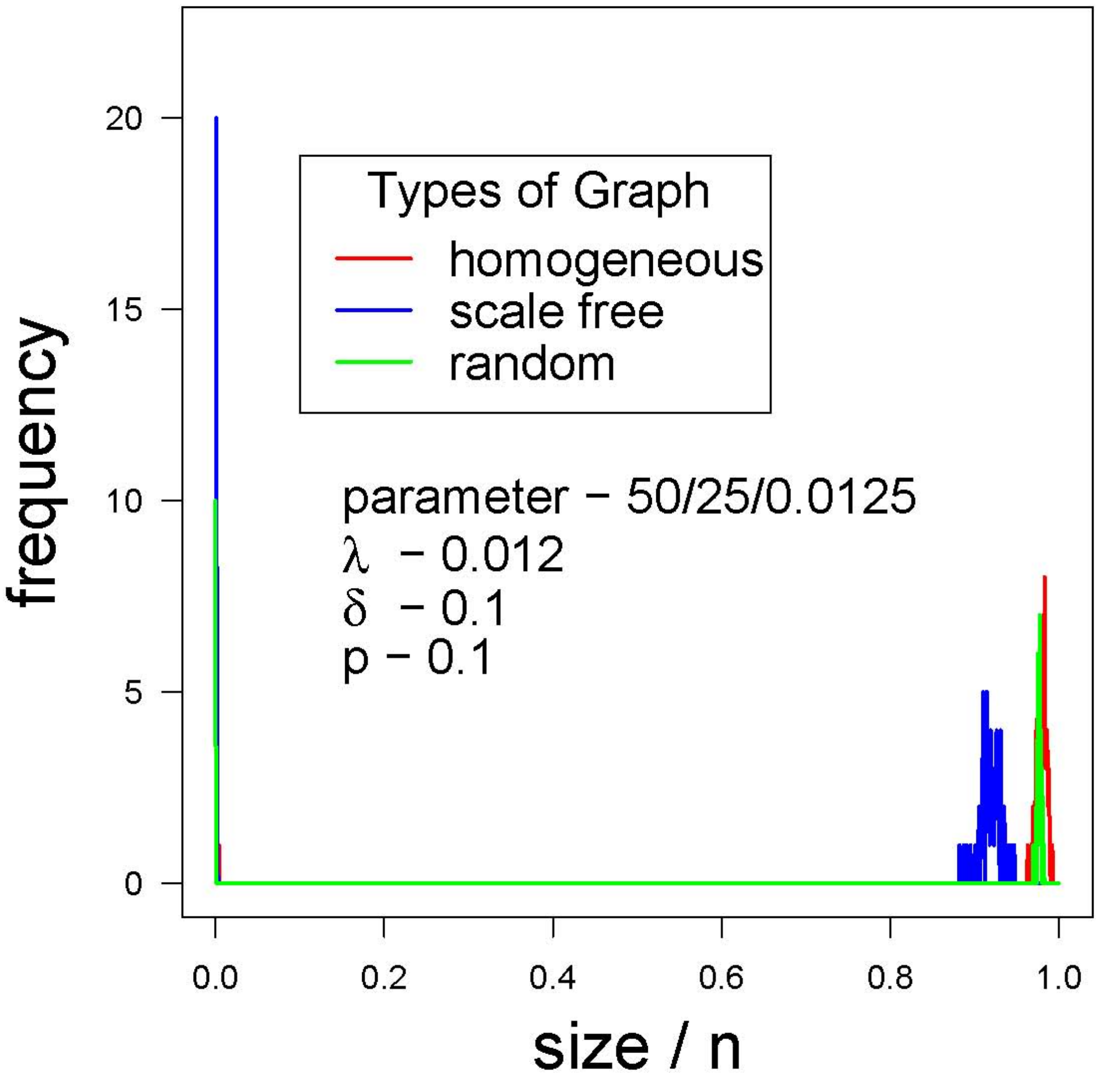
random

$\text{prob} = 0.1$

scale-free

$\text{add 5 edges per node}$

# *Effect of network structure on shape of final size distribution*



## *Effect of network structure on thresholds*

Thresholds in $\psi$ seem similar for homogeneous and random graphs, but higher for scale-free networks

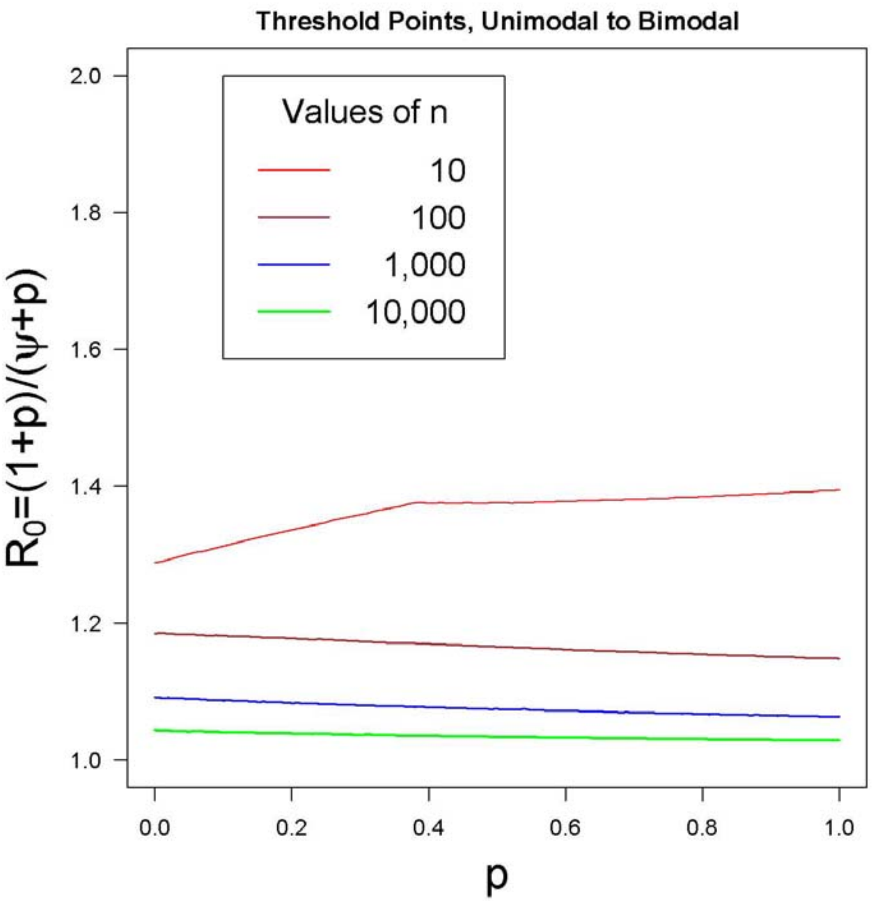
(deterministic approx threshold $\delta = \lambda \left( \mu_K + \sigma_K^2 / \mu_K \right)$ )

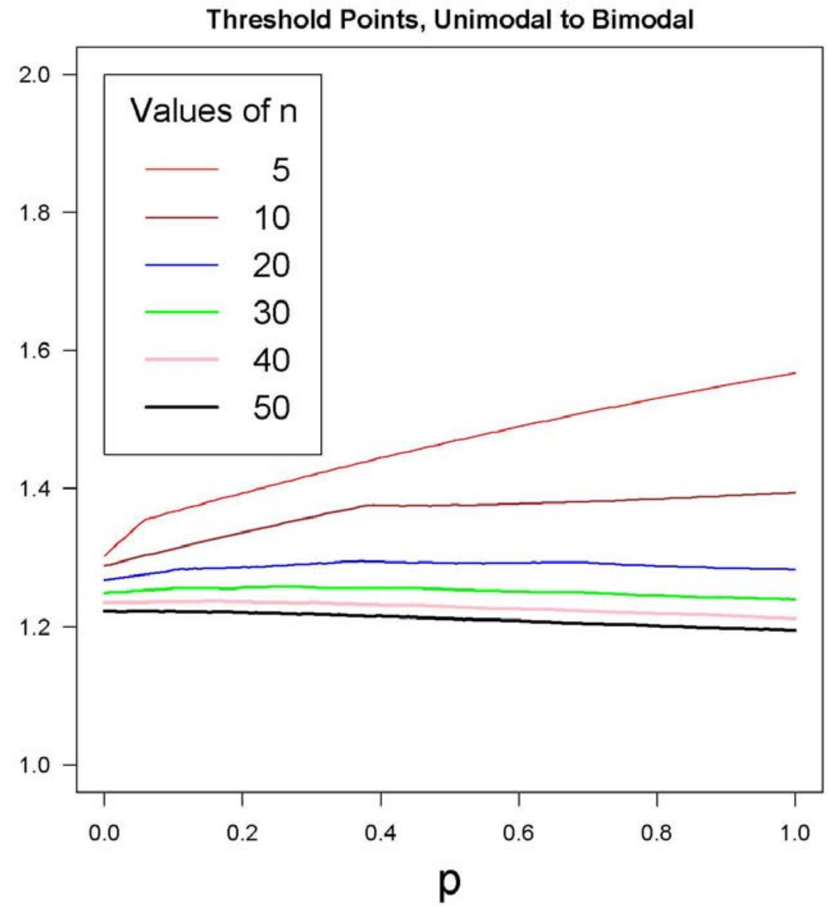
## *Serial vs broadcast spread*

The rumour is much less likely to spread from an initial spreader in broadcast mode

If rumour takes off, then most of the population get to hear it,

# Thresholds for rumour and SIR models

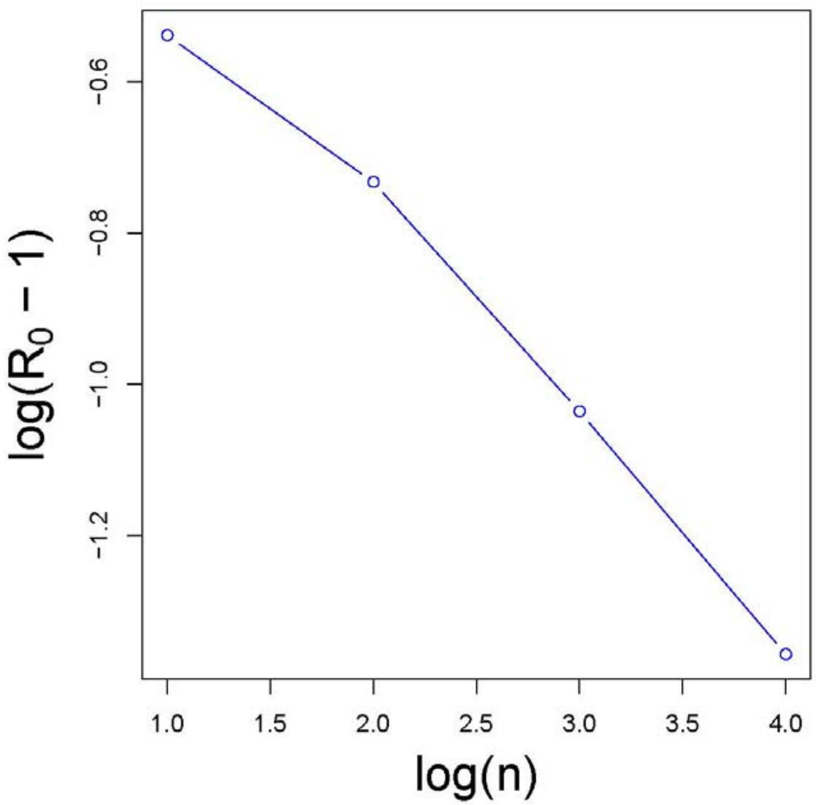


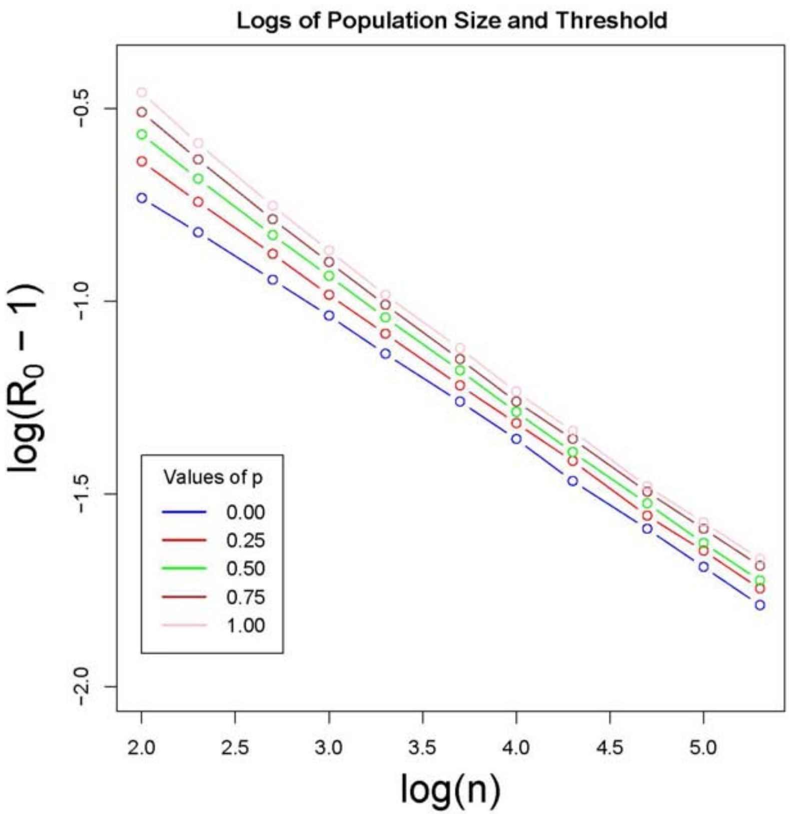




*rumours*: $R$ vs $p$

Threshold becomes independent of $p$ as $n$ increases

# *Thresholds for rumour and SIR models*




*SIR*: $p = 0 \; (R_0 = 1/\psi)$

# For general audience

- Add background on simple det/stoch epidemic models

- Could add slide with det threshold for a general audience

## *Threshold behaviour - SIR model*

Branching process approximation for small $Y(0)/n$ (Whittle, 1955)
ie assume all contacts of infectives are with susceptibles

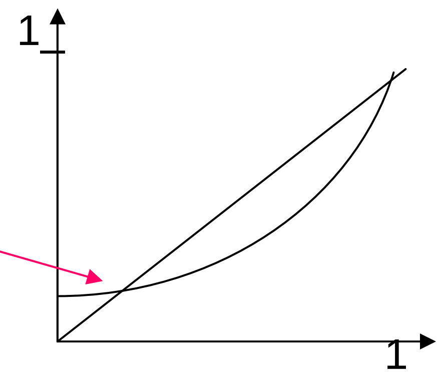
Poisson infectious contacts for exponential infectious period
$\Longrightarrow$ geometric distribution of offspring, pgf $G(z)$, mean $R_0 = \alpha/\gamma$

Starting from single infective

extinction probability is

solution of $z = G(z)$

ie $1/R_0$ for $R_0 > 1$




$\Longrightarrow$ major outbreak with prob. $1-(1/R_0)^{Y(0)}$ $(R_0 > 1)$

U-shaped *final size* distribution ( J -shaped for $R_0 < 1$ )

time to extinction $O(\ln n)$ (Barbour, 1975)

If the outbreak *does* take off….

Central Limit effect as $Y(t)$ increases:

The SIR process tends to a Gaussian diffusion about the deterministic solution.

$\Rightarrow$

Formally, look at a sequence of models as $n \longrightarrow \infty$, where

$x(0)/n, y(0)/n$ are kept fixed (Whittle 1957, Kurtz 1970, ….)

Means and variances scale with *n*

(and thus the *proportions* tend to the deterministic solution with probability one)

# Could add description /results for rumour model from D&G

Deterministic analysis

Results  for embedded chain

Diffusion approxn

TexPoint fonts used in EMF.

Read the TexPoint manual before you delete this box.: AAAAA