Is R_0 compatible with spatial epidemics?
- new results from long-range percolation

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What is R_0?

Textbook definition:

The basic reproduction number R_0 is the number of secondary infections per initial infective individual in a further susceptible large population.

Properties of R_0 in homogeneous randomly mixing populations:

- In the large population limit, R_0 corresponds to the offspring mean of a Galton Watson branching process.
- A large outbreak occurs with positive probability if and only if $R_0 > 1$.
- Let X_k^n be the set of individuals in the k-th infection generation in a randomly mixing population of size n and $\mathcal{B}_k^n := \cup_{j=0}^k X_j^n$,

$$\begin{aligned} \lim_{k\to\infty} \lim_{n\to\infty} (\mathbb{E}(|X_k^n|))^{1/k} &= R_0, \\ \lim_{k\to\infty} \lim_{n\to\infty} (\mathbb{E}(|\mathcal{B}_k^n|))^{1/k} &= \max(R_0, 1). \end{aligned}$$

- Stronger result: if $R_0 > 1$, then $|X_k^n|$ and $|\mathcal{B}_k^n|$ grow exponentially (with base R_0) with positive probability.

Useful properties of R_0 (previous slide), only hold for randomly mixing populations and very special networks.
New definition:

$$R_* = \lim_{k\to\infty} \lim_{n\to\infty} (\mathbb{E}(|\mathcal{B}_k^n|))^{1/k}. \quad (1)$$

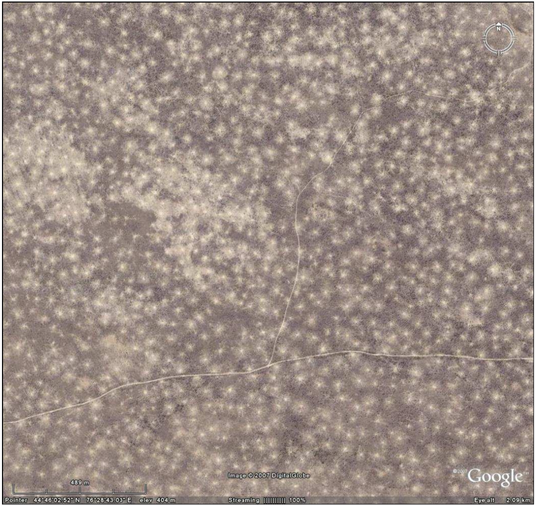
This quantity is easy to compute in randomly mixing populations, Molloy-Reed and Erdös-Rényi, Barabási-Albert random networks and in general on all networks that are locally tree-like (possibly with individuals replaced by "super-individuals"/cliques).
It is hard to compute if structure is more involved and if there is positive clustering.

Consider simplest (toy) model for a spatial *SIR* epidemic

- Individuals are located at the vertices of the square lattice.
- contacts are only possible with nearest neighbours.
- In this model $R_* = 1$: The number of individuals that can be reached within k steps grows at most quadratically, so $(\mathbb{E}(|\mathcal{B}_k^n|))^{1/k} \to 1$.
- Models that use some kind of R_0, and give it the usual interpretation in spatial epidemics are fated to fail.
- Pair approximation techniques for spatial epidemics are dangerous here.

An Example: Plague in Kazakhstan

- Great gerbils live in burrow systems and usually do not wander too far away from their homes.
- Plague is transmitted via fleas "jumping" from one gerbil to another one that is "passing by".
- Great gerbils eat all vegetation in a circle of radius about 20 metres of their burrow system, this leads to the following picture:


489 m
Image © 2007 DigitalGlobe
Google
Pointer 44°46'02.52" N 76°28'43.03" E elev 404 m
Streaming |||||||||| 100%
Eye alt 2.09 km

The burrows of the gerbils are located in a rather regular pattern. Because of the nature of the spread R_0 does not seem to be the thing to consider.
See Davis et al. *Nature* **454**, 634-637.

Long-range percolation

- In reality not only nearest neighbour contacts are made.
- Consider individuals living at the vertices of $\mathbb{Z}^d$.
- Assume that during the infectious period of fixed length 1, an individual make contacts with an individual at distance r at rate $\lambda(r)$ independent of other contacts made in the population. $\lambda(r)$ is non-increasing.
- In a SIR epidemic: The probability of a contact between individuals at distance r is given by $p(r) := 1 - e^{-\lambda(r)}$.

Question:

Is it possible to find a function $\lambda(r)$ such that R_* is non trivial?

- If for some $R > 0$, $\lambda(r) = 0$ for all $r > R$, then $|\mathcal{B}_k|$ is growing at most quadratically.
- If $\lambda(r)$ is decaying exponentially, then a large outbreak spreads like a travelling wave.
  (see e.g. Mollison, *J. Roy. Statist. Soc.B* **39**, 283-326).
- If $\lambda(r)$ is such that $\sum_{x \in \mathbb{Z}^d} p(\|x\|) = \infty$, then $|\mathcal{B}_1| = \infty$ and $R_* = \infty$. An example $\lambda(r) = \alpha r^{-\beta}$ with $0 < \beta \leq d$.

New results:

- If $\lambda(r) = e^{-\beta(r)}$ and $\liminf \beta(r) > d$, then for $k \to \infty$, $|\mathcal{B}_k|^{1/k} \to 1$ a.s. So, $R_* = 1$.
  Marek Biskup already showed that if $\lambda(r) = \alpha e^{-\beta}$ with $d < \beta < 2d$, then the graph distance between x and y (if they are in the same component) scales like $(\log \|x-y\|)^{\Delta}$, where $\Delta = \frac{\log[2]}{\log[2]+\log[d]-\log[\beta]} > 1$.
- If $\lambda(r) = e^{-(d+g(r))}$ and $g(r) \to 0$, then with some minor extra conditions on $g(r)$,
  there exist constants $a_1 > 1$ and $a_2 < \infty$, such that

$$\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} < a_2 | \text{large outbreak}) = 1.$$

  So if R_* exists, then $1 < R_* < \infty$.

Discussion

We showed that it is possible to construct a spatial epidemic for which with positive probability $|\mathcal{B}_k| > c(a_1)^k$ for all $k \in \mathbb{N}$ and some $c > 0$ and $a_1 > 1$.
We did not prove yet whether $|\mathcal{B}_k|^{1/k}$ converges or not and we did not have a clue about how to deduce R_* from $\lambda(r)$.