# Macroscopic Degeneracy and FSS at 1st Order Phase Transitions

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# Plan of talk

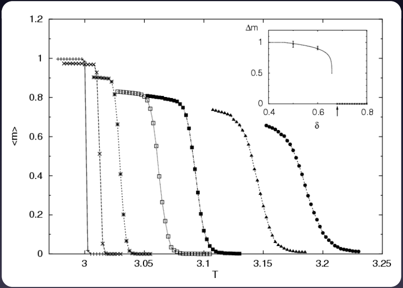
"Standard" 1st order FSS

A Problem with 1st order FSS for a plaquette $3D$ Ising model

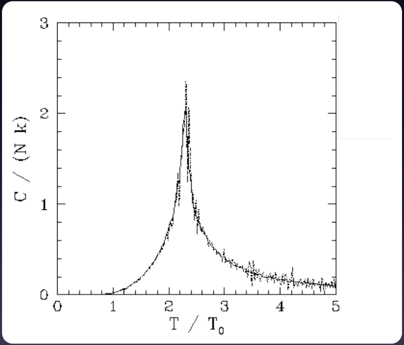
A solution

# First and Second Order Transitions

First order - discontinuities in magnetization, energy (latent heat)




Second order - divergences in specific heat, susceptibility



# The $q$-state Potts model

Hamiltonian

$$\mathcal{H}_q = -\sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j}$$

Evaluate the partition function, derivatives give observables

$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta \mathcal{H}_q)$$

# 1st Order FSS

Pirogov-Sinai Theory (Borgs/Kotecký)

$$Z(\beta) = \left[e^{-\beta L^d f_d} + q e^{-\beta L^d f_o}\right]\left[1 + \mathcal{O}(L^d e^{-L/L_0})\right]$$

# FSS: Specific Heat

Probability of being in any of the states

$$p_{\mathrm{o}} \propto e^{-\beta L^d \hat{f}_{\mathrm{o}}} \text{ and } p_{\mathrm{d}} \propto e^{-\beta L^d \hat{f}_{\mathrm{d}}}$$

Time spent in the ordered states $\propto q p_{\mathrm{o}}$

$$W_{\mathrm{o}}/W_{\mathrm{d}} \simeq q e^{-L^d \beta \hat{f}_{\mathrm{o}}} / e^{-\beta L^d \hat{f}_{\mathrm{d}}}$$

Expand around $\beta^{\infty}$, solve for specific heat peak

$$\beta^{C_V^{\max}}(L) = \beta^{\infty} - \frac{\ln q}{L^d \Delta \hat{e}} + \dots$$

# 1st Order FSS

Peaks grow as $L^d$

Critical points shift as $1/L^d$

Except …

Fixed boundaries ($1/L$ leading term)

$$Z(\beta) = \left[ e^{-\beta(L^d f_d + L^{d-1} \hat{f}_o)} + q e^{-\beta(L^d f_o + L^{d-1} \hat{f}_d)} \right]$$

# A $3D$ Plaquette Ising action

$3D$ cubic lattice, spins on *vertices*

$$H = -\sum_{\Box} \sigma_i \sigma_j \sigma_k \sigma_l$$

Not edges

$$H = -\sum_{\Box} U_{ij} U_{jk} U_{kl} U_{li}$$

Strong 1st order transition

# Dualizing the Model

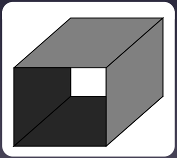
High-T expansion

$$Z(\beta) = \sum_{\{\sigma\}} \prod_{[ijkl]} \cosh(\beta)\left[1 + \tanh(\beta)\left(\sigma_i \sigma_j \sigma_k \sigma_l\right)\right]$$

which can be written as

$$Z(\beta) = \left[2\cosh(\beta)\right]^{3L^3} \sum_{\{S\}} \left[\tanh(\beta)\right]^{n(S)}$$

“Matchbox” spins



# The Dual Hamiltonian

An anisotropically coupled Ashkin-Teller model

$$H_{dual} = -\frac{1}{2}\sum_{\langle ij\rangle_x}\sigma_i\sigma_j - \frac{1}{2}\sum_{\langle ij\rangle_y}\tau_i\tau_j - \frac{1}{2}\sum_{\langle ij\rangle_z}\sigma_i\sigma_j\tau_i\tau_j\ ,$$

Standard duality relation

$$\tanh\beta = e^{-2\beta^*}$$

# The Problem

High precision multicanonical simulation, determine critical point(s) - a nice exercise for a PhD student (Marco)

Original model: $L = 8, 9, ..., 26, 27$, periodic bc, $1/L^3$ fits

$$\beta^{\infty} = 0.549994(30)$$

Dual model: $L = 8, 10, ..., 22, 24$, periodic bc, $1/L^3$ fits

$$\beta^{\infty}_{dual} = 1.31029(19)$$

$$\beta^{\infty} = 0.55317(11)$$

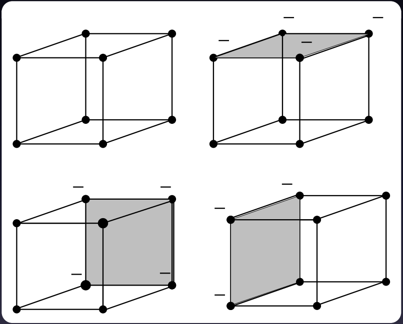
Estimates are about 30 error bars apart.

# The Solution

Degeneracy

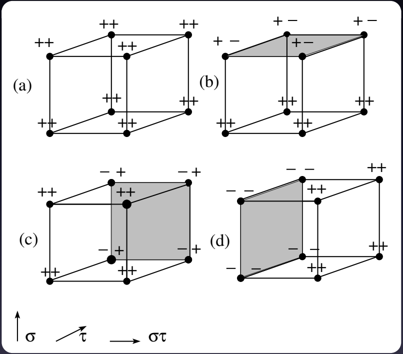
Modified FSS

# Groundstates: Plaquette



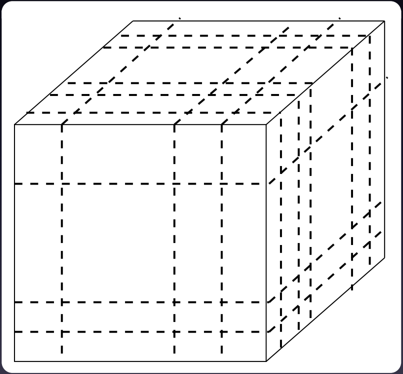

Persists into low temperature phase: degeneracy $2^{3L}$

# Groundstates: Dual




Dual degeneracy

# Ground state



# 1st Order FSS with Exponential Degeneracy

Normally $q$ is constant

Suppose instead $q \propto e^L$

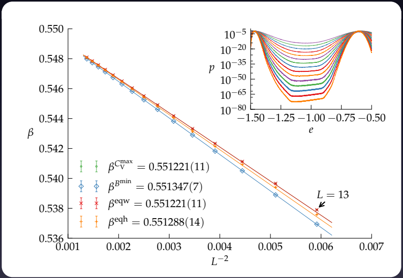
$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta\hat{e}} + \ldots$$

becomes

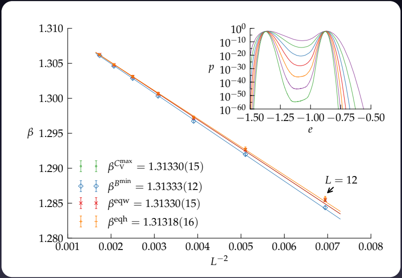
$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{1}{L^{d-1} \Delta\hat{e}} + \ldots$$

# FSS

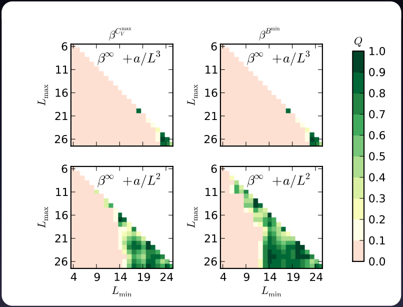
## Plaquette Hamiltonian fits




## Dual Hamiltonian fits



# Quality of fits




Forcing a fit to $1/L^3$ gives much poorer quality

# Conclusions

Standard 1st order FSS: $1/L^3$ corrections in 3D

Fixed BC: $1/L$ (surface tension)

Exponential degeneracy: $1/L^2$ in 3D

Quantum case?

# References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