The Gonihedric Ising model and (probably not) its dual(s)

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No books to flog, but....Springer Lecture Notes in Physics 736
"Rugged Free Energy Landscapes", ed. W. Janke

Gonihedric?

Gonia: angles

Hedra: face (seat)

Plan of Talk

A spin model on a 3D cubic lattice

History of the Gonihedric model

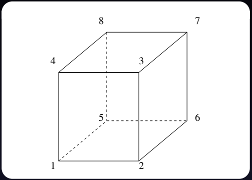
Phase Diagram

Non-equilibrium properties

(probably not) Dual(s)

A spin model on a 3D cubic lattice

A spin model on a 3D cubic lattice



$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \quad , \quad \sigma_i = \pm 1$$

$$Z = \sum_{\{\sigma\}} \exp(-\beta H) \quad , \quad \beta = 1/T$$

$$H = -\sum_{[ijkl]} U_{ij} U_{jk} U_{kl} U_{li} \quad , \quad U_{ij} = \pm 1$$

What to do with such a Model

Calculate or measure: magnetization, energy...

$$M = \frac{1}{N_i} \sum_i \sigma_i \quad , \quad E = \frac{1}{N_e} \sum_{ij} \sigma_i \sigma_j$$

Specific heat, susceptibility

$$C \simeq \langle E^2 \rangle - \langle E \rangle^2 \quad , \quad \chi \simeq \langle M^2 \rangle - \langle M \rangle^2$$

Look for phase transitions

Having done that...

Identify phase transition β_c, order parameter M

Define a reduced temperature $t = |(\beta - \beta_c)/\beta_c|$

Extract critical exponents (continuous transition)

$$M \sim t^{\beta} \ , \ C \sim t^{-\alpha} \ , \ \chi \sim t^{-\gamma}$$

Correlation length:

$$\xi \sim t^{-\nu}$$

- Or observe a “jump” in order parameter (first order)

History of the Gonihedric Model

History of the Gonihedric (String) Model

String Theory

Polyakov Action

$$S = \int d^2\sigma \sqrt{detg} g^{ab} \partial_a X_\mu \partial_b X^\mu$$

Partition function

$$Z = \int [Dg][DX] \exp(-S)$$

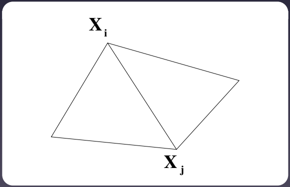
Triangulated Surfaces

Discretize worldsheet with triangles (Eynard lectures)

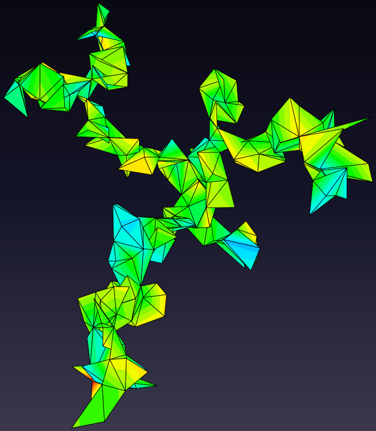
Gaussian Action

$$S \sim \sum_{ij} (X^{\mu}(i) - X^{\mu}(j))^2$$

$$Z = \sum_{T} \int \prod_{i} dX_i \exp{(-S)}$$



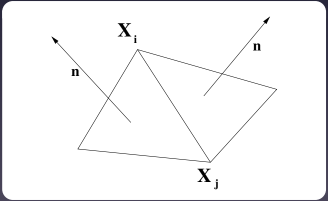
Typical Surfaces



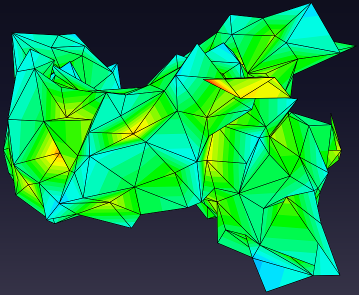
Modifying the Gaussian Action

Add extrinsic curvature term

$$
\begin{aligned}
S &= \sum_{ij} \left(X^\mu(i) - X^\mu(j)\right)^2 \\
&+ \lambda \sum_{\Delta_i,\Delta_j} (1 - \vec{n}_i \cdot \vec{n}_j)
\end{aligned}
$$



Smoothed Surfaces

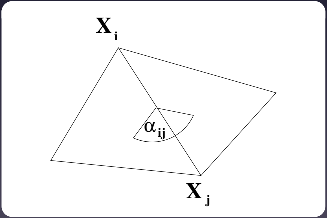


The Gonihedric action

Savvidy "Gonihedric" action

$$
\begin{aligned}
S &= \sum_{ij} |X^\mu(i) - X^\mu(j)| \theta_{ij} \\
\theta_{ij} &= |\pi - \alpha_{ij}|
\end{aligned}
$$

Gonia: angle
Hedra: face

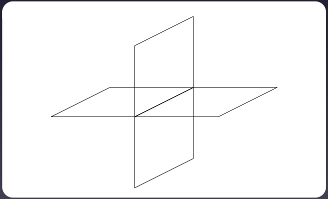


George Savvidy, Franz Wegner and the end of the Soviet Union

Gonihedric Surfaces on a Cubic Lattice

Edge lengths the same

$$
\begin{aligned}
S &= \sum_{ij} \left| X^{\mu}(i) - X^{\mu}(j) \right| \theta_{ij} \\
\theta_{ij} &= |\pi - \alpha_{ij}|
\end{aligned}
$$



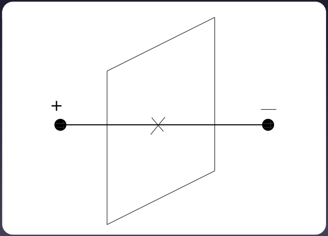
Gonihedric Surfaces on a Cubic Lattice II

Spins on dual lattice $\rightarrow$ plaquettes

Spin cluster boundaries $\rightarrow$ surfaces

Edge spins: $U_{ij} = -1$

Vertex spins: $\sigma_i \sigma_j = -1$



Counting configurations with spins

Spin gadgets to count area, bends and intersections

Area

$$S(\sigma) = \sum_{\langle ij \rangle} \frac{1}{2}(1 - \sigma_i \sigma_j)$$

Self intersection

$$I = -\frac{1}{8}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1)$$
$$+\frac{1}{8}(\sigma_1\sigma_3 + \sigma_2\sigma_4) + \frac{1}{8}(\sigma_1\sigma_2\sigma_3\sigma_4) + \frac{1}{8}$$

Still Counting configurations with spins

Bends (and two from a crossing)

$$\begin{aligned} C = & - \frac{1}{4}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1) \\ & + \frac{1}{4}(\sigma_1\sigma_3 + \sigma_2\sigma_4) - \frac{1}{4}(\sigma_1\sigma_2\sigma_3\sigma_4) + \frac{3}{4} \end{aligned}$$

A Generalized Ising action

In general energy comes from areas, edges and intersections (*A. Cappi, P Colangelo, G. Gonella and A. Maritan*)

$$H = J_1 \sum_{<ij>} \sigma_i \sigma_j + J_2 \sum_{<<ij>>} \sigma_i \sigma_j + J_3 \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

$$\beta_A = 2J_1 + 8J_2, \quad \beta_C = 2J_3 - 2J_2, \quad \beta_I = -4J_2 - 4J_3$$

$$H = \sum (\beta_A n_A + \beta_C n_C + \beta_I n_I)$$

A Particular Generalized Ising action

For the Gonihedric model, we want *no* area weight

$$H = 2\kappa \sum_{<ij>} \sigma_i \sigma_j - \frac{\kappa}{2} \sum_{<<ij>>} \sigma_i \sigma_j + \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

$\kappa = 0$ simple (and interesting)

$$H = \frac{1}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

Phase Diagram

Phase Diagram: Known knowns

What can we say analytically $\kappa = 0$ plaquette?

2D, solvable

In 3D, no exact solution for isotropic model

3D *an*isotropic, solvable (Suzuki)

Phase Diagram: 2D Knowns

In 2D weight corners (rather than edges)

$$H = \kappa \sum_{<ij>} \sigma_i \sigma_j - \frac{\kappa}{2} \sum_{<<ij>>} \sigma_i \sigma_j + \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

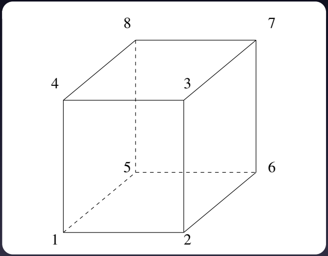
$\kappa = 0$ is a 5-vertex model in disordered regime

No transition

Also appears to be true for $\kappa \neq 0$

Phase Diagram: 3D Anisotropic Knowns $\kappa = 0$

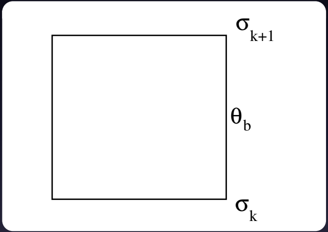
Switch off coupling



Vertical - Stack of 2D Gonihedric Ising

Horizontal (“Fuki-nuke”)- Stack of 2D Ising

Phase Diagram: Stack of 2D Ising Models



$$b = (i, j, k + 1/2)$$

- $$H = -J \sum_b (\theta_b \theta_{b+\hat{x}} + \theta_b \theta_{b+\hat{y}})$$

What can we say in 3D?: Isotropic Ground States

Table: *Elementary cube ground state energies*

Top	Bottom	Energy	Multiplicity
+ + + +	+ + + +	$-3/2 - 3\kappa/2$	2
- + + -	+ - - +	$-3/2 + 21\kappa/2$	2
- - - -	+ + + +	$-3/2 - 3\kappa/2$	6
+ + - -	- - + +	$-3/2 + 5\kappa/2$	6

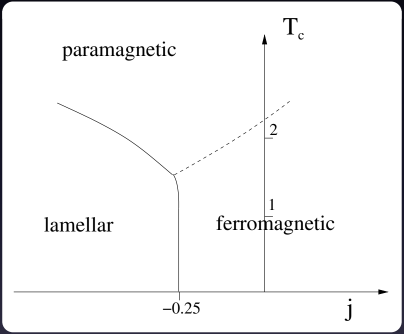
Known Unkowns: Mean Field/Series/Monte Carlo

- Layered ground state

  Series expansions (Pietig, Wegner) and Monte-Carlo suggest this has higher energy at finite temperature when $\kappa \neq 0$

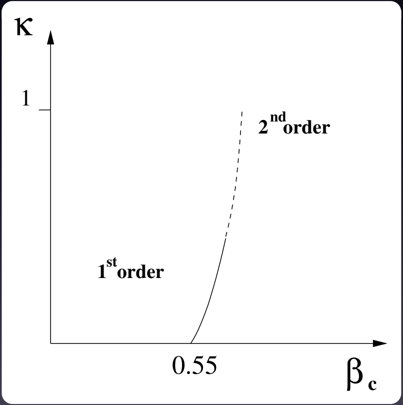
- *But* layered phase still present at $T \neq 0$ when $\kappa = 0$

Known Unknowns: Phase Diagram by Monte-Carlo



$$H = -\sum \sigma_i \sigma_j - j \sum \sigma_i \sigma_j - \frac{1-\kappa}{4\kappa} \sum \sigma_i \sigma_j \sigma_k \sigma_l$$

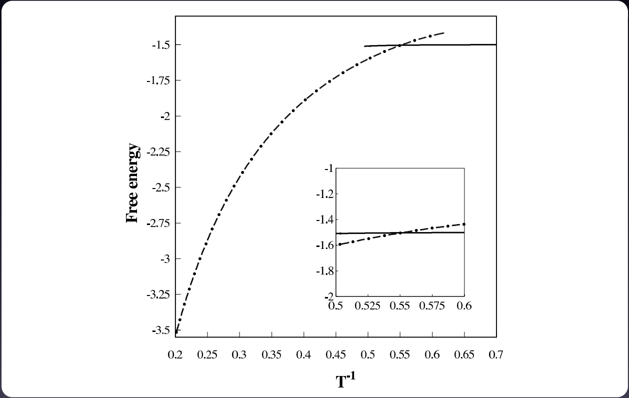
Known Unknowns: Phase Diagram by Monte-Carlo II



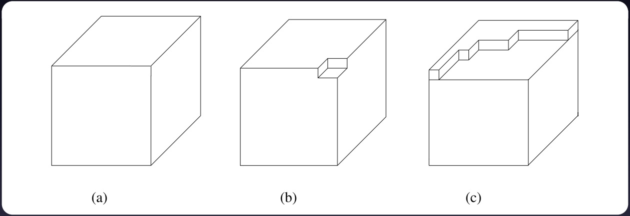
Non-equilibrium Properties

Known Unknowns: $\kappa = 0$

- Metastability



Dynamics - Heuristics



(Spin) glasses

Similar Hamiltonian to standard Ising model

$$H = \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j$$

But couplings are now random

$$J_{ij} = \pm 1$$

Frustration

Glassy phase

Dynamics

- Usual glassy observables

$$C(t, t_w) = \langle \frac{1}{N} \sum_i \sigma_i(t_w) \sigma_i(t + t_w) \rangle$$

$$Q(t_w + t, t_w + t) = \langle \frac{1}{N} \sum_i \sigma_i^{(1)}(t + t_w) \sigma_i^{(2)}(t + t_w) \rangle$$

- $T_g < T < T_C$ stretched exponential

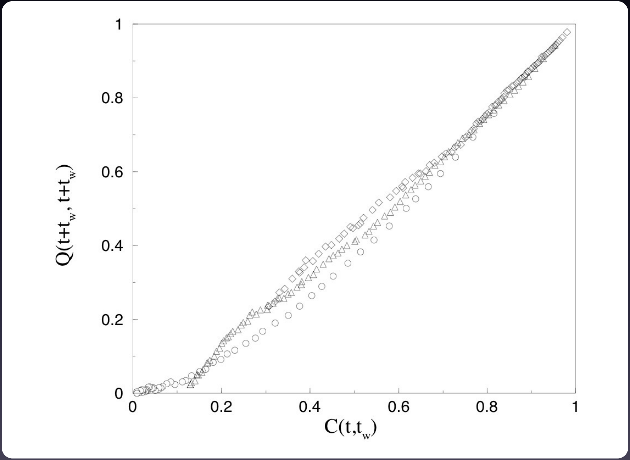
$$A(t, t_w) = \langle E(t_w) E(t + t_w) \rangle$$

$$A(t, t_w) \sim A_0 \exp \left(- \left(\frac{t}{\tau} \right)^\beta \right)$$

- $\tau \sim \frac{A}{T - T^*}$

Aging (Type II)

- $T < T_g$



Summary

Glassy behaviour *without* quenched disorder

Conclusions

- 3D Gonihedric model has novel ground state "flip" symmetry
- Persists at finite T in $\kappa = 0$ (plaquette-only) model
- At $\kappa = 0$ first order phase transition, metastability
- Glassy features in dynamics (with *no* quenched disorder)

(Probably not) Dual(s)

Dual(s)

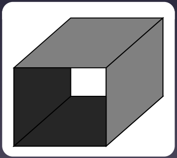
Savvidy/Wegner $\kappa = 0$ dual, by hand

$$Z(\beta) = \sum_{\{\sigma\}} \prod_{[ijkl]} \cosh(\beta) \left[1 + \tanh(\beta) \left(\sigma_i \sigma_j \sigma_k \sigma_l\right)\right]$$

which can be written as

$$Z(\beta) = \left[2\cosh(\beta)\right]^{3L^3} \sum_{\{S\}} \left[\tanh(\beta)\right]^{n(S)}$$

"Matchbox" spins



Dual(s)

Hamiltonian

$$H_{dual0} = -\sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_{\langle ik \rangle} \tau_i \tau_k - \sum_{\langle jk \rangle} \eta_j \eta_k$$

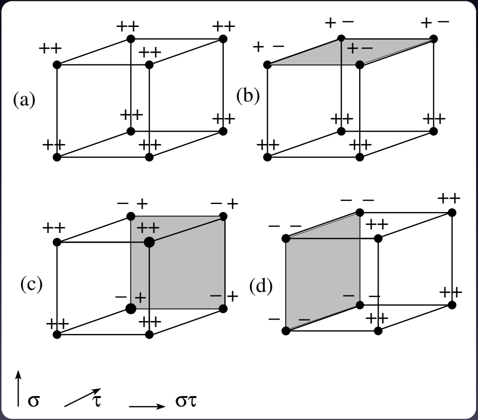
Spins

$$\begin{aligned} e\sigma &= \sigma\ ,\ \ e\tau = \tau\ ,\ \ e\eta = \eta \\ \sigma^2 &= \tau^2 = \eta^2 = e \\ \sigma\tau &= \eta\ ,\ \ \tau\eta = \sigma\ ,\ \ \eta\sigma = \tau \end{aligned}$$

Ashkin-Teller style dual

$$H_{dual1} = -\sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_{\langle ik \rangle} \tau_i \tau_k - \sum_{\langle jk \rangle} \sigma_j \sigma_k \tau_j \tau_k\ .$$

Dual ground states (Ashkin-Teller style)



Other (?) Duals

Wegner wrote a general framework for duality in D dimensional spin models

Two ways of thinking about surfaces in 3D Gonihedric models

2D surfaces, codimension 1 surfaces

$$
\begin{aligned}
H_{dual3} &= -\sum_{\langle ij \rangle} \left(\sigma_i U^1_{ij} \sigma_j + \mu_i U^1_{ij} \mu_j \right) - \sum_{\langle ik \rangle} \left(\tau_i U^2_{ik} \tau_k + \mu_i U^2_{ik} \mu_k \right) \\
&- \sum_{\langle jk \rangle} \left(\sigma_j U^3_{jk} \sigma_k + \tau_j U^3_{jk} \tau_k \right)
\end{aligned}
$$

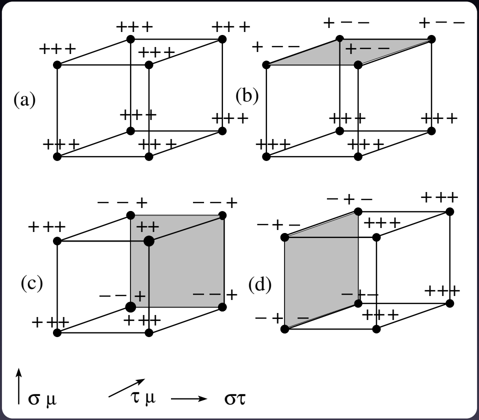
$\longrightarrow$ (Un)Decoration $\longrightarrow$

$$
H_{dual2} = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \mu_i \mu_j - \sum_{\langle ik \rangle} \tau_i \tau_k \mu_i \mu_k - \sum_{\langle jk \rangle} \sigma_j \sigma_k \tau_j \tau_k
$$

$\longrightarrow$ Gauge-Fixing $\longrightarrow$

$$
H_{dual1} = -\sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_{\langle ik \rangle} \tau_i \tau_k - \sum_{\langle jk \rangle} \sigma_j \sigma_k \tau_j \tau_k
$$

Dual ground states



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- Persists at finite T in $\kappa = 0$ (plaquette-only) model
- At $\kappa = 0$ first order phase transition, metastability
- Glassy features in dynamics (with *no* quenched disorder)
- Dual(s) are anisotropic, interesting beasties in their own right