Traffic Jams to Corruption: mathematical models of condensation

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Plan of Talk

ASEP - ASymmetric Exclusion Process

ZRP - Zero Range Process

Definition of the models

Solutions

Applications

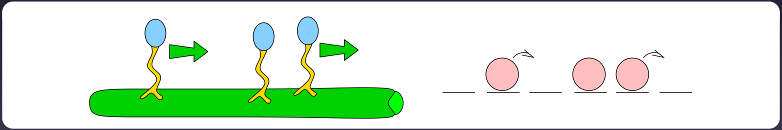
The first model: ASEP

Definition: ASEP

Originally came from biology

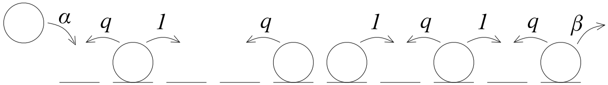
Model for transport in cells

Kinesins moving along a microtubule



Abstracting the mathematical model

- One dimensional lattice, N sites
- "Hard" particles
- Forcing



Setup for solution

- Configuration

$$\mathcal{C}$$

- Statistical weights for configuration

$$f(\mathcal{C})$$

- Normalized probability

$$P(\mathcal{C}) = f(\mathcal{C})/Z$$

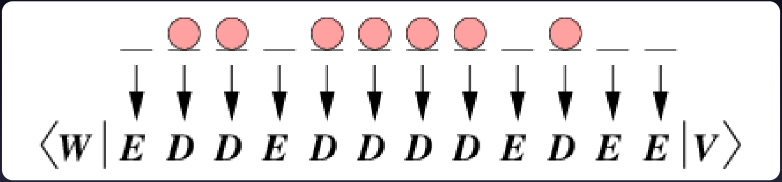
- Normalization

$$Z = \sum_{\mathcal{C}} f(\mathcal{C})$$

- Master Equation

$$\frac{\partial P(\mathcal{C},t)}{\partial t} = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[P(\mathcal{C}',t) W(\mathcal{C}' \to \mathcal{C}) - P(\mathcal{C},t) W(\mathcal{C} \to \mathcal{C}') \right]$$

Particles to Matrices



Matrix Product solution

- Represent ball with

$$D$$

- Represent space with

$$E$$

- Represent $P(\mathcal{C})$ as

$$P(\mathcal{C}) = \frac{\langle W|DDE\dots E|V\rangle}{Z_N}$$

- Make sure behaviour of D, E is compatible with dynamics:

$$\begin{aligned} DE - qED &= D + E \\ \alpha\langle W|E &= \langle W| \\ \beta D|V\rangle &= |V\rangle \end{aligned}$$

Why does this work?

$$\frac{\partial f}{\partial t}(\tau_1, \tau_2, \ldots, \tau_N) = \left[\hat{h}_L + \sum_{i=1}^{N-1} \hat{h}_{i,i+1} + \hat{h}_R\right] f(\tau_1, \tau_2, \ldots, \tau_N)$$

Find $\tilde{X}, X$ such that

$$\begin{aligned}
\hat{h}_{i,i+1}\langle W|\cdots X_{\tau_i} X_{\tau_{i+1}} \cdots |V\rangle &= \langle W|\cdots[\tilde{X}_{\tau_i} X_{\tau_{i+1}} - X_{\tau_i}\tilde{X}_{\tau_{i+1}}]\cdots|V\rangle \\
\hat{h}_L\langle W|X_{\tau_1}\cdots|V\rangle &= \left[-\langle W|\tilde{X}_{\tau_1}\right]\cdots|V\rangle \\
\hat{h}_R\langle W|\cdots X_{\tau_N}|V\rangle &= \langle W|\cdots\left[\tilde{X}_{\tau_N}|V\rangle\right]
\end{aligned}$$

Everything cancels: $\tilde{D} = -1$, $\tilde{E} = 1$

Use generating function

Sum up different lengths N

$$\mathcal{Z}(z) = \sum_{N=0}^{\infty} z^N Z_N \ .$$

Consider the formal series, $C = D + E$

$$\langle W | \frac{1}{1 - zC} | V \rangle = \sum_{n=0}^{\infty} z^n \langle W | C^n | V \rangle = \mathcal{Z}(z)$$

And notice that

$$(1 - \eta D)(1 - \eta E) = 1 - \eta(D + E) + \eta^2 DE = 1 - \eta(1 - \eta)C$$

suggests taking $z = \eta(1 - \eta)$

Use generating function

Factorize

$$\langle W|\frac{1}{1-zC}|V\rangle = \langle W|\frac{1}{1-\eta E}\frac{1}{1-\eta D}|V\rangle$$

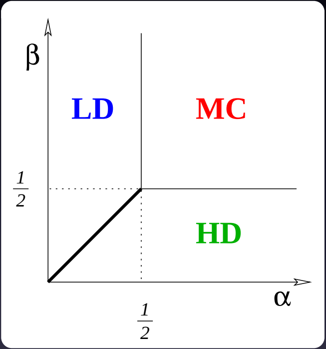
Act on vectors

$$\mathcal{Z}(z) = \left(1-\frac{\eta(z)}{\alpha}\right)^{-1}\left(1-\frac{\eta(z)}{\beta}\right)^{-1}$$

Where

$$\eta(z) = \frac{1}{2}\left(1-\sqrt{1-4z}\right)$$

The Phase Diagram ($q = 0$)



Roll of honour (TASEP $q = 0$): Derrida, Evans, Hakim Pasquier
Roll of honour (PASEP $q \neq 0$): Blythe, Evans, Colaiori, Essler

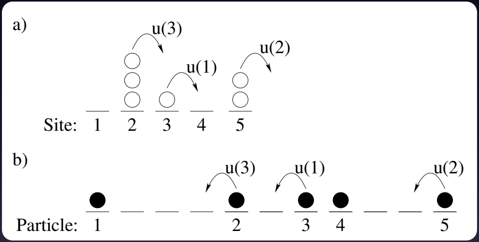
In Real Life

Traffic jams

The second model: ZRP

Multiple particles per site

Jump rate depends on occupation number



$$p(n) = \prod_{i=1}^{n} u(i)^{-1} \qquad \text{for} \quad n > 0 \; , \qquad p(0) = 1 \; .$$

Finding a steady state

N balls in L boxes (so $\rho = N/L$)

A *factorized* steady state is possible

$$P(\{n_l\}) = Z_{L,N}^{-1} \prod_{l=1}^{L} p(n_l)$$

Where

$$Z_{L,N} = \sum_{\{n_l\}} \prod_{l=1}^{L} p(n_l)\, \delta\left(\sum_{l=1}^{L} n_l - N\right)$$

Finding a steady state

$$
\begin{aligned}
Z(N,\rho) = & \sum_{\{n_l\}} \prod_{l=1}^{L} p(n_l) \\
& \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{d}\lambda e^{-i\lambda(n_1+\cdots+n_L-\rho L)} \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{d}\lambda e^{i\lambda\rho L} \left(\sum_n p(n) e^{-i\lambda n} \right)^L \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{d}\lambda \exp\left(L(i\lambda\rho + K(i\lambda)\right))
\end{aligned}
$$

Not Finding a steady state!

Take $p(n) \sim n^{-\beta}$

As ρ increases $\lambda_*(\rho) \to 0$

Solutions vanishes for some ρ_c when $\lambda_*(\rho_c) = 0$

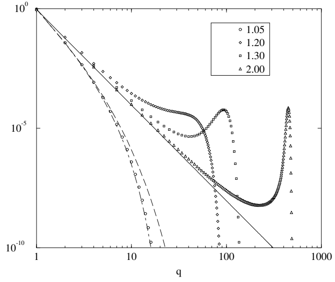
$$\rho_c = \frac{\zeta(\beta - 1)}{\zeta(\beta)}$$

What gives?

Condensation

Consider the “dressed” probability

$$\pi(n) \;\; = \;\; n^{-4} \, \frac{Z(L-1, N-n)}{Z(L, N)} \;\; ; \;\; (\rho_c \sim 1.11)$$



Wealt Condensation

The rich really are different....

Application: Wealth Condensation

Pareto 1897 - $p(n)$ of the personal income n for a rich guy

$$p(n) \sim n^{-\beta}$$

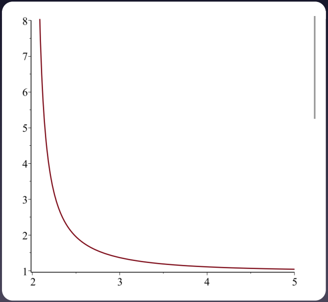
Gibrat 1931 - $p(n)$ for most is log-normal

$$p(n) = \frac{1}{n\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\log^2(n/n_0)}{2\sigma^2}\right]$$

Wealth Condensation

$$p(n) = n^{-\beta}$$

$$\rho_c = \frac{\zeta(\beta - 1)}{\zeta(\beta)}$$



Wealth Condensation

Simple, exactly solvable models can give insight

ASEP - flow/jamming

ZRP - condensation

References

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THE END :)