ASEPs and ZRPs

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Plan of Talk

ASEP - ASymmetric Exclusion Process

ZRP - Zero Range Process

Definition of the models

Solutions

Applications

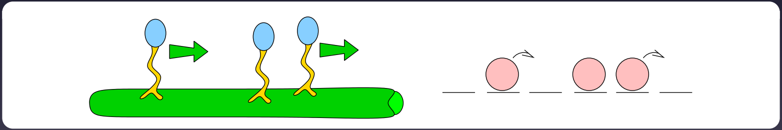
The first model: ASEP

Definition: ASEP

Originally came from biology

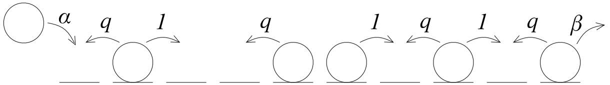
Model for transport in cells

Kinesins moving along a microtubule



Abstracting the mathematical model

- One dimensional lattice, N sites
- "Hard" particles
- Forcing



Setup for solution

- Configuration

$$\mathcal{C}$$

- Statistical weights for configuration

$$f(\mathcal{C})$$

- Normalized probability

$$P(\mathcal{C}) = f(\mathcal{C})/Z$$

- Normalization

$$Z = \sum_{\mathcal{C}} f(\mathcal{C})$$

- Master Equation

$$\frac{\partial P(\mathcal{C},t)}{\partial t} = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[P(\mathcal{C}',t) W(\mathcal{C}' \to \mathcal{C}) - P(\mathcal{C},t) W(\mathcal{C} \to \mathcal{C}') \right]$$

Matrix Product solution ($q = 0$)

- Represent ball with
$$X_i = D$$
- Represent space with
$$X_i = E$$
- Represent $P(\mathcal{C})$ as
$$P(\mathcal{C}) = \frac{\langle W | X_1 X_2 \dots X_N | V \rangle}{Z_N}$$
- Make sure behaviour of D, E is compatible with dynamics:
$$\begin{aligned} DE &= D + E \\ \alpha \langle W | E &= \langle W | \\ \beta D | V \rangle &= | V \rangle \end{aligned}$$

Use generating function

Sum up different lengths

$$\mathcal{Z}(z) = \sum_{N=0}^{\infty} z^N Z_N \ .$$

Consider the formal series

$$\langle W| \frac{1}{1 - zC} |V\rangle = \sum_{n=0}^{\infty} z^n \langle W|C^n|V\rangle = \mathcal{Z}(z)$$

And notice that

$$(1 - \eta D)(1 - \eta E) = 1 - \eta(D + E) + \eta^2 DE = 1 - \eta(1 - \eta)C$$

suggests taking $z = \eta(1 - \eta)$

Use generating function

Factorize

$$\langle W|\frac{1}{1-zC}|V\rangle = \langle W|\frac{1}{1-\eta E}\frac{1}{1-\eta D}|V\rangle$$

Act on vectors

$$\mathcal{Z}(z) = \left(1 - \frac{\eta(z)}{\alpha}\right)^{-1}\left(1 - \frac{\eta(z)}{\beta}\right)^{-1}$$

Where

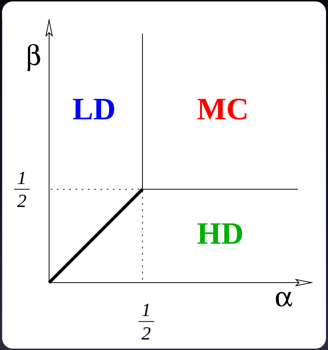
$$\eta(z) = \frac{1}{2}\left(1 - \sqrt{1-4z}\right)$$

Use generating function

Extract asymptotics (low density)

$$
\begin{aligned}
Z_N(\alpha,\beta) &\sim \left[\lim_{z\to z_0}\left(1-\frac{z}{z_0}\right)\mathcal{Z}(z;\alpha,\beta)\right] z_0^{-N} \\
&\sim \frac{\alpha\beta}{z_0\eta'(z_0)\left[\beta-\eta(z_0)\right]} z_0^{-N} \\
&\sim \frac{\alpha\beta(1-2\alpha)}{(\beta-\alpha)}\left[\alpha(1-\alpha)\right]^{-N-1}
\end{aligned}
$$

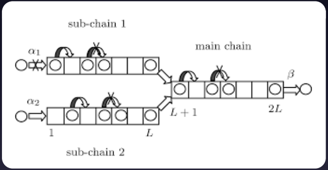
The Phase Diagram ($q = 0$)



Roll of honour (TASEP $q = 0$): Derrida, Evans, Hakim Pasquier
Roll of honour (PASEP $q \neq 0$): Blythe, Evans, Colaiori, Essler

Another Application

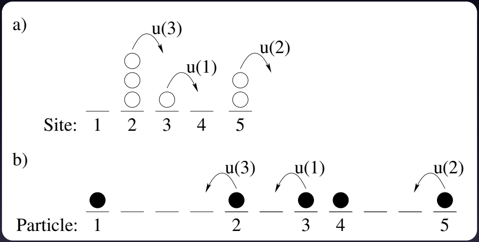
Traffic jams!



The second model: ZRP

Multiple particles per site

Jump rate depends on occupation number



$$p(n) = \prod_{i=1}^{n} u(i)^{-1} \qquad \text{for} \quad n > 0 \, , \qquad p(0) = 1 \, .$$

Finding a steady state

N balls in L boxes (so $\rho = N/L$)

A *factorized* steady state is possible

$$P(\{n_l\}) = Z_{L,N}^{-1} \prod_{l=1}^{L} p(n_l)$$

Where

$$Z_{L,N} = \sum_{\{n_l\}} \prod_{l=1}^{L} p(n_l) \, \delta\left(\sum_{l=1}^{L} n_l - N \right)$$

Finding a steady state

$$
\begin{aligned}
Z(N,\rho) = \quad & \sum_{\{n_l\}} \prod_{l=1}^{L} p(n_l) \\
& \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{d}\lambda e^{-i\lambda(n_1+\cdots+n_L-\rho L)} \\
= \quad & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{d}\lambda e^{i\lambda\rho L} \left(\sum_n p(n) e^{-i\lambda n} \right)^L \\
= \quad & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{d}\lambda \exp\left(L(i\lambda\rho + K(i\lambda)\right))
\end{aligned}
$$

Finding a steady state II

Generating function

$$K(\sigma) = \ln \sum_{n=1}^{\infty} p(n) e^{-\sigma n}$$

Exponential term

$$f(\rho) = \sigma_*(\rho)\rho + K(\sigma_*(\rho))$$

where $\sigma_*(\rho)$ is a solution of

$$\rho + K'(\sigma_*) = 0$$

Giving

$$Z(L, \rho) = e^{L f(\rho) + \dots}$$

Not Finding a steady state!

As ρ increases $\sigma_*(\rho) \to 0$

Solutions vanishes for ρ_c when $\sigma_*(\rho) = 0$

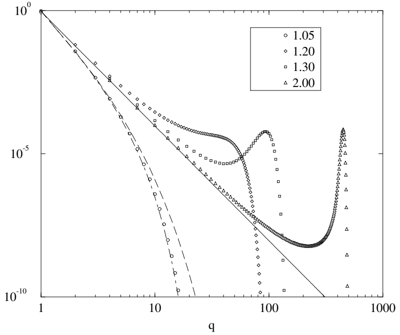
What gives?

Take $p(n) \sim n^{-\beta}$

Condensation

Consider the “dressed” probability

$$\pi(n) \;\; = \;\; p(n)\frac{Z(L-1,N-q)}{Z(L,N)}$$



Application: Wealth Condensation

The rich really are different - Pareto 1897

Probability density function $p(n)$ of the personal income n for a rich guy

$$p(n) \sim n^{-\beta}$$

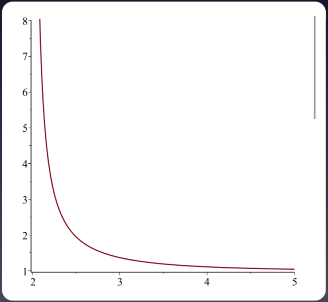
For most - Gibrat 1931

$$p(n) = \frac{1}{n\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\log^2(n/n_0)}{2\sigma^2}\right]$$

Wealth Condensation

$$p(n) = n^{-\beta}$$

$$\rho_c = \frac{\zeta(\beta - 1)}{\zeta(\beta)}$$



Wealth Condensation

Simple, exactly solvable models can give insight

ASEP - flow

ZRP - condensation

THE END :)