

# Spin Chain SUSY (A short history of failure...)

Des Johnston  
Maxwell Institute for Mathematical Sciences  
Heriot-Watt University, Edinburgh  
CompPhys17, Nov 2017

# Plan of talk

SUSY

Stuff that works, other people - Fendley, Hagendorf - (XXZ)  
Spin Chain SUSY

Stuff that doesn't work - me - ASEPs (ASAPs)

# SUSY

Originally.....

$$Q|Boson\rangle = |Fermion\rangle$$

$$Q^\dagger|Fermion\rangle = |Boson\rangle$$

$$Q^2 = 0, \quad (Q^\dagger)^2 = 0$$

In particular, Super-Poincaré algebra in  $4d$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu$$

Getting rid of divergences in QFT

Lattice discretisation difficult (but see Catterall et.al.)

# SUSY QM

Hamiltonian

$$H\psi_n(x) = \left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2 \right) \psi_n(x) = E_n \psi_n(x)$$

Define

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x); \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x)$$

Two SUSY Hamiltonians

$$H^{(1)} = A^\dagger A = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}} W'(x) + W^2(x)$$

$$H^{(2)} = AA^\dagger = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}} W'(x) + W^2(x)$$

# SUSY QM II

## SUSY algebra

$$\begin{aligned}[H, Q] &= [H, Q^\dagger] = 0 \\ \{Q, Q^\dagger\} &= H \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0\end{aligned}$$

$$\begin{aligned}H &= \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(2)} \end{pmatrix} \\ Q &= \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \\ Q^\dagger &= \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}\end{aligned}$$

# XXZ Spin Chain

XXZ Hamiltonian acts on  $V^{\otimes N}$  where  $V \simeq \mathbb{C}^2$ .

$$H^{(N)} = -\frac{1}{2} \sum_{i=1}^{N-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \frac{1}{2} \sigma_i^z \sigma_{i+1}^z \right) - \frac{1}{4} (\sigma_1^z + \sigma_N^z) + \frac{3N-1}{4} \mathbb{I}$$

Write as sum of local terms,  $h_{ij} : V \otimes V \rightarrow V \otimes V$  in bulk

$$H^{(N)} = \sum_{\langle ij \rangle} h_{ij} + h_1 + h_N$$

# XXZ Spin Chain II

Local Hamiltonian (in the bulk)

$$h_{ij} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{1}{2} \sigma^z \otimes \sigma^z + \frac{3}{4} \mathbb{I}$$

In matrix form, standard basis for  $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# XXZ Spin Chain/SUSY

Possible to define (Fendley, Yang)

$$Q^{(N)} : V^{\otimes N} \rightarrow V^{\otimes N-1}; \quad Q^{(N)\dagger} : V^{(N-1)} \rightarrow V^{(N)}$$

Properties of  $Q$ : nilpotent

$$Q^{(N-1)}Q^{(N)} = 0, \quad Q^{\dagger(N+1)}Q^{\dagger(N)} = 0,$$

Hamiltonian length  $N$

$$H^{(N)} = Q^{\dagger(N)}Q^{(N)} + Q^{(N+1)}Q^{(N+1)\dagger}.$$



# XXZ Spin Chain/Supercharge I

Length-changing operator

$$H^{(N-1)}Q^{(N)} = Q^{(N)}H^{(N)}, \quad H^{(N)}Q^{\dagger(N)} = Q^{\dagger(N)}H^{(N-1)}.$$

Local expressions (Hagendorf)

$$Q^{(N)} = \sum_{i=1}^{N-1} (-1)^{i+1} q_{i,i+1}, \quad Q^{\dagger(N)} = \sum_i^{N-1} (-1)^{i+1} q_i^{\dagger}$$

Local action - “dynamical SUSY”

$$q_i^{\dagger} : V \rightarrow V \otimes V; \quad q_{i,i+1} : V \otimes V \rightarrow V$$

# XXZ Spin Chain/Supercharge II

In terms of the local supercharge, nilpotency  $(Q^\dagger)^2 = 0$  becomes

$$(q^\dagger \otimes 1 - 1 \otimes q^\dagger)q^\dagger = 0$$

Or, more accurately - including a sort of “gauge” freedom

$$\left[ (q^\dagger \otimes 1)q^\dagger - (1 \otimes q^\dagger)q^\dagger \right] |m\rangle = |\chi\rangle \otimes |m\rangle - |m\rangle \otimes |\chi\rangle$$

For all  $|m\rangle \in V$ , for a particular  $|\chi\rangle \in V \rightarrow V \otimes V$

# XXZ Spin Chain/Supercharge III

Substitute  $Q, Q^\dagger$  expression into Hamiltonian, most terms vanish:

$$q_i^\dagger q_j + q_{j+1} q_i^\dagger = 0, \quad 1 \leq i < j - 1 \leq N - 1$$

Leaving

$$h = -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q + \frac{1}{2} (q q^\dagger \otimes 1 + 1 \otimes q q^\dagger)$$

With boundary terms

$$h_B = \frac{1}{2} q q^\dagger$$

# XXZ Spin Chain/Supercharge IIbis

The game of “does my Hamiltonian have dynamical supersymmetry”?

1) Pick a  $q^\dagger$  -  $(4 \times 2)$

2) Construct  $h$  -  $(4 \times 4)$  using:

$$h = -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q + \frac{1}{2} (q q^\dagger \otimes 1 + 1 \otimes q q^\dagger)$$

# XXZ Spin Chain/Supercharge IV

$q^\dagger$  is remarkably simple for XYZ (at combinatorial point)

$$q^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Action

$$q^\dagger|0\rangle = \emptyset; \quad q^\dagger|1\rangle = |00\rangle$$

# XXZ Spin Chain/Supercharge V

Could start with  $|1\rangle$  instead of  $|0\rangle$ , i.e.

$$\bar{q}^\dagger|1\rangle = \emptyset ; \quad \bar{q}^\dagger|0\rangle = |11\rangle$$

$\bar{q}^\dagger$  is still simple

$$\bar{q}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stitch together  $q, \bar{q}$ , using “gauge” freedom

$$q^\dagger(y) = x \begin{pmatrix} -2y & 1 \\ -y^2 & -y \\ -y^2 & -y \\ y^3 & -2y^2 \end{pmatrix} ; \quad x = (1 + |y|^6)^{-1/2}$$

# XXZ Spin Chain/Supercharge VI

With  $q^\dagger(y)$  still true that

$$h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Boundaries** become ( $y = \rho e^{i\theta}$ )

$$h_B(y) = \left( \frac{1 + 5\rho^2 + \rho^4}{4(1 - \rho^2 + \rho^4)} \right) \mathbb{I} + \sum_{j=1}^3 \lambda_j \sigma^j,$$

where

$$\lambda_1 = -\frac{\rho \cos \theta}{1 + \rho^2}, \quad \lambda_2 = -\frac{\rho \sin \theta}{1 + \rho^2}, \quad \lambda_3 = -\frac{1}{4} \left( \frac{1 - \rho^2}{1 + \rho^2} \right).$$

# XXZ Spin Chain/Supercharge VII

So far - SUSY for XXZ with off-diagonal BCs **same** at both ends (at  $\Delta = 1/2$ )

$$h_{ij} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \frac{1}{2} \sigma^z \otimes \sigma^z + \frac{3}{4} \mathbb{I}$$

$$h_B(y) = \left( \frac{1 + 5\rho^2 + \rho^4}{4(1 - \rho^2 + \rho^4)} \right) \mathbb{I} + \sum_{j=1}^3 \lambda_j \sigma^j,$$

Also possible to have **different** BCs ( $y$ 's related by roots of unity) again at  $\Delta = 1/2$



# Story so Far

XXZ (and XYZ) spin chain has a “dynamical” supersymmetry

Relates chains of different length

Can be formulated for both open and closed chains

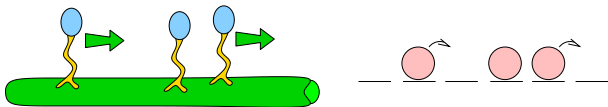
Can be formulated for diagonal, off-diagonal, unequal BCs

# ASEPs

Originally came from biology

Model for transport in cells

Kinesins moving along a microtubule



# What's this got to do with $XXZ$ ?

Markov matrix  $M$

$$\frac{d}{dt}|P(t)\rangle = M|P(t)\rangle,$$

Related to  $H_{XXZ}$

$$M = \sqrt{pq}U_{\mu}^{-1}H_{XXZ}U_{\mu}$$

Where

$$U_{\mu} = \bigotimes_{i=1}^L \begin{pmatrix} 1 & 0 \\ 0 & \mu Q^{j-1} \end{pmatrix},$$
$$H_{XXZ} = -\frac{1}{2} \sum_{j=1}^{L-1} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \Delta \sigma_j^z \sigma_{j+1}^z \right. \\ \left. + h(\sigma_{j+1}^z - \sigma_j^z) + \Delta \right] + B_1 + B_L$$

# What's this got to do with $XXZ$ ??

Various  $XXZ$  parameters related to (P)ASEP

$$B_1 = \frac{1}{2\sqrt{\rho q}} \left( \alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha\mu\sigma_1^- - 2\gamma\mu^{-1}\sigma_1^+ \right)$$

$$B_L = \frac{1}{2\sqrt{\rho q}} \left( \beta + \delta - (\beta - \delta)\sigma_L^z - 2\delta\mu Q^{L-1}\sigma_L^- - 2\beta\mu^{-1}Q^{-L+1}\sigma_L^+ \right)$$

$$\Delta = -\frac{1}{2}(Q + Q^{-1}), \quad h = \frac{1}{2}(Q - Q^{-1}), \quad Q = \sqrt{\frac{q}{p}}.$$

# And SUSY??

Well.....“transfer matrix” symmetry (Ayyer, Mallick)

For ASEP and ASAP

$$T_{L,L+1} M_L = M_{L+1} T_{L,L+1}$$

Relates  $M$ s of different length

$$M_L = \sum_{i=1}^{L-1} M_{i,i+1} + R + L$$

Where, for ASAP

10  $\rightarrow$  01 with rate 1

11  $\rightarrow$  00 with rate  $\lambda$

# So??

Is the transfer matrix symmetry related to SUSY?

Related spin chains have unequal, off-diagonal BCs

**NOT** at combinatorial point  $\lllll$

# References

Xiao Yang and Paul Fendley, Non-local spacetime supersymmetry on the lattice, *J. Phys. A*, 37(38):8937–8948, 2004.

Christian Hagendorf and Jean Liénardy, Open spin chains with dynamic lattice supersymmetry, *J. Phys. A*, 50(18):185202, 32, 2017.

Robert Weston and Junye Yang, Lattice Supersymmetry in the Open XXZ Model: An Algebraic Bethe Ansatz Analysis, [arXiv:1709.00442]