Lattice SUSY and the DiSSEP

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Acronyms

SUSY: supersymmetry

QM: Quantum Mechanics

TASEP: Totally Asymmetric Simple Exclusion Process

PASEP: Partially Asymmetric Simple Exclusion Process

(Di)SSEP: (Dissipative) Symmetric Simple Exclusion Process

ASAP: Asymmetric Annihilation Process

Plan

- QM SUSY
- Spin Chain SUSY
- DiSSEP SUSY, what's different
- Things undone

SUSY QM

Hamiltonian

$$H\psi_n(x) = \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2\right)\psi_n(x) = E_n\psi_n(x)$$

Define

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x); \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$

Two SUSY Hamiltonians

$$H^{(1)} = A^\dagger A = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x)$$

$$H^{(2)} = AA^\dagger = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x)$$

SUSY Algebra

H Hamiltonian, Q supercharge

$$
\begin{aligned}
[H, Q] &= [H, Q^\dagger] = 0 \\
\{Q, Q^\dagger\} &= H \\
\{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0
\end{aligned}
$$

$$
\begin{aligned}
H &= \begin{pmatrix} H^{(1)} & 0 \\ 0 & H^{(2)} \end{pmatrix} \\
Q &= \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \\
Q^\dagger &= \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

XXZ Spin Chain

XXZ Hamiltonian acts on $V^{\otimes N}$ where $V \simeq \mathbb{C}^2$.

$$\begin{aligned} H^{(N)} &= -\frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}\sigma_i^z\sigma_{i+1}^z\right) \\ &\quad -\frac{1}{4}(\sigma_1^z + \sigma_N^z) \end{aligned}$$

Write as sum of local terms, $h_{ij} : V \otimes V \to V \otimes V$ in bulk

$$H^{(N)} = \sum_{\langle ij \rangle} h_{ij} + h_1 + h_N$$

Spin Chain Supercharge

Supercharge: Length-changing operator

$$H^{(N-1)}Q^{(N)} = Q^{(N)}H^{(N)}, \quad H^{(N)}Q^{\dagger(N)} = Q^{\dagger(N)}H^{(N-1)}.$$

Local expressions (Hagendorf)

$$Q^{(N)} = \sum_{i=1}^{N-1}(-1)^{i+1}q_{i,i+1}, \quad Q^{\dagger(N)} = \sum_{i}^{N-1}(-1)^{i+1}q_i^{\dagger}$$

Local action - "dynamical SUSY"

$$q_i^{\dagger} : V \to V \otimes V; \quad q_{i,i+1} : V \otimes V \to V$$

Local Supercharge

The game of “construct a Hamiltonian with dynamical SUSY”?

1) Pick a $q^\dagger$ - (4×2)

2) In terms of the local supercharge, nilpotency $(Q^\dagger)^2 = 0$ becomes

$$(q^\dagger \otimes 1 - 1 \otimes q^\dagger) q^\dagger = 0$$

3) Construct h - (4×4) using:

$$\begin{aligned} h &= -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q \\ &\quad + \frac{1}{2}\left(q\, q^\dagger \otimes 1 + 1 \otimes q\, q^\dagger\right) \end{aligned}$$

Local Supercharge - XXZ

Pick a $q^\dagger$, construct h

$$q^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad h_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using (previous slide)

$$\begin{aligned} h &= -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q \\ &\quad + \frac{1}{2}\left(q\, q^\dagger \otimes 1 + 1 \otimes q\, q^\dagger\right) \end{aligned}$$

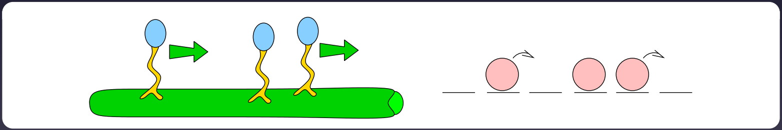
Local SUSY exists for *XXZ* (and XYZ, and...) for particular parameter values

ASEPs/PASEPS/DiSSEPs etc etc

Originally came from biology

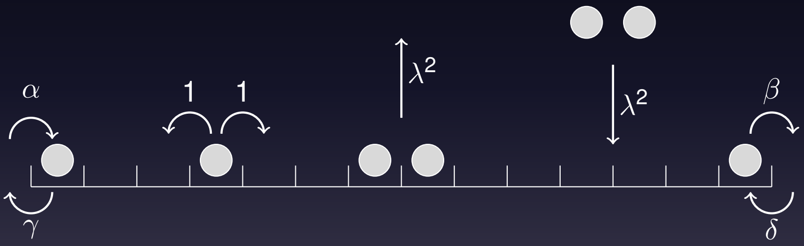
Model for transport in cells

Kinesins moving along a microtubule



DiSSEP

Rates for the DiSSEP



DiSSEP

Markov matrix M - bulk m, boundaries B

$$M(\lambda^2) = B_1 + \sum_{k=1}^{L-1} m_{k,k+1} + \overline{B}_L$$

$$B_1 = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \quad , \qquad m = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix}$$

Column sum zero - NEW feature of Markov matrix

DiSSEP, *XXZ* in disguise

Hamiltonian $\langle = \rangle$ Markov Matrix

$$H_{XXZ}(\lambda^2) = -U \otimes U \cdots \otimes U \, M(\lambda^2) \, U^{-1} \otimes U^{-1} \cdots \otimes U^{-1}$$

Conjugation matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Hamiltonian

$$\begin{aligned} H_{XXZ}(\lambda^2) &= -(\alpha - \gamma)\sigma_1^+ + \frac{\alpha + \gamma}{2}(\sigma_1^z + \mathbb{I}) - (\delta - \beta)\sigma_L^+ + \frac{\delta + \beta}{2}(\sigma_L^z + \mathbb{I}) \\ &\quad + \frac{\lambda^2 - 1}{2} \sum_{k=1}^{L-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y - \frac{\lambda^2 + 1}{\lambda^2 - 1} (\sigma_k^z \sigma_{k+1}^z - \mathbb{I}) \right) \end{aligned}$$

Local Supercharge - DiSSEP

Pick a $q^\dagger$, construct h

$$q^\dagger = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad h_{ij} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Using (previous slide)

$$\begin{aligned} h &= -(q \otimes 1)(1 \otimes q^\dagger) - (1 \otimes q)(q^\dagger \otimes 1) + q^\dagger q \\ &\quad + \frac{1}{2}\left(q\, q^\dagger \otimes 1 + 1 \otimes q\, q^\dagger\right) \end{aligned}$$

Local SUSY exists for DISSEP for particular (boring) parameter values: $\alpha = \beta = \gamma = \delta = \lambda^2 = 1$

Observations

SUSY exists in a $1D$ non-equilibrium model

BUT at parameter values that correspond to zero drive

HOWEVER... Spectrum is known for general $\alpha, \beta \ldots$

Still shows SUSY features - zero energy singlet, energies pair off with “superpartners”

SUSY? Spectrum

$\alpha = \beta = \gamma = \delta = 1$

$$M^2 = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}$$

Generic

$$\begin{pmatrix} -1-\alpha-\delta & \beta & \gamma & 1 \\ \delta & -1-\alpha-\beta & 1 & \gamma \\ \alpha & 1 & -1-\gamma-\delta & \beta \\ 1 & \alpha & \delta & -1-\beta-\gamma \end{pmatrix}$$

$$\Lambda(\epsilon) = \frac{\alpha+\gamma}{2}(\epsilon_0\epsilon_1 - 1) + \sum_{j=1}^{L-1}(\epsilon_j\epsilon_{j+1} - 1) + \frac{\beta+\delta}{2}(\epsilon_L\epsilon_{L+1} - 1)$$

SUSY? Other *SEPs*, *SAPs

"Transfer matrix" symmetry (Ayyer, Mallick)

For ASEP and ASAP, length changing symmetry discovered by combinatorial means

$$T_{L,L+1} M_L = M_{L+1} T_{L,L+1}$$

Relates Ms of different length

$$M_L = \sum_{i=1}^{L-1} M_{i,i+1} + R + L$$

References

Xiao Yang and Paul Fendley, Non-local spacetime supersymmetry on the lattice, *J. Phys. A*, 37(38):8937–8948, 2004.

Christian Hagendorf and Jean Liénardy, Open spin chains with dynamic lattice supersymmetry, *J. Phys. A*, 50(18):185202, 32, 2017.

Robert Weston and Junye Yang, Lattice Supersymmetry in the Open XXZ Model: An Algebraic Bethe Ansatz Analysis, *J. Stat. Mech.* 123104, 2017.

Me, Lattice SUSY for the DiSSEP at $\lambda^2 = 1$ (and $\lambda^2 = 3$), *J. Phys. Commun. 3*, 105011, 2019.