Macroscopic Degeneracy and Scaling at a First Order Transition

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Plan of talk

A family of $3D$ Ising models motivated by random surface simulations

A problem with the 1st order FSS for a plaquette $3D$ Ising model in this family

A solution

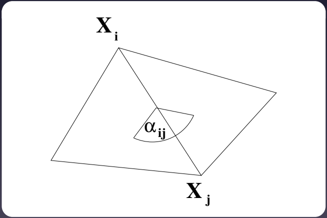
Quantum version of the model - fractons

The Gonihedric action

- Savvidy "Gonihedric" surface action

$$
\begin{aligned}
H &= \sum_{ij} |X^{\mu}(i) - X^{\mu}(j)| \theta_{ij} \\
\theta_{ij} &= ||\pi - \alpha_{ij}||
\end{aligned}
$$

Gonia: angle
Hedra: face

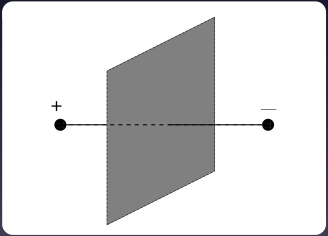


Spins Cluster Boundaries as Surface Models

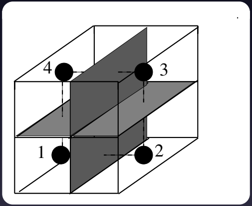
Spin cluster boundaries $\leftrightarrow$ surfaces

Edge spins: $U_{ij} = -1$

Vertex spins: $\sigma_i \sigma_j = -1$



Counting configurations with spins (areas and intersections)



Ising/Surface correspondence

Allow energy from areas, edges and intersections (*A. Cappi, P Colangelo, G. Gonella and A. Maritan*)

$$H = \sum(\beta_A n_A + \beta_C n_C + \beta_I n_I)$$

$$\beta_A = 2J_1 + 8J_2, \quad \beta_C = 2J_3 - 2J_2, \quad \beta_I = -4J_2 - 4J_3$$

$$H = J_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \sigma_i \sigma_j + J_3 \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

Gonihedric $\rightarrow$ Tune Out Area Term

One parameter family of "Gonihedric" Ising models (Savvidy, Wegner)

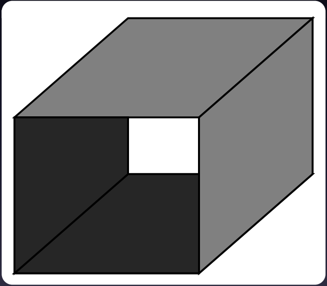
$$\mathcal{H}^{\kappa} = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

$\kappa = 0$ pure plaquette

$$\mathcal{H} \;=\; -\sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

The Dual ($\kappa = 0$)

Constraint on dual spins at centre of plaquettes:



Double up spins, anisotropic interactions

The Dual ($\kappa = 0$)

An anisotropically coupled Ashkin-Teller model

$$\mathcal{H}_{dual} = -\sum_{\langle ij \rangle_x} \sigma_i \sigma_j - \sum_{\langle ij \rangle_y} \tau_i \tau_j - \sum_{\langle ij \rangle_z} \sigma_i \sigma_j \tau_i \tau_j \,,$$

Standard duality relation

$$\tanh \beta = e^{-2\beta^*}$$

Known: First Order Transition

Strong first order transition

$$\mathcal{H} = -\sum_{\square} \sigma_i \sigma_j \sigma_k \sigma_l$$

Only inaccurate (Metropolis, yours truly, bad ...) determination of transition point

Same for dual model - (Metropolis, yours truly, bad ...) determination of transition point

Exercise for a starting PhD student (Marco Mueller)

Simulate $3d$ plaquette model and dual, using multicanonical methods

Do a better job than, ahem, before at determining transition point of both

Obtain consistent estimates of transition point

“Standard” First Order Transitions

The q-state Potts model

Hamiltonian

$$\mathcal{H}_q = -\sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j}$$

Evaluate the partition function, derivatives give observables

$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta \mathcal{H}_q)$$

1st Order FSS: Heuristic two-phase model

A fraction W_{o} in q ordered phase(s), energy e_{o}

A fraction $W_{\mathrm{d}} = 1 - W_{\mathrm{o}}$ in disordered phase, energy e_{d}

Ignore transits

1st Order FSS: Energy moments

Energy moments become

$$\langle e^n \rangle = W_\mathrm{o} e_\mathrm{o}^n + (1 - W_\mathrm{o}) e_\mathrm{d}^n$$

And the specific heat then reads:

$$C_V(\beta, L) = L^d \beta^2 \left(\left\langle e^2 \right\rangle - \langle e \rangle^2 \right) = L^d \beta^2 W_\mathrm{o}(1 - W_\mathrm{o}) \Delta e^2$$

Max of $C_V^{\mathrm{max}} = L^d \, (\beta^\infty \Delta e/2)^2$ at $W_\mathrm{o} = W_\mathrm{d} = 0.5$

Volume scaling

1st Order FSS: Specific Heat peak shift

Probability of being in any of the states

$$W_o \sim q \exp(-\beta L^d f_o)\,, \; W_d \sim \exp(-\beta L^d f_d)$$

Take logs, expand around β^∞

$$\begin{aligned} \ln(W_o/W_d) &= \ln q + \beta L^d (f_d - f_o) \\ &= \ln q + L^d \Delta e(\beta - \beta^\infty) \end{aligned}$$

Solve for specific heat peak $W_o = W_d$, $\ln(W_o/W_d) = 0$

$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta e} + \dots$$

1st Order FSS: summary

Peaks grow as L^d

Critical points shift as $1/L^d$

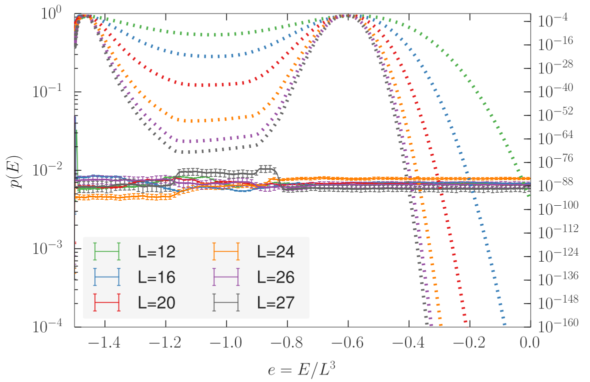
Except …

Fixed boundaries ($1/L$ leading term)

$$Z(\beta) = \left[e^{-\beta(L^d f_d + L^{d-1} \tilde{f}_o)} + q e^{-\beta(L^d f_o + L^{d-1} \tilde{f}_d)} \right] [1 + \ldots]$$

A Problem

Plaquette Model: Careful Multicanonical Simulations



Multicanonical histograms

Scaling of the Plaquette Model and Dual

Determine critical point(s) $L = 8 \ldots 27$, periodic bc, $1/L^3$ fits - the nice exercise for a PhD student (Marco)

Original model:

$$\beta^{\infty} = 0.549994(30)$$

Dual model:

$$\beta^{\infty}_{dual} = 1.31029(19)$$

$$\beta^{\infty} = 0.55317(11)$$

Estimates are about 30 error bars apart

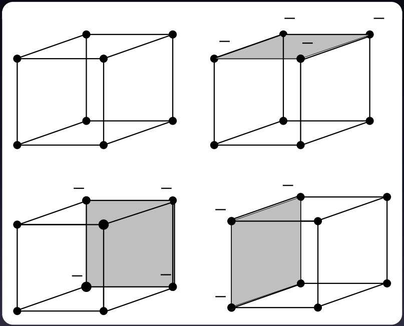
Potential Solutions

Blame the student (yours truly, bad....)

Incorrect, try again (good guys)

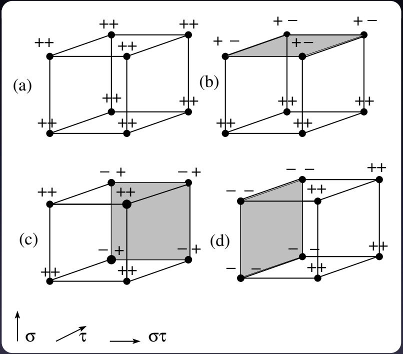
What is special about plaquette model?

Groundstates: Plaquette



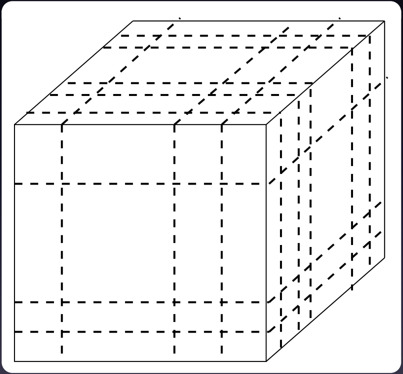
Persists into low temperature phase: degeneracy 2^{3L}

Groundstates: Dual



Dual degeneracy

Ground state



1st Order FSS with Exponential Degeneracy

Normally q is constant

Suppose instead $q \propto e^L$

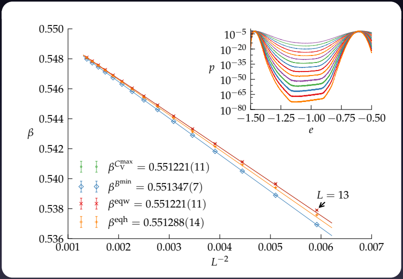
$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta e} + \dots$$

becomes

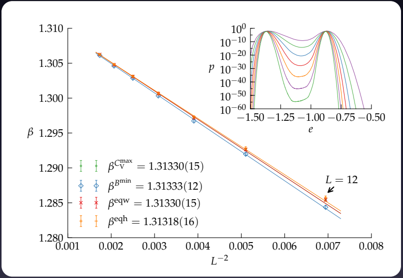
$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{1}{L^{d-1} \Delta e} + \dots$$

FSS

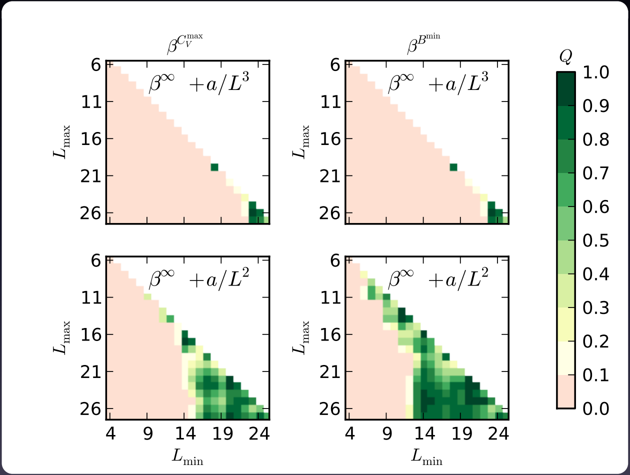
Plaquette Hamiltonian fits



Dual Hamiltonian fits



Quality of fits



Forcing a fit to $1/L^3$ gives much poorer quality

Conclusions

Standard 1st order FSS: $1/L^3$ corrections in 3D

Fixed BC: $1/L$ (surface tension)

Exponential degeneracy: $1/L^2$ in 3D

A Quantum Postscript

Quantum Plaquette Hamiltonian

We can write down a quantum version

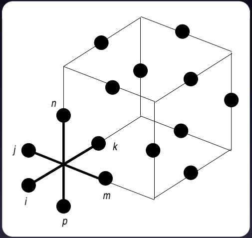
$$H_0 = -t \sum_{[i,j,k,l]} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z - h \sum_i \sigma_i^x \ .$$

....and its dual

$$H_{\mathrm{nexus}} = -t \sum_i \tau_i^z - h \sum_i A_i \ ,$$

Dual Hamiltonian

$$A_i \equiv \prod_{j \in P(i)} \tau_j^x$$



$$\begin{aligned} B_i^{(xz)} &= \tau_i^z \tau_p^z \tau_n^z \tau_k^z \ , \\ B_i^{(xy)} &= \tau_i^z \tau_j^z \tau_k^z \tau_m^z \ , \\ B_i^{(yz)} &= \tau_j^z \tau_n^z \tau_m^z \tau_p^z \end{aligned}$$

The X-Cube fracton Hamiltonian

Construct a “fracton” Hamiltonian

$$H_{\text{fracton}} = -\sum_i B_i - \sum_i A_i$$

c.f. Ising gauge $\rightarrow$ Toric Code

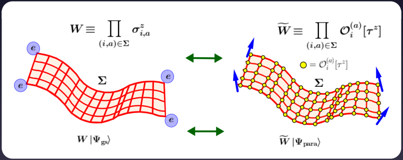
Planar flip symmetry has conseqences for behaviour of excitations

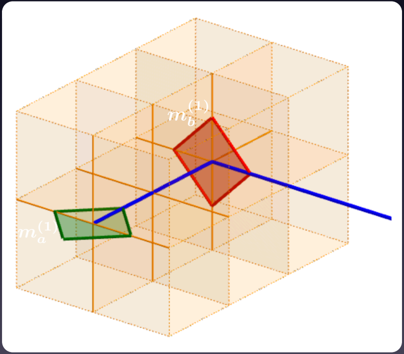
Fractons

Excitations have restricted mobility (Pics plagiarized from arXiv:1603.04442, Vijay, Haah and Fu)

Linear motion (magnetic) excitations

Pinned (electric) excitations





References

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