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CQT
Computational Quantum Field Theory

Planar (“fuki-nuke”) ordering and finite-size effects for a model with four-spin interactions

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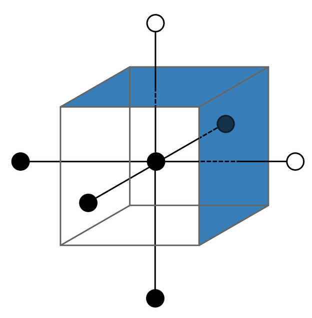
Outline

We analyse a model with local four spin interaction of Ising spins, which appears as a special, plaquette-only case of the so-called gonihedric Ising model, a discrete variant when describing interacting surfaces (1). In three dimensions, it shows a strong first-order phase transition between phases with exponentially degenerate ground states and this degeneracy gives rise to a nonstandard finite-size scaling of the transition temperature. Our multicanonical simulations that confirmed this unusual finite-size scaling in the first place also provide a way of measuring planar order parameters. These come from considering an exactly solvable anisotropic limit of the model and can distinguish between the low- and high-temperature phases in both the anisotropic and isotropic cases. In two dimensions, the model may serve as a pedagogical example on calculating how different finite-size corrections appear from different boundary conditions.

The Gonihedric Ising model

- Spins $\sigma \in \{-1,+1\}$ on each vertex of a 3D cubic lattice, linear lattice size L
- Plaquettes in the dual lattice separate contiguous spins with opposite signs
- Bulk of plaquettes defines a surface



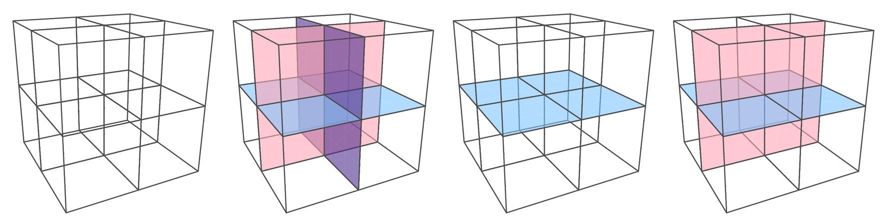
- Hamiltonian:

$$\mathcal{H} = -2\kappa \sum_{\langle i,j \rangle} \sigma_i\sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i\sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i\sigma_j\sigma_k\sigma_l$$

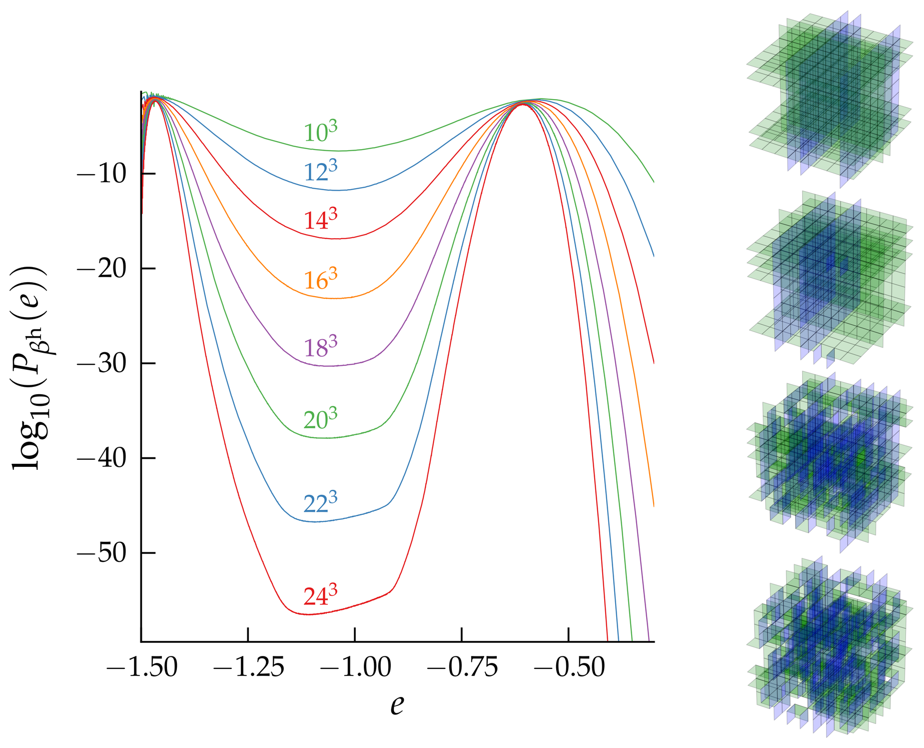
  - Only *linear size* instead of *surface area* contributes to the partition function (2)
  - κ. . . surface self-avoidance control parameter, complete self-avoidance for $\kappa \to \infty$
- Special Case $\kappa = 0$:

$$\mathcal{H} = -\frac{1}{2}\sum_{\Box} \sigma\sigma\sigma\sigma$$

  - Zero-temperature, elementary ground state composed of:



  - Flip of whole planes parallel to either one of the xy, yz, zx-planes allowed
  - Bulk ground state degeneracy $q = 2^{3L}$, with L being the linear lattice size
  - Strong first order phase transition: energy probability density at β^{eqh}:



  - reduced interface tension $\hat{\sigma} = \beta\sigma$ can be extracted from several lattice sizes

$$\hat{\sigma}(L) = \frac{1}{2L^2} \ln\left(\frac{\max[P_{\beta^{\mathrm{eqh}}}(L)]}{\min[P_{\beta^{\mathrm{eqh}}}(L)]}\right); \ \hat{\sigma}(L \to \infty) = 0.1183(6)$$

Degeneracy and Finite-Size Scaling

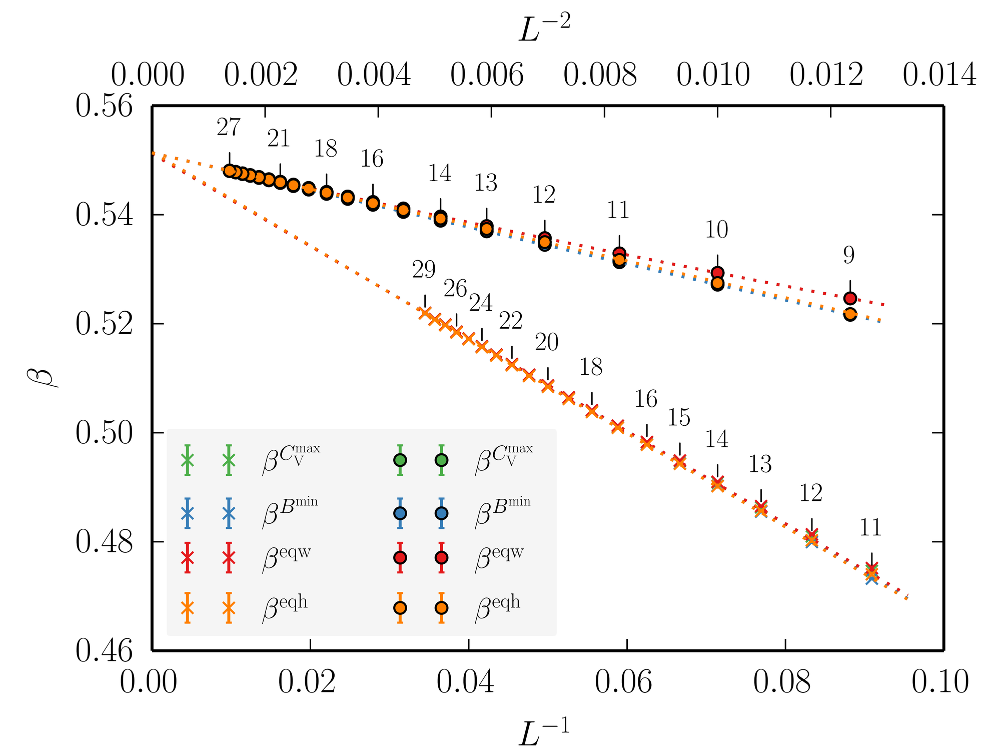
- Nonstandard for periodic boundary conditions (3, 4): respect ground-state degeneracy (5)

$$\beta_0(L) - \beta_0^\infty \propto q(L)L^{-3} \propto L^{-2}$$

- Standard scaling for fixed boundary conditions:

$$\beta_0(L) - \beta_0^\infty \propto L^{-1}$$

- Comparison between periodic and fixed boundary conditions (6):



- Great consistency in all amplitudes and with a dual model (7)
- Standard magnetization $< m >= 0$ analytically, how to find an order parameter?

Planar order parameters

- anisotropic variant:
  - Hamiltonian:

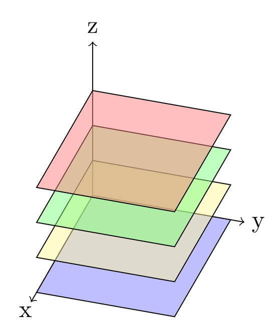
$$\mathcal{H} = -\frac{1}{2}\left(\sum_{\Box_{xz}} \sigma\sigma\sigma\sigma + \sum_{\Box_{yz}} \sigma\sigma\sigma\sigma\right)$$

  - anisotropic variant with free boundaries and no interaction in z-direction, dubbed fuki-nuke model (“without roof”) is exactly solvable (8) by a transformation $\tau = \sigma\sigma$
  - layers of independent $2d$ Ising models of τ-variables emerge, individual order parameters in planes:

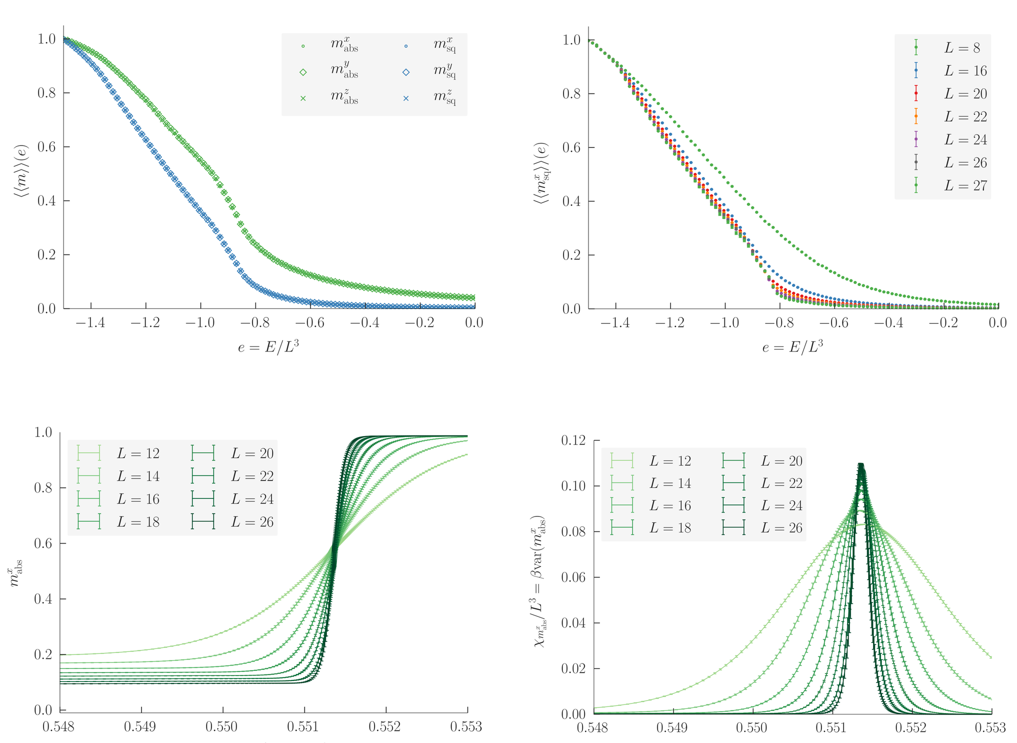
$$m_{2d,z} = \left\langle \frac{1}{L^2}\sum_x\sum_y \tau \right\rangle = \left\langle \frac{1}{L^2}\sum_x\sum_y \sigma\sigma \right\rangle$$

  - Average over layers

$$m_{3d} = \left\langle \frac{1}{L^3}\sum_z \left| \sum_x\sum_y \sigma\sigma \right| \right\rangle$$



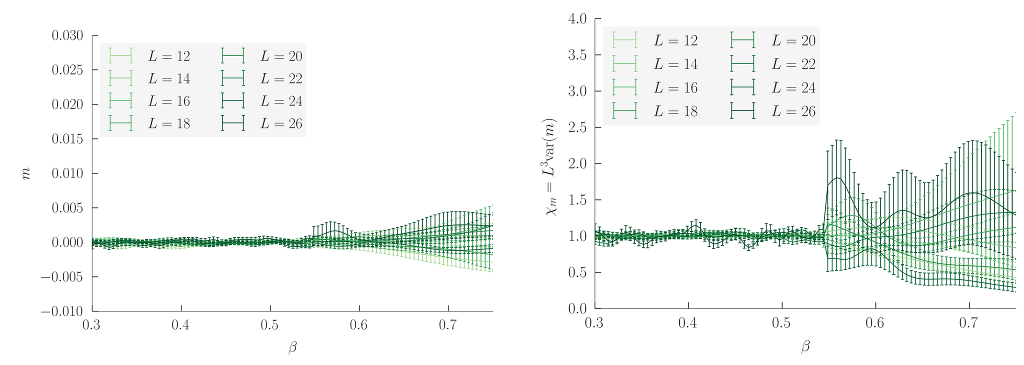
- Isotropic model



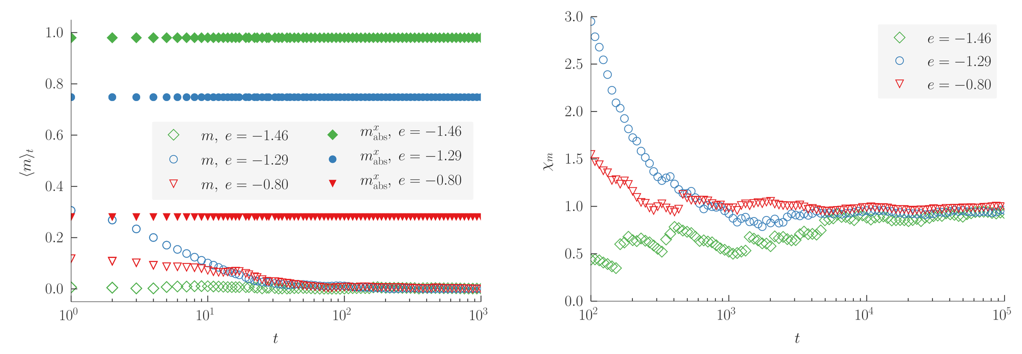
→ Fuki-nuke order parameters also applicable to isotropic model

Nature of the planar symmetry

- Magnetization and susceptibility show “signal” near phase transition

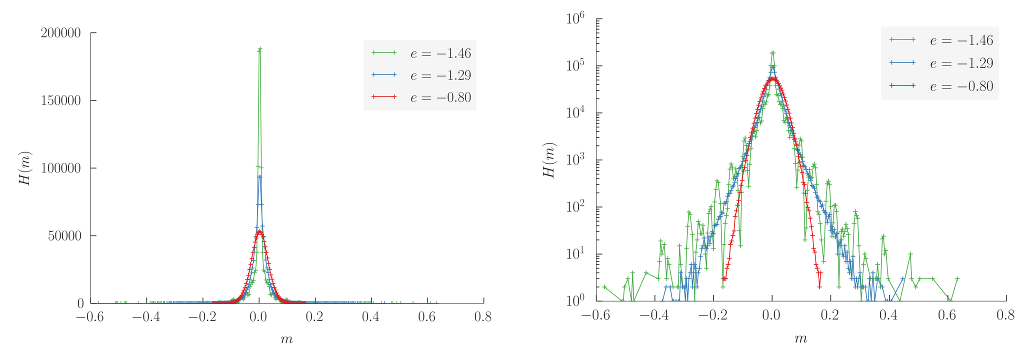


- Behaviour is even worse w/o multicanonical algorithm
- Fix system in one state and flipping planes:



→ Failure of Metropolis and multicanonical algorithms of exploring the exponentially degenerate ground-state

- $\mathbb{Z}_2$-symmetry of the spins vs. planar symmetry



Two-dimensional model

- Free boundary conditions - relation to $1d$-Ising chain

$$Z = 2^{L_x}(Z_{1d,\,\mathrm{Ising}})^{L_y-1} = 2^{L_xL_y}\cosh(\beta)^{(L_x-1)(L_y-1)}.$$

- y-free, x-periodic boundary conditions

$$Z = 2^{L_xL_y}\cosh(\beta)^{L_x(L_y-1)}\left(1+\tanh(\beta)^{L_x}\right)^{L_y-1}$$

- Periodic boundary conditions

$$Z = (2\cosh(\beta))^{L_xL_y}\frac{1}{2}\sum_{v=0}^{L_x}\sum_{h=0}^{L_y}\binom{L_x}{v}\binom{L_y}{h}\tanh(\beta)^{vL_y+hL_x-2vh}$$

Conclusions

The isotropic gonihedric Ising model shows the same planar, fuki-nuke order seen in the strongly anisotropic limit. The finite-size scaling of the fuki-nuke order parameters is in good agreement with the scaling of energetic quantities. Standard magnetization and susceptibility is difficult to sample for the multicanonical algorithm (and impossible for Metropolis).
Albeit the fuki-nuke model is exactly solvable for finite lattices under free boundary conditions almost trivially, we argue that the solution under periodic boundaries may very well be much more complicated, judging from the expressions in the two-dimensional case.

References

(1) G. K. Savvidy and F. J. Wegner, Nucl. Phys. B **413**, 605 (1994).

(2) G. K. Savvidy and K. G. Savvidy, Int. J. Mod. Phys. A **8**, 3993 (1993).

(3) C. Borgs and R. Kotecký, J. Stat. Phys. **61**, 79 (1990).

(4) W. Janke, Phys. Rev. B **47**, 14757 (1993).

(5) M. Mueller, W. Janke and D.A. Johnston, Phys. Rev. Lett. **112** 200601 (2014).

(6) W. Janke, M. Mueller, and D. A. Johnston, J. Phys.: Conf. Ser. **640**, 012002 (2015).

(7) M. Mueller, D. A. Johnston and W. Janke 2014, Nucl. Phys. B **888** 214 (2014).

(8) M. Suzuki, Phys. Rev. Lett. **28** 507 (1972); Y. Hashizume and M. Suzuki, Int. J. Mod. Phys. B **25** 73 (2011); Y. Hashizume and M. Suzuki, Int. J. Mod. Phys. B **25** 3529 (2011).

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