

Exact solutions to plaquette Ising models with free and periodic boundaries

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Motivation

- ▶ your *first Monte Carlo simulation* of spin-lattices was (is) erroneous almost surely
- ▶ compare to enumeration, exact solutions for *finite lattices*

Exact solutions:

- ▶ 1d Ising model: {free, fixed, (anti)periodic}-boundary conditions
- ▶ 2d Ising model: {(anti)periodic, Brascamp-Kunz, . . . }-boundary conditions, *no solution* for free boundaries

Spin-Bond transformation: solving the 1d Ising chain

- for free boundary conditions:

$$H = - \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}, \quad \sigma_i \in \{+1, -1\},$$

$$Z_{1d, \text{free}} = \sum_{\{\sigma\}} \exp \left(\beta \sum_{i=1}^{L-1} \sigma_i \sigma_{i+1} \right)$$

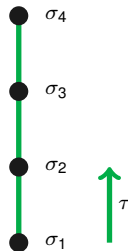
- spin-bond transformation:

$$\{\sigma_1, \sigma_2, \dots, \sigma_L\} \rightarrow \{\tau_1, \tau_2, \dots, \tau_L\}$$

where $\tau_1 = \sigma_1 \sigma_2$, $\tau_2 = \sigma_2 \sigma_3$, $\dots$, $\tau_{L-1} = \sigma_{L-1} \sigma_L$ and setting $\tau_L = \sigma_L$, the mapping $\{\sigma\} \rightarrow \{\tau\}$ with an inverse relation of the form $\sigma_i = \tau_L \tau_{L-1} \tau_{L-2} \dots \tau_i$ is *one-to-one*

- partition function factorises:

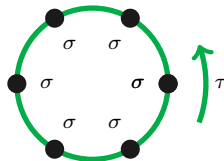
$$Z_{1d, \text{free}} = \sum_{\{\tau\}} \exp \left(\beta \sum_{i=1}^{L-1} \tau_i \right) = 2 \prod_{i=1}^{L-1} \sum_{\tau_i = \pm 1} \exp(\beta \tau_i) = 2^L \text{ch}(\beta)^{L-1}$$



Spin-Bond transformation: solving the 1d Ising chain (again)

- for periodic boundary conditions:

$$H = - \sum_{i=1}^L \sigma_i \sigma_{i+1}, \quad \sigma_i \in \{+1, -1\}$$



- spin-bond transformation:

$$\{\sigma_1, \sigma_2, \dots, \sigma_L\} \rightarrow \{\tau_1, \tau_2, \dots, \tau_L\}$$

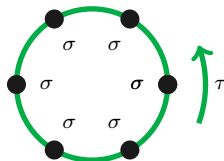
$\tau_1 = \sigma_1 \sigma_2, \tau_2 = \sigma_2 \sigma_3, \dots, \tau_L = \sigma_L \sigma_{L+1} = \sigma_L \sigma_1$, with an inverse relation of the form $\sigma_i = \sigma_1 \times \tau_1 \tau_2 \tau_3 \dots \tau_{i-1}$, mapping is *two-to-one* and we have the *constraint*

$$\prod_{i=1}^L \tau_i = \prod_{i=1}^L \sigma_i^2 = 1$$

Spin-Bond transformation: solving the 1d Ising chain (again), cont'd

- partition function:

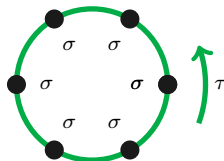
$$\begin{aligned} Z_{1d, \text{periodic}} &= \sum_{\{\sigma\}} \exp \left(\beta \sum_{i=1}^L \sigma_i \sigma_{i+1} \right) \\ &= 2 \sum_{\{\tau\}} \exp \left(\beta \sum_{i=1}^L \tau_i \right) \delta \left(\prod_{i=1}^L \tau_i, 1 \right) \end{aligned}$$



Spin-Bond transformation: solving the 1d Ising chain (again), cont'd

- partition function:

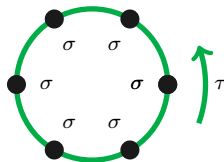
$$\begin{aligned} Z_{1d, \text{periodic}} &= \sum_{\{\sigma\}} \exp \left(\beta \sum_{i=1}^L \sigma_i \sigma_{i+1} \right) \\ &= 2 \sum_{\{\tau\}} \exp \left(\beta \sum_{i=1}^L \tau_i \right) \delta \left(\prod_{i=1}^L \tau_i, 1 \right) \\ &= \sum_{\{\tau\}} \exp \left(\beta \sum_{i=1}^L \tau_i \right) \left(1 + \prod_{i=1}^L \tau_i \right) \end{aligned}$$



Spin-Bond transformation: solving the 1d Ising chain (again), cont'd

- partition function:

$$\begin{aligned} Z_{1d, \text{periodic}} &= \sum_{\{\sigma\}} \exp \left(\beta \sum_{i=1}^L \sigma_i \sigma_{i+1} \right) \\ &= \sum_{\{\tau\}} \exp \left(\beta \sum_{i=1}^L \tau_i \right) \delta \left(\prod_{i=1}^L \tau_i, 1 \right) \\ &= \sum_{\{\tau\}} \exp \left(\beta \sum_{i=1}^L \tau_i \right) \left(1 + \prod_{i=1}^L \tau_i \right) \\ &= \left[\prod_{i=1}^L \sum_{\tau_i=\pm 1} \exp(\beta \tau_i) + \prod_{i=1}^L \sum_{\tau_i=\pm 1} \tau_i \exp(\beta \tau_i) \right] \\ &= 2^L \text{ch}(\beta)^L \left[1 + \text{th}(\beta)^L \right]. \end{aligned}$$

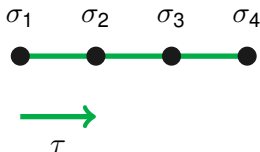


Spin-Bond transformation, highlights

solving the 1d Ising chain

free

periodic

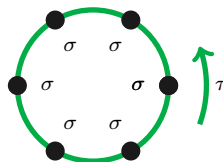


last spin, σ_L
cause of "2"

remains untransformed
summing over σ_L

Z

$$2^L \text{ch}(\beta)^{L-1}$$

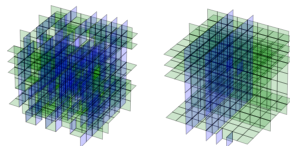


is transformed
two-to-one transformation
additional constraint

$$2^L \text{ch}(\beta)^L [1 + \text{th}(\beta)^L]$$

Plaquette model (in 3d)

$$\mathcal{H} = - \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$



- particular limit of 3d model of the gonihedric string

$$\mathcal{H} = -2\kappa \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

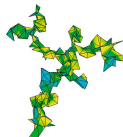


Fig. 7.1. A typical rougher triangulated surface resulting from a simulation of the Gaussian curvature in $\mathbb{R}^3$.

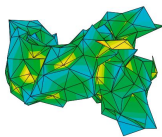
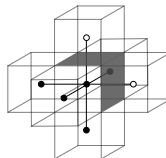


Fig. 7.2. A typical smoother triangulated surface resulting from a simulation of the Gaussian plus curvature-curvature interaction in $\mathbb{R}^3$ with $\lambda = 1.1$.



D. A. Johnston, A. Lipowski, and R. P. K. C. Malmini, in *Rugged Free Energy Landscapes*, Vol. 736 of *Lecture Notes in Physics*, Berlin Springer Verlag, edited by W. Janke (2008), pp. 173–199.

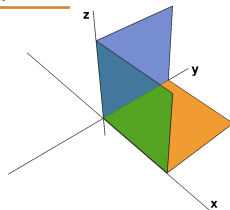
Anisotropic plaquette model

$$H_{\text{aniso}}(\{\sigma\}) = \underbrace{-J_x \sum_{\square_{yz}} \sigma\sigma\sigma\sigma}_{\text{blue}} \underbrace{-J_y \sum_{\square_{zx}} \sigma\sigma\sigma\sigma}_{\text{green}} \underbrace{-J_z \sum_{\square_{xy}} \sigma\sigma\sigma\sigma}_{\text{orange}}$$

$$H_{\text{aniso}}^{J_x=J_y=0}(\{\sigma\}) = -J_z \sum_{z=1}^{L_z} \left[\sum_{2d \square} \sigma\sigma\sigma\sigma \right]$$

$$Z_{\text{aniso}}^{J_x=J_y=0} = \sum_{\{\sigma\}} \exp \left(-\beta H_{\text{aniso}}^{J_x=J_y=0}(\{\sigma\}) \right)$$

$$= (Z_{2d, \text{gonihedric}})^{L_z}$$



Two dimensional plaquette model: free boundaries in y -direction

- ▶ Spin-bond-transformation in y -direction, $\tau_{x,y} = \sigma_{x,y}\sigma_{x,y+1}$, with the condition $\tau_{x,L_y} = \sigma_{x,L_y}$
- ▶ partition function factorises:

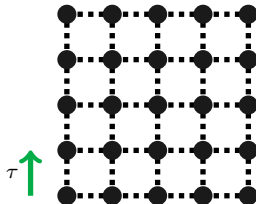
Z_{2d} , gonihedric, free

$$= \sum_{\{\sigma\}} \exp \left(\beta \sum_{x=1}^{L_x-1} \sum_{y=1}^{L_y-1} \sigma_{x,y} \sigma_{x,y+1} \sigma_{x+1,y} \sigma_{x+1,y+1} \right)$$

$$= \sum_{\{\tau\}} \exp \left(\beta \sum_{x=1}^{L_x-1} \sum_{y=1}^{L_y-1} \tau_{x,y} \tau_{x+1,y} \right)$$

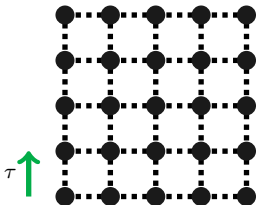
$$= 2^{L_x} (Z_{1d, \text{Ising}})^{L_y-1}$$

- ▶ the factor 2^{L_x} comes from the L_x sums over $\tau_{x,L_y} = \sigma_{x,L_x} = \pm 1$ which do not appear in the exponent

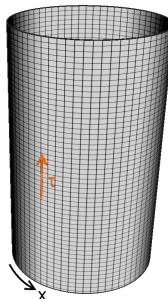


Two dimensional plaquette model: mixed boundary conditions

$$2^{L_x L_y} \text{ch}(\beta)^{(L_x-1)(L_y-1)}$$



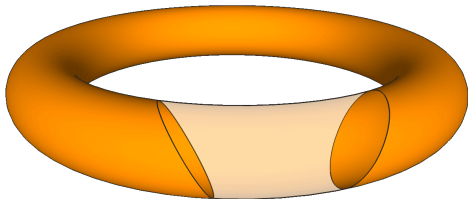
$$2^{L_x L_y} \text{ch}(\beta)^{L_x(L_y-1)} \left(1 + \text{th}(\beta)^{L_x}\right)^{L_y-1}$$



Two dimensional plaquette model: periodic boundaries

- ▶ Consider periodic boundary conditions in y -direction: $\sigma_{x,L_y+1} = \sigma_{x,1}$, here also in x -direction $\sigma_{L_x+1,y} = \sigma_{1,y}$
- ▶ Spin-bond-transformation in y -direction, $\tau_{x,y} = \sigma_{x,y}\sigma_{x,y+1}$ is *two-to-one* and imposes L_x constraints $\prod_y \tau_{x,y} = 1$

$$Z_{2d, \text{gonihedric, periodic}} = 2^{L_x} \sum_{\{\tau\}} \exp \left(\beta \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} \tau_{x,y} \tau_{x+1,y} \right) \prod_{x=1}^{L_x} \delta \left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right)$$



- ▶ the funny “trick” of rewriting the δ -constraints leads to complicated products $\rightarrow$ we go straight to the high-temperature representation

Two dimensional plaquette model: periodic boundaries

- high-temperature representation

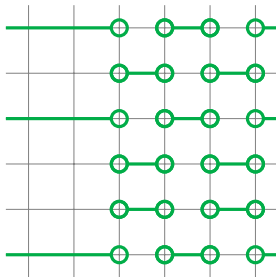
$Z_{2d, \text{gonihedric, periodic}}$

$$\begin{aligned} &= 2^{L_x} \sum_{\{\tau\}} \exp \left(\beta \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} \tau_{x,y} \tau_{x+1,y} \right) \prod_{x=1}^{L_x} \delta \left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right) \\ &= 2^{L_x} \text{ch}(\beta)^{L_x L_y} \sum_{\{\tau\}} \left[\prod_{y=1}^{L_y} \prod_{x=1}^{L_x} (1 + \text{th}(\beta) \tau_{x,y} \tau_{x+1,y}) \right] \prod_{x=1}^{L_x} \delta \left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right) \end{aligned}$$

- similar to counting loops in the $2d$ Ising model, but simpler: only coupling in x -direction

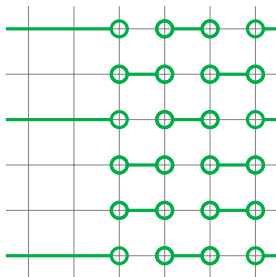
Two dimensional plaquette model: periodic boundaries

$$2^{L_x} \text{ch}(\beta)^{L_x L_y} \sum_{\{\tau\}} \left[\prod_{y=1}^{L_y} \prod_{x=1}^{L_x} (1 + \text{th}(\beta) \tau_{x,y} \tau_{x+1,y}) \right] \prod_{x=1}^{L_x} \delta \left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right)$$



Two dimensional plaquette model: periodic boundaries

$$2^{L_x} \text{ch}(\beta)^{L_x L_y} \sum_{\{\tau\}} \left[\prod_{y=1}^{L_y} \prod_{x=1}^{L_x} (1 + \text{th}(\beta) \tau_{x,y} \tau_{x+1,y}) \right] \prod_{x=1}^{L_x} \delta \left(\prod_{y=1}^{L_y} \tau_{x,y}, 1 \right)$$



$$\left(\frac{1}{2}\right) 2^{L_x L_y} \text{ch}(\beta)^{L_x L_y} \sum_{v=0}^{L_x} \binom{L_x}{v} \left(\text{th}(\beta)^v + \text{th}(\beta)^{L_x-v} \right)^{L_y}$$

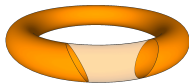
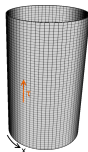
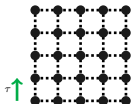
Two dimensional plaquette model

solving the 2d plaquette model

free-free

periodic-free

periodic-periodic



top line, σ_{x,L_y}
cause of " 2^{L_x} "

not transformed
summing over top row

transformed
two-to-one
additional constraint

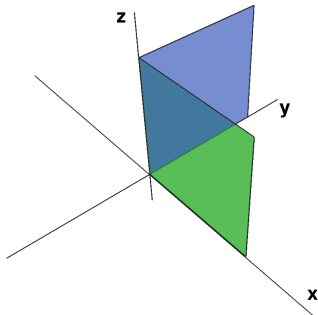
$$Z_{2d, \text{gonihedric, free}} = 2^{L_x L_y} \text{ch}(\beta)^{(L_x-1)(L_y-1)}$$

$$Z_{2d, \text{gonihedric, mixed}} = 2^{L_x L_y} \text{ch}(\beta)^{L_x(L_y-1)} \left(1 + \text{th}(\beta)^{L_x}\right)^{L_y-1}$$

$$Z_{2d, \text{gonihedric, periodic}} = \left(\frac{1}{2}\right) 2^{L_x L_y} \text{ch}(\beta)^{L_x L_y} \sum_{v=0}^{L_x} \binom{L_x}{v} \left(\text{th}(\beta)^v + \text{th}(\beta)^{L_x-v}\right)^{L_y}$$

Anisotropic plaquette model (again) - “fuki-nuke”

$$H_{\text{fuki-nuke}}(\{\sigma\}) = \underbrace{-J_x \sum_{\square_{yz}} \sigma\sigma\sigma\sigma}_{\text{blue}} \underbrace{-J_y \sum_{\square_{zx}} \sigma\sigma\sigma\sigma}_{\text{green}}$$



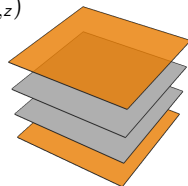
Three dimensional plaquette model: free boundaries in z-direction

- ▶ Spin-bond-transformation in z-direction $\tau_{x,y,z} = \sigma_{x,y,z} \sigma_{x,y,z+1}$ in a cuboidal $L \times L \times L_z$, for *one-to-one* correspondence: equality on one plane $\tau_{x,y,L_z} = \sigma_{x,y,L_z}$
- ▶ partition function factorises:

$$H_{\text{fuki-nuke}}(\{\tau\}) = - \sum_{x=1}^L \sum_{y=1}^L \sum_{z=1}^{L_z-1} (\tau_{x,y,z} \tau_{x+1,y,z} + \tau_{x,y,z} \tau_{x,y+1,z})$$

$$Z_{\text{fuki-nuke}} = \sum_{\{\tau\}} \exp(-\beta H_{\text{fuki-nuke}}(\{\tau\}))$$

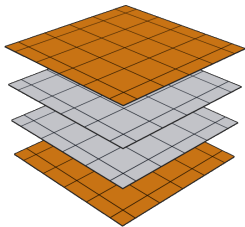
$$= 2^{L^2} (Z_{2d \text{ Ising}})^{L_z-1}$$



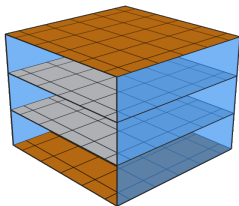
- ▶ the factor 2^{L^2} comes from the $L \times L$ sums over $\tau_{x,y,L_z} = \sigma_{x,y,L_z} = \pm 1$ which do not appear in the exponent
- ▶ free energy contributions

$$\beta f_{\text{fuki-nuke}} \equiv - \lim_{L \rightarrow \infty} \frac{1}{L^2 L_z} \ln Z_{\text{fuki-nuke}} = \beta f_{2d \text{ Ising}} - \frac{\ln 2 + \beta f_{2d \text{ Ising}}}{L_z}$$

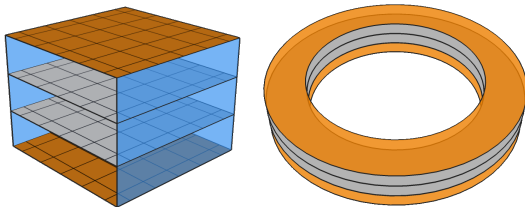
Interlude: topology



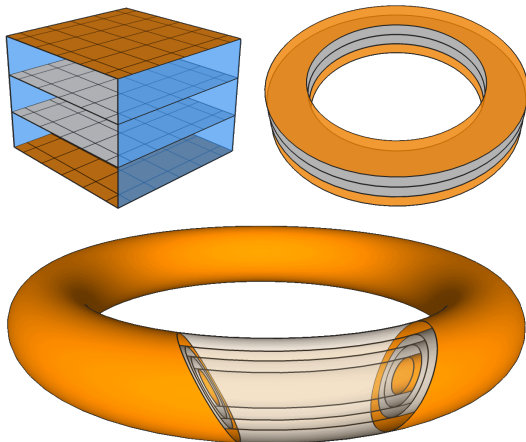
Interlude: topology



Interlude: topology



Interlude: topology

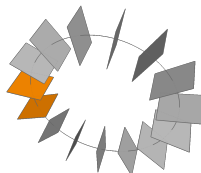


Three dimensional plaquette model: periodic boundaries in z-direction

- Spin-bond-transformation in z-direction $\tau_{x,y,z} = \sigma_{x,y,z} \sigma_{x,y,z+1}$ is *two-to-one* and imposes $L \times L$ constraints $\prod_z \tau_{x,y,z} = 1$

$$H_{\text{fuki-nuke}}(\{\tau\}) = - \sum_{x=1}^L \sum_{y=1}^L \sum_{z=1}^{L_z} (\tau_{x,y,z} \tau_{x+1,y,z} + \tau_{x,y,z} \tau_{x,y+1,z}) ,$$

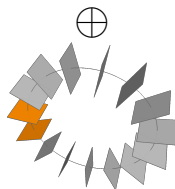
$$\begin{aligned} Z_{\text{fuki-nuke}} &= 2^{L^2} \sum_{\{\tau\}} \exp(-\beta H_{\text{fuki-nuke}}(\{\tau\})) \prod_{x=1}^L \prod_{y=1}^L \delta\left(\prod_{z=1}^{L_z} \tau_{x,y,z}, 1\right) \\ &= \sum_{\{\tau\}} \exp(-\beta H_{\text{fuki-nuke}}(\{\tau\})) \prod_{x=1}^L \prod_{y=1}^L \left(1 + \prod_{z=1}^{L_z} \tau_{x,y,z}\right) \end{aligned}$$



Three dimensional plaquette model: periodic boundaries in z-direction

$Z_{\text{fuki-nuke}}$

$$\begin{aligned}
 &= \sum_{\{\tau\}} \exp(-\beta H_{\text{fuki-nuke}}(\{\tau\})) \left(1 + \sum_{x=1}^L \sum_{y=1}^L \prod_{z=1}^{L_z} \tau_{x,y,z} + \mathcal{O}(\tau\tau) \right) \\
 &= (Z_{2d \text{ Ising}})^{L_z} \left(1 + \sum_{x=1}^L \sum_{y=1}^L \left(\langle \tau_{x,y} \rangle_{Z_{2d \text{ Ising}}} \right)^{L_z} + \mathcal{O}(\tau\tau) \right)
 \end{aligned}$$



- ▶ assuming translational invariance in each layer ($2d$ periodic Ising model)

$$Z_{\text{fuki-nuke}} = (Z_{2d, \text{ Ising}})^{L_z} \left(1 + L^2 C_1^{L_z} + \mathcal{O}(\tau\tau) \right)$$

- ▶ $C_1 = \langle \tau_{1,1} \rangle_{Z_{2d, \text{ Ising}}}$ is the normalized one-point function (magnetization)
- ▶ $\mathcal{O}(\tau\tau) =$

$$\frac{1}{2} \left(\sum_{x_1=1}^L \sum_{y_1=1}^L \sum_{x_2=1}^L \sum_{y_2=1}^L \left(\langle \tau_{x_1,y_1} \tau_{x_2,y_2} \rangle_{Z_{2d \text{ Ising}}} \right)^{L_z} - 1 \right) + \mathcal{O}(\tau\tau\tau)$$

Three dimensional plaquette model: periodic boundaries in z-direction

$Z_{\text{fuki-nuke}}$

$$= \sum_{\{\tau\}} \exp(-\beta H_{\text{fuki-nuke}}(\{\tau\})) \left(1 + \sum_{x=1}^L \sum_{y=1}^L \prod_{z=1}^{L_z} \tau_{x,y,z} + \mathcal{O}(\tau\tau) \right)$$

$$= (Z_{2d \text{ Ising}})^{L_z} \left(1 + \sum_{x=1}^L \sum_{y=1}^L \left(\langle \tau_{x,y} \rangle_{Z_{2d \text{ Ising}}} \right)^{L_z} + \mathcal{O}(\tau\tau) \right)$$

- ▶ assuming translational invariance in each layer ($2d$ periodic Ising model)

$$Z_{\text{fuki-nuke}} = (Z_{2d, \text{ Ising}})^{L_z} \left(1 + L^2 C_1^{L_z} + \mathcal{O}(\tau\tau) \right)$$

- ▶ $C_1 = \langle \tau_{1,1} \rangle_{Z_{2d, \text{ Ising}}}$ is the normalized one-point function (magnetization)

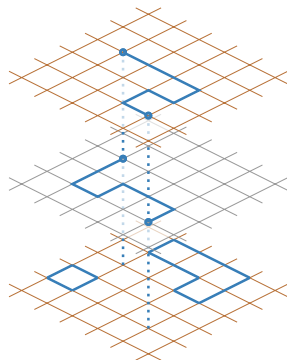
- ▶ $\mathcal{O}(\tau\tau) =$

$$\frac{1}{2} \left(\sum_{x_1=1}^L \sum_{y_1=1}^L \sum_{x_2=1}^L \sum_{y_2=1}^L \left(\langle \tau_{x_1,y_1} \tau_{x_2,y_2} \rangle_{Z_{2d \text{ Ising}}} \right)^{L_z} - 1 \right) + \mathcal{O}(\tau\tau\tau)$$



Fuki-Nuke: full-periodic

$$(Z_{2d \text{ Ising}})^{L_z} \left(1 + \sum_{x=1}^L \sum_{y=1}^L \left(\langle \tau_{x,y} \rangle_{Z_{2d \text{ Ising}}} \right)^{L_z} + \mathcal{O}(\tau\tau) \right)$$

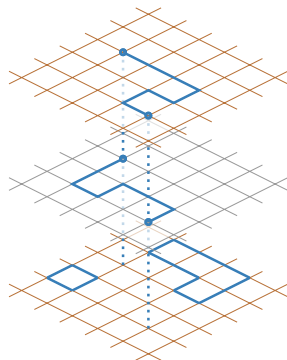


Fuki-Nuke: full-periodic

$$(Z_{2d \text{ Ising}})^{L_z} \left(1 + \sum_{x=1}^L \sum_{y=1}^L \left(\langle \tau_{x,y} \rangle_{Z_{2d \text{ Ising}}} \right)^{L_z} + \mathcal{O}(\tau\tau) \right)$$

- ▶ without the power L_z in $\mathcal{O}(\tau\tau) \rightarrow$ (high-temperature) susceptibility of the $2d$ Ising model, no closed-form expression
- ▶ too late, discovered in loop-matrix calculations already

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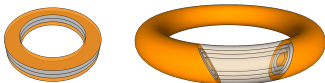


Conclusion

- ▶ identical spin-bond transformation can be treated explicitly for the 1d Ising and 2d plaquette models



- ▶ the 3d fuki-nuke model: explicit closed-form solution, as long as one boundary is free and 2d Ising model boundary is known



- ▶ the 3d fuki-nuke model: fully-periodic lattice creates sum over non-trivial n -point correlation functions
- ▶ the (full) 3d plaquette model: to be investigated (or maybe not)

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Exact solutions to plaquette Ising models with free and periodic boundaries
<http://dx.doi.org/10.1016/j.nuclphysb.2016.11.005>
arXiv:1601.03997

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