Convolution spline approximations of Volterra integral equations

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April 6, 2012

Abstract We derive a new “convolution spline” approximation method for convolution Volterra integral equations. This shares some properties of convolution quadrature, but instead of being based on an underlying ODE solver is explicitly constructed in terms of basis functions which have compact support. At time step $t_n = nh > 0$, the solution is approximated in a “backward time” manner in terms of basis functions $\phi_j$ by $u(t_n - t) \approx \sum_{j=0}^{n} u_{n-j} \phi_j(t/h)$ for $t \in [0, t_n]$. We carry out a detailed analysis for B-spline basis functions, but note that the framework is more general than this.

For B-splines of degree $m \geq 1$ we show that the schemes converge at the rate $O(h^2)$ when the kernel is sufficiently smooth. We also establish a methodology for their stability analysis and obtain new stability results for several non-smooth kernels, including the case of a highly oscillatory Bessel function kernel (in which the oscillation frequency can be $O(1/h)$). This is related to convergence analysis for approximation of time domain boundary integral equations (TDBIEs), and provides evidence that the new convolution spline approach could provide a useful time-stepping mechanism for TDBIE problems. In particular, using compactly supported basis functions would give sparse system matrices.

Keywords Convolution quadrature · Volterra integral equations · time dependent boundary integral equations

Mathematics Subject Classification (2000) 65R20 · 65M12

1 Introduction

Although the focus of this paper is approximation methods for convolution–kernel Volterra integral equations, our motivation and goal is to develop new “convolution spline” time-stepping methods for the time-dependent boundary integral equation (TDBIE)

\[
\frac{1}{4\pi} \int_{\Gamma} \frac{u(x', t - |x' - x|)}{|x' - x|} \, dx' = a(x, t) \quad \text{for } x \in \Gamma, \ t > 0 .
\]

(1.1)

This is the single layer potential equation for acoustic scattering from the surface $\Gamma \subset \mathbb{R}^3$ with zero Dirichlet boundary conditions: $-a(x, t)$ is the (known) incident field, and the problem is to compute the induced surface potential $u$ (see e.g. [19] for a full derivation of (1.1)).

Equation (1.1) is well-posed when $\Gamma$ is a smooth, closed or open, flat surface [1,18,25], but it is very hard to obtain numerically stable approximations using conventional finite element or collocation methods. Bamberger and Ha Duong [1] showed that a full Galerkin approximation in time and space is stable and convergent for smooth, closed $\Gamma$ (this was extended to the case of open, flat $\Gamma$ in [18]), but the stability of the method relies on all the integrals being evaluated extremely accurately (the key insight on how

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to do this was provided by Terrasse [31]). We have previously analysed collocation approximations of (1.1) (using Fourier methods analogous to those of von Neumann stability for PDE approximations) in [12] and demonstrated their stability when \( T = \mathbb{R}^2 \), but our analysis also requires all surface integrals to be evaluated very accurately. Furthermore, it obviously cannot take edge effects or corner singularities into consideration, and these may well affect the stability of the numerical approximation. Good survey articles on these and other approximation methods are given in [9,19].

Approximations of (1.1) typically have the convolution sum form 

\[
\sum_{m=0}^{n} Q^m U^{n-m} = a^n, 
\]

(1.2)

for \( U^n \in \mathbb{R}^{N_S} \), the representation of the spatial approximation of \( u \) at or near time \( t^n = n \Delta t \), where the right-hand side vector \( a^n \) is derived from \( a(x,t) \). In the case of both Galerkin and collocation approximations the matrices \( Q^m \in \mathbb{R}^{N_S \times N_S} \) are sparse – the number of nonzero elements per row of matrix \( Q^m \) is \( \mathcal{O}(\min\{m, N_S^{1/2}\}) \). In particular this means that (1.2) can be solved in \( \mathcal{O}(N_S^{3/2}) \) operations once the right-hand side is known, and the overall computational complexity to obtain the approximate solution up to time \( N_T \) is \( \mathcal{O}(\min(N_S^3, N_S, N_S^{3/2})) \) operations. For these hyperbolic problems it is usual to use a timestep \( h \) commensurate with the side \( \Delta x \) of a typical space mesh element, and in this case \( N_T \approx N \) and \( N_S \approx N^2 \) for \( N = 1/\Delta x \), and the total computational complexity is \( \mathcal{O}(N^3) \). Although this compares somewhat unfavourably with the \( \mathcal{O}(N^4) \) computational complexity of a finite difference or finite element approximation of the PDE formulation of the acoustic wave equation in \( \mathbb{R}^3 \), the plane wave “fast” methods developed by Michielssen and co-workers [15,16,23] reduces the complexity to \( \mathcal{O}(N^3 \log^2 N) \). However numerical stability remains a significant issue with these methods.

Lubich proved in [25] that an approximation of (1.1) which is piecewise linear Galerkin in space and uses convolution quadrature (CQ) in time (based on an underlying linear multistep ODE solver) is convergent. A key difference is that he also showed that this method is stable when the inner product integrals are approximated, which means that time-stepping numerical schemes which use CQ in time are inherently far more stable than those which use Galerkin or collocation time approximations. Unfortunately the drawback of this approach is that all the matrices \( Q^m \) in the corresponding solution algorithm (1.2) are now dense, which increases the computational complexity. The issue is not solving (1.2) for \( U^n \) which can typically be done efficiently by approximating \( Q^0 \) appropriately, but in performing the matrix–vector products needed to calculate the right-hand side. Lubich explains that the technique of [21] can be used to reduce the overall complexity to \( \mathcal{O}(N_S^3 N_T \log^2 N_T) \), i.e. \( \mathcal{O}(N^5 \log^2 N) \).

CQ time–stepping methods for (1.1) give dense \( Q^m \) matrices because the underlying basis functions are global (see e.g. [2,20] or Sec. 2 for more details). A systematic cut-off strategy to replace small matrix entries by zero is described and carefully analysed in [20], and this reduces the storage costs of the method. This is combined with panel clustering in [22] to further reduce the storage costs.

CQ methods which are based on underlying Runge–Kutta ODE solvers have been developed and analysed for TDBIEs in [3,4]. There are several advantages of these methods over linear multistep CQ methods: the basis functions are more highly concentrated (see [2, Figs 1–2]), which makes sparsifying the \( Q^m \) matrices more straightforward; and higher order accurate methods in time are possible. Banjai uses this approach in [2] to develop a practical, parallelizable solution algorithm for (1.1) which he illustrates with a number of realistic large-scale numerical examples.

Here we consider an alternative approach, that of constructing a new “convolution spline” method which shares some properties of CQ, but instead of being based on an underlying ODE solver is explicitly constructed in terms of basis functions which have compact support. Although our aim is to develop time-stepping methods for (1.1), we restrict attention here to numerical methods for Volterra integral equations. The motivation for this is that if \( T = \mathbb{R}^2 \), then the spatial Fourier transform of (1.1) at frequency \( \omega \in \mathbb{R}^2 \) is

\[
\int_0^t J_0(\omega t') \tilde{u}(\omega, t-t') dt' = 2\tilde{u}(\omega, t),
\]

(1.3)

where \( \omega = |\omega| \) and \( J_0 \) is the first kind Bessel function of order zero. As noted in [10], instabilities of approximation schemes for (1.1) are typically exhibited at the highest spatial frequency which can be represented on the mesh. Hence it is important to ensure that any prototype numerical scheme for an equation like (1.3) is stable and convergent at values of \( \omega \approx 1/h \) (assuming \( h \approx \Delta x \)). We discuss this in detail in Section 5.
Throughout the rest of the paper we shall consider the numerical approximation of the convolution kernel Volterra integral equation (VIE)

\[ \int_0^t K(t') u(t-t') \, dt' = a(t), \quad t \in [0, T], \]

where

\[ a \in C^{d+1}[0, T], \quad K \in C^{d+1}[0, T], \quad a(0) = 0 \quad \text{and} \quad K(0) = 1 \]

for some \( d \geq 0 \) to be specified. Under these assumptions, equation (1.4) possesses a unique solution \( u \in C^d[0, T] \) (see, e.g., [6, Theorem 2.1.9]).

We begin in Section 2 with an alternative derivation of Lubich’s [24] CQ method for (1.4) in terms of basis functions, and show how they can be calculated either explicitly or in terms of a recurrence relation. A new “convolution spline” approximation of (1.4) is described in Section 3 in terms of basis functions which have compact support and are (essentially) all translates, and we give sufficient conditions for this approximation to be stable. We then consider the special case in which the basis functions are B-splines, proving convergence for (1.4) in Section 4 when \( K \) and \( a \) are smooth, and showing in Section 5 how Laplace transform techniques can be used to prove stability for highly oscillatory problems like (1.3).

2 Convolution quadrature based on linear multistep methods

We begin by outlining the derivation of the CQ method for (1.4) from [24], in order to show how it can be reinterpreted in terms of CQ basis functions. For simplicity we restrict attention to the case for which the extension of the solution \( u \) by zero to the negative real axis is in \( C^d(-\infty, T] \) (otherwise the CQ method needs to be ‘corrected’ as described in [24, Sec. 3] in order to attain optimal convergence). This is guaranteed by requiring \( a^{(p)}(0) = 0 \) for \( p = 0 : d + 1 \) (2.1) because \( u^{(p)}(0) = a^{(p+1)}(0) - \sum_{\ell=0}^{p-1} K^{(p-\ell)}(0) u^{(\ell)}(0) \). We also assume that the Laplace transform \( \overline{K}(s) \) of the kernel \( K \) is sufficiently well-behaved for all the formal manipulations in the next subsection to be rigorous. For details see for example [25, Sec. 1] or [2, App].

2.1 Lubich’s CQ method

We follow [24] and substitute the Laplace inversion formula for \( \overline{K}(s) \) into (1.4) to obtain

\[ a(t) = \frac{1}{2\pi i} \int_\gamma \overline{K}(s) y(t, s) \, ds, \]

where \( \gamma \) is an infinite contour within the region of analyticity of \( \overline{K}(s) \) and

\[ y(t, s) = \int_0^t e^{st'} u(t-t') \, dt'. \]

Treating the Laplace variable \( s \) as a parameter, \( y(t) \) solves the ODE:

\[ \dot{y}(t) = sy(t) + u(t), \quad y(0) = 0, \]

and this is approximated by the \( k \)-step (\( k \leq d \)) linear multistep method with timestep \( h \)

\[ \sum_{j=0}^{k} \alpha_j y_{n+j-k} = h \sum_{j=0}^{k} \beta_j f_{n+j-k}, \]

where \( t_n = nh, \ y_n \approx y(t_n) \) and \( f_n = s y_n + u(t_n) \). The starting values are \( y_{-k} = \ldots = y_{-1} = 0 \) because of the assumption (2.1). Multiplying (2.4) by \( \xi^n \) and summing over \( n \) (for \( \xi \in \mathbb{C} \) for which the sum converges) gives

\[ \left( \frac{\delta(\xi)}{h} - s \right) \sum_{n=0}^{\infty} y_n \xi^n = \sum_{n=0}^{\infty} u(t_n) \xi^n, \]
where the symbol $\delta$ of the multistep method is

$$\delta(\xi) = \sum_{j=0}^{k} \alpha_j \xi^{k-j} / \sum_{j=0}^{k} \beta_j \xi^{k-j}. $$

Hence $y_n$ is the coefficient of $\xi^n$ in the expansion of $\left(\delta(\xi)/h - s\right)^{-1} \sum_{k=0}^{\infty} u(t_k) \xi^k$. Substituting $y_n$ for $y(t_n)$ in (2.2) shows that $a(t_n)$ is approximated by the coefficient of $\xi^n$ in

$$\frac{1}{2\pi i} \int_{\gamma} \left(\delta(\xi)/h - s\right)^{-1} K(s) ds \sum_{k=0}^{\infty} u(t_k) \xi^k = K(\delta(\xi)/h) \sum_{k=0}^{\infty} u(t_k) \xi^k$$

using Cauchy’s integral formula. Hence, defining the CQ weights $q_k = q_k(h)$ to be the coefficients in the expansion

$$K(\delta(\xi)/h) = \sum_{k=0}^{\infty} q_k \xi^k$$

(2.5) gives the CQ approximation of (1.4)

$$a(t_n) = \sum_{j=0}^{n} q_j u_{n-j}. $$

(2.6)

This can be rearranged to give the time-stepping approximate solution $u_n \approx u(t_n)$

$$u_n = \frac{1}{q_0} \left( a(t_n) - \sum_{j=1}^{n-1} q_j u_{n-j} \right) \quad \text{for } n \geq 1, $$

(2.7)

since by assumption $u_0 = u(0) = 0$.

### 2.2 Alternative derivation in terms of CQ basis functions

It is illuminating to express the CQ approximation scheme for $u$ in terms of temporal basis functions on a uniform grid with spacing $h$. This is the framework which we use for the derivation of the new “convolution spline” method in Section 3.

At $t = t_n$ (1.4) can be written as

$$a(t_n) = \int_{0}^{\infty} K(t') u(t_n - t') dt', $$

(2.8)

because $u(t) = 0$ for $t \leq 0$. We show below that the standard CQ method is equivalent to approximating $u$ in (2.8) by

$$u(t_n - t') \approx \sum_{j=0}^{n} u_{n-j} \phi_j(t'/h) \quad \text{for } t' \geq 0 $$

(2.9)

where $\phi_j$ are basis functions. This is fundamentally different from standard finite–element type approximations which have the form $u(t) \approx \sum_{k=0}^{N} u_k \psi_k(t/h)$ – i.e. the basis function $\psi_k$ is always associated with the same “unknown” $u_k$.

Substituting (2.9) into (2.8) and comparing the resulting expression with (2.6) gives the relationship between the standard CQ weights and basis functions:

$$q_j = \int_{0}^{\infty} K(t) \phi_j(t/h) dt. $$

(2.10)

Comparing this with the standard CQ definition of $q_j$ in (2.5) gives (see [2, Eq. (3.1)])

$$e^{-\delta(\xi)t} = \sum_{j=0}^{\infty} \phi_j(t) \xi^j. $$

(2.11)

An immediate consequence is that the basis functions satisfy the sum to unity property

$$\sum_{j=0}^{\infty} \phi_j(t) = 1, $$

(2.12)

provided the underlying multistep ODE solver is consistent, because in this case $\delta(1) = 0$. This is a crucial property which we use in Section 3.
2.3 Examples of CQ basis functions for LMMs

In principle (2.11) can be used directly to find the basis functions \( \phi_j(t) \) corresponding to any underlying linear multistep ODE method for (2.3), although this may not be easy in practice. We now illustrate that direct process for BDF1 and the trapezoidal rule for which the calculation is relatively straightforward. The standard backward difference formula (BDF) of order \( m \leq 6 \) has symbol [24] \( \delta(\xi) = \sum_{k=1}^{m} (1-\xi)^k/k \). When \( m = 1 \) it follows from (2.11) that \( \phi_j(t) = e^{-t^j/j!} \), i.e. the basis functions are Erlang functions, used in statistics as probability density functions and have the properties \( \phi_j(t) \geq 0 \) and \( \int_0^\infty \phi_j(t)dt = 1 \). These are plotted in Figure 2.1.

The symbol for the trapezoidal rule ODE solver is [2] \( \delta(\xi) = 2(1-\xi)/(1+\xi) \) and in this case (2.11) is

\[
\sum_{j=0}^{\infty} \phi_j(t) \xi^j = e^{-2t} f(\xi), \quad \text{where } f(\xi) = \exp(4t/(1+\xi)).
\]

This gives \( f(\xi) = \sum_{j=0}^{\infty} f_j \xi^j \) where

\[
f_j = \frac{1}{j!} \frac{d^j}{d\xi^j} f(\xi) \bigg|_{\xi=0} = \frac{1}{j!(4t)^j} \frac{d^j}{dz^j} e^{-1/z} \bigg|_{z=1/(4t)}
\]

using the change of variables \( z = (1+\xi)/(4t) \). It follows from [26, eq. 18.5.6] that \( f_j = (-1)^j e^{-4t} L_j^{-1}(4t), \) where \( L_j^{-1}(x) \) is a Laguerre polynomial. The identity \( L_j^{-1}(x) = L_j(x) - L_{j-1}(x) \) [17, eq. 8.971–5] gives the trapezoidal rule basis functions \( \phi_j(t) = (-1)^j \{ \ell_j(4t) - \ell_{j-1}(4t) \} \), where \( \ell_j(x) = e^{-x/2} L_j(x) \) is the \( j \)th Laguerre function. They are oscillatory, as shown in Figure 2.1 and [2, Fig. 1], but do not satisfy \( \int_0^\infty \phi_j(t)dt = 1 \).

Deriving the basis functions for BDF2 can be done in a similar way to the trapezoidal rule, but is more complicated. The explicit formula

\[
\phi_j(t) = \frac{1}{j!} H_j(\sqrt{2t}) \left( \frac{t}{2} \right)^{j/2} e^{-3t/2},
\]

where \( H_j \) is the \( j \)th Hermite polynomial, is given in [20]. Note that the properties of \( H_j \) imply that \( \phi_j(t) \) involves a \( j \)th degree polynomial and an exponential in \( t \) with no fractional powers of \( t \). The BDF2 basis functions are also illustrated in Figure 2.1 and [2, Fig. 1].

This direct approach appears intractable for more complicated schemes (even for BDF3), and we now give an alternative.

2.4 Recurrence relations for CQ basis functions generated by LMMs

A more general approach to finding the CQ basis functions is as follows. Differentiating the generating function (2.11) formally with respect to \( \xi \) and rearranging gives

\[
\sum_{j=1}^{\infty} j \phi_j(t) \xi^{j-1} + t \delta'(\xi) \sum_{j=0}^{\infty} \phi_j(t) \xi^j = 0. \tag{2.13}
\]

If \( \delta(\xi) \) is a polynomial (as it is for the BDF schemes) or a rational polynomial (as it is for the trapezoidal rule), then collecting terms in \( \xi \) gives a recurrence for the functions \( \phi_j \). In the polynomial case \( \delta(\xi) = \delta_0 + \delta_1 \xi + \cdots + \delta_k \xi^k \), and matching terms in powers of \( \xi \) in (2.13) gives

\[
\sum_{j=1}^{\infty} j \phi_j(t) + t \left[ \delta_1 \phi_{j-1}(t) + 2 \delta_2 \phi_{j-2}(t) + \cdots + k \delta_k \phi_{j-k}(t) \right] = 0
\]

where \( \phi_n(t) \equiv 0 \) for \( n < 0 \), and note that the first term of the Taylor expansion of (2.11) gives \( \phi_0(t) = e^{-\delta(0)t} = e^{-\delta_0 t} \). The recurrence relations for BDF1–4 are given in Table 2.1.

In the rational polynomial case \( \delta(\xi) = \alpha(\xi)/\beta(\xi) \) where \( \alpha(\xi) \) and \( \beta(\xi) \) are polynomials of degree up to \( k \) in \( \xi \) and the generating function relationship (2.13) becomes

\[
\beta(\xi)^2 \sum_{j=1}^{\infty} j \phi_j(t) \xi^{j-1} + t \left[ (\alpha' (\xi) \beta(\xi) - \alpha(\xi) \beta'(\xi)) \right] \sum_{j=0}^{\infty} \phi_j(t) \xi^j = 0.
\]

In general, this defines a \( 2k \)-step recurrence for the \( \phi_j(t) \) and in particular, the 2-step trapezoidal rule recurrence is listed in Table 2.1.
3 Convolution spline approach

As discussed in Sec. 1, basis functions with global support (such as those described above) will give rise to dense matrices $Q^n$ in the TDBIE scheme (1.2), and this has storage and computational cost implications. In this section we explore the use of compactly supported basis functions, which although not derived via standard CQ, nevertheless do fit into the CQ form (2.9). Our main proofs and numerical results in Secs. 4–5 are for B-spline basis functions, but in this Section we set up a more general framework for basis functions which are (mainly) translates.

3.1 Construction of a convolution spline scheme

We consider approximations of the form (2.9), but where all the basis functions $\phi_j$ have compact support of width $O(h)$ and almost all are translates of a standard, compactly supported basis function $\phi_m$, i.e.

$$\phi_j(t/h) = \phi_m(t/h + m - j) \quad \text{for } j \geq m. \quad (3.1)$$

The translate property (3.1) means that the approximation $U(t_n - t) \approx u(t_n - t)$ has the form

$$U(t_n - t) = \sum_{j=0}^{m-1} v_{n-j} \phi_j \left( \frac{t}{h} \right) + \sum_{j=m}^{n} v_{n-j} \phi_m \left( \frac{t}{h} + m - j \right) \quad (3.2)$$

for $t \geq 0$, where $v_j$ approximates $u(t)$ for $t$ near (but not necessarily at) $t_j$, and a sum is defined to be zero if its upper index is less than its lower index. Substituting the approximation (3.2) into the integral equation (1.4) and collocating at each time level as described in Section 2.2 gives

$$\sum_{j=0}^{n} q_j v_{n-j} = a(t_n) \quad (3.3)$$

for $n = 0 : N$ where the weights $q_j$ are defined by (2.10). The unknown coefficients $\{v_j\}_{j=0}^{N}$ are then found by time marching as in (2.7).
3.2 Stability of (3.3)

For TDBIE applications and analysis (see e.g. [12]), we require the scheme (3.3) to be stable in the following sense, independent of the input function \( a(t) \).

**Definition 3.1 (Stability)** The scheme (3.3) is said to be stable when the impulse response sequence \( \{p_n\} \) defined by

\[
p_0 = 1, \quad p_n = -\frac{1}{q_0} \sum_{j=1}^{n} q_j p_{n-j} \quad \text{for } n \geq 1
\]  

(3.4)

satisfies \( |p_n| \leq C \) for all \( n \) such that \( nh \leq T \), where the constant \( C \) is independent of \( h \).

This is weaker than BIBO (bounded input bounded output) stability in the signal processing literature (see e.g. [28]), which requires boundedness of the absolute sum

\[
\sum_{n=0}^{\infty} |p_n| < \infty.
\]

Stability properties of the scheme (3.3) can be established by using the Z-transform, defined as follows.

**Definition 3.2** The Z-transform of a sequence \( \{f_n\}_{n=0}^{\infty} \) is the function \( F \) given by

\[
F(\xi) = \mathcal{Z}\{f_n\}(\xi) = \sum_{n=0}^{\infty} f_n \xi^n
\]  

(3.5)

where \( \xi \in \mathbb{C} \) with \( |\xi| \leq 1 \) is such that the sum converges.

The Z-transform of the scheme (3.3) is

\[
Q(\xi) V(\xi) = A(\xi),
\]  

(3.6)

where

\[
Q(\xi) = \sum_{j=0}^{\infty} \xi^j \int_{0}^{\infty} K(t) \phi_j(t/h) \, dt
\]  

(3.7)

and we take \( a_n = a(t_n) \). Similarly, the Z-transform of (3.4) is \( Q(\xi) P(\xi) = q_0 \) and so

\[
P(\xi) = \frac{q_0}{Q(\xi)}.
\]

When \( Q(\xi) \) is a rational polynomial, stability is guaranteed by a variant of the “root condition” familiar (after the change of variable \( z = 1/\xi \)) from zero stability analysis of numerical methods for ODEs.

**Lemma 3.1 (Root condition for stability)** If the Z-transform \( Q(\xi) \) of \( \{q_n\} \) is a rational polynomial in \( \xi \), then the approximation (3.3) is stable in the sense of Definition 3.1 if the roots \( \xi_k \) of \( Q(\xi) \) satisfy the following for any constant \( c \geq 0 \) (independent of \( h \)): \( |\xi_k| \geq 1/(1+ch) \) and any with \( 1/(1+ch) \leq |\xi_k| \leq 1 \) are simple.

The proof follows standard arguments familiar from the numerical ODE literature. Note that simple roots with \( |\xi_k| = 1/(1+ch) \) make a bounded contribution to \( p_n \) as \( n \) increases by the standard result

\[
|\xi_k|^{-n} = (1+ch)^n \leq e^{cT}
\]

for \( t_n \leq T \), but roots of this size with multiplicity \( \mu \geq 2 \) grow like \( n^{\mu-1} \) and hence violate the stability definition 3.1.

Verifying the stability condition Definition 3.1 directly or via the root condition above for a general approximation scheme for (1.4) may be very complicated. But as we show below, schemes with the translate property (3.1) can be tackled within the framework of Laplace transforms originally introduced for CQ, and this approach gives a way to extend the scope of stability analysis to a far broader range of kernel functions. We first set up the framework for the analysis for general basis functions satisfying (3.1), and prove specific stability results when the \( \phi_j \) are B-splines in Section 5.

Substituting the Laplace inversion formula for \( K \) into the weight formula (2.10) gives

\[
q_j = \frac{h}{2\pi i} \int_{\gamma} K(s) \Phi_j(-sh) \, ds
\]
where \( \Phi_j(s) \) is the Laplace transform of \( \phi_j \). Hence the approximation scheme (3.3) can be written as
\[
a(t_n) = \frac{1}{2\pi i} \int_{\gamma} \mathcal{K}(s) y_n(sh) \, ds
\]
where
\[
y_n(sh) = h \sum_{j=0}^m v_{n-j} \Phi_j(-sh).
\]
Comparing equations (2.2) and (3.8) we see that \( y_n \) plays the same role here that the approximate solution of the ODE (2.3) does in standard CQ. The translate property (3.1) and the compact support of \( C \) implies
\[
\Phi_j(-sh) = e^{sh(j-m)} \Phi_m(-sh) \quad \text{for } j \geq m
\]
and so
\[
y_n(sh) - e^{sh} y_{n-1}(sh) = h v_n \Phi_0(-sh) + h \sum_{j=1}^m v_{n-j} \left( \Phi_j(-sh) - e^{sh} \Phi_{j-1}(-sh) \right),
\]
(using \( v_j \equiv 0, j \leq 0 \)). Taking the Z-transform of this expression gives
\[
Y(\xi,sh) = h \left( \frac{B(\xi,sh)}{1 - e^{sh} \xi} \right) V(\xi)
\]
when \( \xi \neq e^{-sh} \), where
\[
B(\xi,sh) = \Phi_0(-sh) + \sum_{j=1}^m \left[ \Phi_j(-sh) - e^{sh} \Phi_{j-1}(-sh) \right] \xi^j.
\]
It hence follows from (3.8) that
\[
A(\xi) = V(\xi) \frac{h}{2\pi i} \int_{\gamma} \mathcal{K}(s) \left( \frac{B(\xi,sh)}{1 - e^{sh} \xi} \right) \, ds
\]
and comparison with (3.6) yields the alternative representation
\[
Q(\xi) = \frac{h}{2\pi i} \int_{\gamma} \mathcal{K}(s) \left( \frac{B(\xi,sh)}{1 - e^{sh} \xi} \right) \, ds
\]
(3.10)
of the Z-transform of the weights \( q_j \). The expression \( B(\xi,sh)/(1 - e^{sh} \xi) \) plays a role similar to that of \( (\delta(\xi)/h - s)^{-1} \) in standard CQ analysis, and it is the key quantity in determining whether the scheme is stable or not. Unfortunately it has a more complicated structure: it has an infinite vertical line of simple poles at \( s = s_k \) for \( k \in \mathbb{Z} \), where
\[
s_k := \frac{1}{h} (-\ln|\xi| - i \text{ Arg}(\xi) + i 2\pi k)
\]
(3.11)
and the principal argument \( \text{Arg}(\xi) \in (-\pi, \pi] \). Note that if \( |\xi| < 1 \) then \( \text{Re}(s_k) > 0 \).

To evaluate \( Q(\xi) \) defined by (3.10) for a given kernel function \( K(t) \) we can use either the left or right “D” contours illustrated in Figure 3.1, taking the limit \( R \to \infty \) and setting \( \gamma = \lim_{R \to \infty} \gamma_R \). Using the right contour gives
\[
Q(\xi) = \sum_{k=\infty}^{\infty} \mathcal{K}(s_k) B(\xi, s_k h) - \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R^+} \mathcal{K}(s) \left( \frac{B(\xi,sh)}{1 - e^{sh} \xi} \right) \, ds.
\]
The integral round \( C_R^+ \) does not necessarily vanish as \( R \to \infty \) since for some basis functions (including higher order B-splines) the quantity
\[
\frac{B(\xi,sh)}{1 - e^{sh} \xi} = \mathcal{O}(e^{c|\xi|}) \quad \text{as } \text{Re}(s) \to \infty
\]
for \( c \geq 1 \). We may also use the left contour when \( \mathcal{K}(s) \) has simple poles at \( s = \kappa_j \) with \( \text{Re}(\kappa_j) \leq 0 \) and obtain the analogous result
\[
Q(\xi) = \sum_{j} \frac{B(\xi, s_j) h}{1 - e^{s_j} \xi} \lim_{s \to \kappa_j} (s - \kappa_j) \mathcal{K}(s) - \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R^-} \mathcal{K}(s) \left( \frac{B(\xi,sh)}{1 - e^{sh} \xi} \right) \, ds.
\]
The asymptotic behaviour of the integral \( C_R^- \) as \( R \to \infty \) is determined primarily by \( \mathcal{K}(s) \). The extension of this left contour approach to poles with higher multiplicity is straightforward. We illustrate the use of these formulae in Sec. 5 for various kernels \( K \) when the basis functions are B-splines.
Fig. 3.1 The left and right “D” contours of radius $R$ used for the stability and Z-transform calculations. The crosses are the poles (3.11) and the vertical line of length (approximately) $2R$ is $\gamma_R$.

4 B-spline basis functions

The main result of this section is that the approximation given by (3.3) when the basis functions $\phi_j$ are B-splines converges to the solution $u$ of (1.4) for general smooth $a$ and $K$. The analysis proceeds in the now standard fashion of proving convergence in the case that $K \equiv 1$ and then using Taylor expansion to show that the same result also holds when $h$ is small enough (see e.g. [6]). Thus it does not apply to the important case of an oscillatory kernel like (1.3), where the oscillation frequency $\omega \approx 1/h$, and stability in this case is considered in Section 5. Discontinuous polynomial collocation or Galerkin approximations of (1.4) converge at optimal order (see e.g. [6–8]), but we show that convergence of the B-spline approximation (3.3) is at most second order, no matter how high the polynomial degree (because quasi-interpolation by the Schoenberg B-spline operator is at most $O(h^2)$ [13]).

We state and prove the convergence result in Section 4.2, and before this list some general properties of B-splines which are needed in this and subsequent analysis.

4.1 Notation and properties

Throughout Sections 4 and 5 we shall assume that the basis functions $\{\phi_j\}_{j \geq 0}$ are B-splines of polynomial degree $m$ based on the nodes (or knots) $t_j$ for $j \geq 0$. We assume uniform spacing, so $t_{j+1} - t_j = h$ for $j \geq 0$. It is necessary to introduce $m$ new knots at $t = 0$ in order for the B-spline basis functions to have the sum to unity property (2.12) in the interval $[0, t_m)$, and we set $t_j = 0$ for $j = -m : -1$. The $m$th degree B-splines are $b_m^j(t)$ for $j \geq -m$, and B-splines of degree $m > 0$ are recursively defined in terms of those of lower degree as follows, using the convention that $0/0$ is interpreted as $0$.

**Definition 4.1** [13] When $m = 0$

$$b_0^j(t) = \begin{cases} 1 & \text{if } t \in [t_j, t_{j+1}) \text{ for } j \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $m > 0$ then

$$b_m^j(t) = \left( \frac{t - t_j}{t_{j+m} - t_j} \right) b_{m-1}^j(t) + \left( \frac{t_{j+m+1} - t}{t_{j+m+1} - t_{j+1}} \right) b_{m-1}^{j+1}(t)$$

where the convention is that $0/0$ is interpreted as $0$.

We shall make use of several B-spline properties (see for example standard references such as [13,30]) in Section 4, which we list here for convenience.
B-spline properties

P1. Compact support. \( b^m_j(t) = 0 \) outwith \([t_j, t_{j+m+1})\), and \( b^m_j(t_j) = 0 \) unless \( j = -m \).

P2. Translate property. If \( j \geq 0 \) then \( b^m_j(t) = b^m(t/h - j) \), where the functions \( b^m \) are defined recursively:

\[
 b^0(\tau) = \begin{cases} 
 1 & \text{if } \tau \in [0, 1), \\
 0 & \text{otherwise} 
\end{cases}
\]

and if \( m \geq 1 \):

\[
 b^m(\tau) = \frac{\tau}{m} b^{m-1}(\tau) + \frac{m+1-\tau}{m} b^{m-1}(\tau-1). 
\]

It follows that \( \phi_j(\tau) = b^m(\tau + m - j) \) for \( j \geq m \).

P3. Sum to unity. \( \sum_{j=-\infty}^{\infty} b^m_j(t) = 1 \) for all \( t \geq 0 \).

P4. Moments.

\[
 \int_{t_j-m}^{t_j+1} b^m_{j-m}(t) \, dt = \frac{t_{j+1} - t_{j-m}}{m+1} \quad \text{and} \quad \int_{t_j-m}^{t_j+1} t b^m_{j-m}(t) \, dt = \frac{t_{j+1} - t_{j-m}}{(m+1)(m+2)} \sum_{k=0}^{m+1} t_{j-m+k}. 
\]

P5. Shoenberg quasi-interpolation. Suppose that \( m \geq 1 \) and set \( t^m_j \equiv h(m+j)(m+j+1)/2m \) for \( j = -m, -1 \) and \( t^m_{j+1} = (t^m_j+m+1)/2 \) for \( j \geq 0 \). Then

\[
 \sum_{j=-m}^{\infty} t^m_j b^m_{j-m}(t) = t \quad \text{when } t \geq 0.
\]

It follows from properties P1, P3 and P5 above that

\[
 f(t) = \sum_{j=-m}^{\infty} f(t^m_j) b^m_j(t) + \mathcal{O}(h^2) \tag{4.1}
\]

for any \( f \in C^2[0, \infty) \), and if \( f \in C^{p+1}[0, \infty) \) for \( p \geq 2 \) and \( t \in [t_\ell, t_{\ell+1}) \) for some \( \ell \geq 0 \), then

\[
 f(t) - \sum_{j=\ell-m}^{\ell} f(t^m_j) b^m_j(t) = \sum_{k=2}^{p} \frac{f^{(k)}(t_\ell)}{k!} \left[ (t-t_\ell)^k - \sum_{j=\ell-m}^{\ell} (t_j^m - t_\ell)^k t_j^m(t) \right] + \mathcal{O}(h^{p+1}). \tag{4.2}
\]

The basis functions are \( \phi_j(t/h) = b^m_{j-m}(t) \) for \( j \geq 0 \), and it follows from P1 that the CQ weights are

\[
 q_j = \int_{t_j-m}^{t_j+1} K(t) b^m_{j-m}(t) \, dt. \tag{4.3}
\]

The convergence analysis of the next subsection relies crucially on knowing the values of the weights when \( K \) is a constant, and this follows immediately from P4: when \( K \equiv 1 \) the weights \( q_j \) of (4.3) are given by

\[
 \frac{q_j}{h} = \begin{cases} 
 \frac{j+1}{m+1} & \text{for } j = 0 : m-1 \\
 1 & \text{if } j \geq m. 
\end{cases}
\]

4.2 Convergence results for (1.4)

When \( m = 0 \) the B-spline approximation (3.2) is the same as using piecewise constant collocation (at the interval endpoints), and this has been fully analysed (see e.g. [6] for details). Here we assume that \( m \geq 1 \), and the main result of this section is that the difference between the solution \( u \) of (1.4) and its approximation \( U \) of (3.2) is at most \( \mathcal{O}(h^2) \) when \( a \) and \( K \) are sufficiently smooth.

The approximation error \( e_n(t) \) for \( n > 0, \ t \geq 0 \) is

\[
 e_n(t) = u(t_n - t) - \sum_{j=0}^{n} v_{n-j} b^m_{j-m}(t), \tag{4.4}
\]

where the coefficients \( v_j \) satisfy

\[
 a(t_n) = \sum_{j=0}^{n} q_{j} v_{n-j}. \tag{4.5}
\]
for weights $q_j$ as defined in (4.3). Note that $v_0 = 0$ (because $a(0) = 0$) and so the sums above can be taken from $j = 0$ to $n - 1$, and it then follows from Property P1 that $e_n(t) = 0$ for $t \geq t_n$ and each weight can be written as $q_j = \int_0^{t_n} K(t) b_{j-m}^m(t) dt$. Hence, multiplying (4.4) by $K(t)$ and integrating gives

$$
\int_0^{t_n} K(t) e_n(t) dt = \int_0^{t_n} K(t) u(t_n - t) dt - \sum_{j=0}^{n-1} q_j v_{n-j} = 0,
$$

by (1.4) and (4.5), i.e. $e_n$ is orthogonal to $K$ on $(0, t_n)$.

We now state the main result of this section.

**Theorem 4.1** Suppose that $m \geq 1$ and the conditions (1.5) and (2.1) hold for $d \geq 4$. Then

$$
|e_n(t)| \leq C h^2 \tag{4.6}
$$

for $t \in [0, T]$, for some $C$ independent of $n$ and $h$.

The restriction to second order convergence for any $m$ is a fundamental aspect of the method (because interpolation by B-splines is at most second order accurate [13]) and not an artefact of the proof, as illustrated in Figure 4.1.

Note that (4.6) trivially holds for $t \geq t_n$ (because $e_n(t) = 0$), and also for $t \leq t_m$ since it follows from the assumptions (1.5) and (2.1) that

$$
u(ch) = O(h^d) \tag{4.7}
$$

for any constant $c$. Hence it is enough to prove the result for $t \in (t_m, t_n)$ for $n \leq T/h$. This is done in stages, the first of which is to show that $e_n(t)$ can be expressed in terms of coefficients $\varepsilon_k = u(t_k+(m-1)/2) - v_k$.

**Lemma 4.1** Under the conditions of Theorem 4.1, the approximation error $e_n$ satisfies

$$
e_n(t) = \sum_{j=m}^{n} \varepsilon_{n-j} b_{j-m}^m(t) + O(h^2) \tag{4.8}
$$

for $t \in [t_m, t_n]$ with $t_n \leq T$.

**Proof** Substituting the quasi-interpolation result (4.1) with $f(t) = u(t_n - t)$ in the definition (4.4) of $e_n(t)$ gives

$$
e_n(t) = \sum_{j=m}^{n} \varepsilon_{n-j} b_{j-m}^m(t) + \sum_{j=n+1}^{n+m} u(t_n - t_{j-m}) b_{j-m}^m(t) + O(h^2),
$$

where we have used $b_{j-m}^m(t) = 0$ for $j < m$ and $j > n + m$. Equation (4.7) implies that the second sum term in the equation above is $O(h^d)$, and hence yields (4.8). \[\square\]
Because there are at most \( m + 1 \) nonzero terms in the sum (4.8) for any \( t \), the following result is sufficient for Theorem 4.1.

**Lemma 4.2** Under the conditions of Theorem 4.1, there exists a constant \( C \) independent of \( h \) such that

\[
|\varepsilon_t| \leq \begin{cases} 
    Ch^d & \text{for } \ell = 0 : m \\
    Ch^2 & \text{when } m < \ell \leq T/h.
\end{cases} 
\tag{4.9}
\]

**Proof** It follows from (4.5) that

\[
\sum_{j=0}^{n} q_j v_{n-j} = \int_0^{t_n} K(t) u(t_n - t) \, dt 
\]

and after some manipulation using (4.11) the second central difference of (4.10) can be written as

\[
\frac{q_j}{h} \varepsilon_{n-j} = \frac{q_j}{h} u(t_n - j/(m-1/2)) - \frac{1}{h} \int_0^{t_n} K(t) u(t_n - t) \, dt := R_n. 
\tag{4.10}
\]

If \( h \) is sufficiently small, then expanding \( K(t) \) and using P4 gives

\[
\frac{q_j}{h} = \begin{cases} 
    \frac{j + 1}{m + 1} + h K'(0) \frac{(j + 1)^2(j + 2)}{2(m + 1)(m + 2)} + O(h^2) & \text{for } j = 0 : m - 1 \\
    K(t_{j-m}) + \frac{1}{h} \frac{1}{2} h (m + 1) K'(t_{j-m}) + O(h^2) & \text{if } j \geq m.
\end{cases} \tag{4.11}
\]

In particular this shows that for sufficiently small \( h \) each \( q_j/h \) is an \( O(1) \) quantity bounded away from zero, and together with (4.7) this gives the top line of (4.9). If \( n \geq m \) then

\[
h R_n = \int_0^{t_n} K(t) \left( \sum_{j=0}^{n-1} u(t_n - j/(m-1/2)) b_{j-m}^m(t) - u(t_n - t) \right) \, dt + O(h^{d+1})
\]

and it follows from the quasi-interpolation result (4.2) with \( p = 3 \) that

\[
h R_n = \sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} \eta_{\ell}^k u^{(k)}(t_n - \ell) + O(h^4),
\]

where

\[
\eta_{\ell}^k = \frac{(-1)^{k+1}}{k!} \int_{t_{\ell}}^{t_{\ell+1}} K(t) \left( (t - t_{\ell})^k - \sum_{j=\ell}^{\ell+m} (t_{j-m} - t_{\ell})^k b_{j-m}^m(t) \right) \, dt.
\]

It is then straightforward to show that

\[
R_{n+1} - 2 R_n + R_{n-1} = O(h^3)
\]

and after some manipulation using (4.11) the second central difference of (4.10) can be written as

\[
(1 + h \mu_0) \varepsilon_{n+m+1} = \varepsilon_n + h \sum_{\ell=0}^{m} \mu_{\ell+1} \varepsilon_{n-\ell+m} + h^2 \sum_{\ell=m+1}^{n} \mu_{\ell+1} \varepsilon_{n-\ell+m} + \gamma_n, 
\tag{4.12}
\]

where \( \gamma_n = O(h^3) \) and all the \( \mu_{\ell} \) are bounded. This can be written as a one-step recurrence for the vector \( \delta^n \in \mathbb{R}^{m+1} \) with components \( \delta_j^n \equiv \varepsilon_{n+j} \) for \( j = 0 : m \). The recurrence is \( \delta_j^{n+1} = \delta_j^n + \gamma_n \) for \( j = 0 : m - 1 \) with \( \delta_m^{n+1} \) given by (4.12), which gives the matrix–vector system

\[
\delta^{n+1} = (M + h W_0) \delta^n + h^2 \sum_{\ell=0}^{n-m} W_{n-\ell} \delta^{n+\ell} + \gamma_m \varepsilon^m
\tag{4.13}
\]

where \( \varepsilon^m = (0, \ldots, 0)^T \), each matrix \( W_\ell \) is bounded and \( M \in \mathbb{R}^{(m+1) \times (m+1)} \) is the circulant matrix whose only nonzero components are \( M_{0,0} = M_{j,j+1} = 1 \) for \( j = 0 : m - 1 \). The eigenvalues of \( M \) are the \( (m + 1) \)-th roots of unity, and are hence distinct. Following Brunner [5] we note that \( M \) belongs to Ortega’s [27, §1.3] Class M, and so there is a vector norm \( \| \cdot \| \), on \( \mathbb{R}^{m+1} \) for
which the induced matrix norm satisfies \( \|M\|_* = \rho(M) = 1 \). Taking this norm of (4.13) then implies that there is a constant \( C \) such that

\[
\|\hat{g}^{n+1}\|_* \leq (1 + C h) \|\hat{g}^n\|_* + C h^2 \sum_{\ell=0}^{n+1-m} \|\hat{g}^{m+\ell}\|_* + C h^3
\]

and the top bound of (4.9) gives \( \|\hat{g}^0\|_* \leq C h^d \). Standard arguments can then be used to show that \( \|\hat{g}^n\|_* \leq \nu_n \) where \( \nu_0 = C h^d \) and

\[
\nu_{n+1} = (1 + C h) \nu_n + C h^2 \sum_{\ell=0}^n \nu_\ell + C h^3 \quad \text{for } n \geq 0.
\]

This has the solution \( \nu_n = A_+ \lambda^n + A_- \lambda^{-n} \) where \( \lambda_\pm = 1 + O(h) \) and \( A_\pm = O(h^2) \). Hence \( \nu_n \leq C_1 h^2 e^{C_2 T} \) for \( n \leq T/h \), which concludes the proof of (4.9).

\[\square\]

### 5 Stability results for convolution B-splines

We now use the theoretical framework introduced in Section 3 to examine the stability of the convolution spline approximation of (1.4) for four different example kernels when the basis functions are B-splines. The kernels are \( K(t) \) equal to a constant, a step function, \( \cos(\omega t) \) and \( \exp(-\xi t) \). Throughout this section we use \( B_m(\xi, s) \) to denote the function defined by (3.9) when the basis functions are \( \phi_j = b_m^{j-m} \), and \( Q_m(\xi) \) to denote the coefficient Z-transform given by (3.10). The first few values of \( B_m(\xi, s) \) are listed in Table 5.1; those for higher values of \( m \) are more complicated, but are easily computed in a standard algebraic manipulation package.

In the first three cases \( Q_m(\xi) \) is a rational polynomial in \( \xi \) and Lemma 3.1 can be used to determine stability. In the final example, the Laplace transform of \( J_0(\omega t) \) has a branch cut and stability is determined from the Z-transform inversion formula by bounding the coefficients \( p_n \) of (3.4) directly. Note that this bound is independent of \( n \), and so is a practically useful stability result, in contrast with the (essentially) uncheckable hypotheses needed in [11].

<table>
<thead>
<tr>
<th>B-spline degree</th>
<th>( B_m(\xi, s) )</th>
<th>( \lim_{s \to 0} B_m(\xi, s) )</th>
<th>( B_m(e^{-s}, s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 0 )</td>
<td>( s^{-1}(e^{-1} - 1) )</td>
<td>1</td>
<td>( s^{-1}(e^{-1} - 1) )</td>
</tr>
<tr>
<td>( m = 1 )</td>
<td>( s^{-2}(e^{1/2} - 1) )</td>
<td>( (1 + \xi)/2 )</td>
<td>( s^{-2}(e^{-1})e^{-\xi} )</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>( s^{-3}((2e^{1/2} - 1) + (1 + \xi - s - 1)) )</td>
<td>( (1 + \xi)/3 )</td>
<td>( s^{-3}(e^{-1})^2e^{-\xi} )</td>
</tr>
<tr>
<td>( m = 3 )</td>
<td>( \cdots )</td>
<td>( (1 + \xi + \xi^2)/4 )</td>
<td>( s^{-3}(e^{-1})^3e^{-\xi} )</td>
</tr>
</tbody>
</table>

Table 5.1 The function \( B_m(\xi, s) \) for the first few B-splines.

#### 5.1 Constant kernel \( K(t) = 1 \), transform \( K(s) = 1/s \)

Integrating (3.10) round the left contour in Figure 3.1 gives

\[
Q_m(\xi) = \lim_{s \to 0} \frac{B_m(\xi, sh)}{1 - e^{sh}\xi} = \frac{1 - \xi^{m+1}}{(m+1)(1 - \xi)^2}.
\]

(5.1)

The function \( Q_m \) has \( m \) simple roots on the unit circle, and stability of the approximation then follows from Lemma 3.1. (Note that for smooth \( a \), stability follows from the convergence result of the previous section.)
5.2 Discontinuous kernel $K(t) = 1$ for $t \in [0, L]$, otherwise 0.

Kernels of this type arise in TDBIE approximations when approximating objects with edges and even on spheres. Examples expressed in terms of the Laplace transform $K(s) = (1 - e^{-sL})/s$ can be found in [3, Sec. 6.1] and [29, Sec. 4.1]. Assume that the duration $L$ is independent of $h$ and set $p = [L/h]$, i.e.

$$L = (p + r)h \quad \text{for } p \in \mathbb{Z}^+ \text{ and } r \in [0, 1).$$

It is simplest to work with the explicit $Z$-transform formula (3.5) using the weights given in (4.3). Results for $m = 0 : 3$ are summarised below.

**Case $m = 0$:**

$$Q_0(\xi) = r \xi^p + \sum_{n=0}^{p-1} \xi^n = \frac{1}{\xi - 1} (r \xi^{p+1} + (1 - r) \xi^p - 1).$$

When $r \in (0, 1)$ it can be shown that the $p$ roots $\xi_j$ of $Q_0$ satisfy $|\xi_j| > 1$ for $j = 1 : p$, and when $r = 0$ there are $p - 1$ simple roots $\xi_j = \exp(i2\pi j/p)$ for $j = 1 : p - 1$. Hence Lemma 3.1 implies that the $m = 0$ scheme is stable for all $L$.

**Case $m = 1$:**

$$2Q_1(\xi) = r^2 \xi^{p+1} + (2r - r^2 + 1) \xi^p + 2 \sum_{n=1}^{p-1} \xi^n + 1 \quad \text{when } r \in (0, 1).$$

In this case the modulus of the product of the $p + 1$ roots of $Q_1(\xi)$ is $1/r^2$ and it can be shown that there is a real negative root with modulus strictly greater than $1/r^2$. Hence the modulus of the product of the remaining roots must be strictly less than 1, implying that one or more of the roots has $|\xi_j| < 1$ and the scheme is unstable. In the case that $r = 0$,

$$Q_1(\xi) = \frac{(1 + \xi)(1 - \xi^p)}{2(1 - \xi)}$$

and the roots of $Q_1$ are $-1$, and $\exp(i2\pi j/p)$ for $j = 1 : p - 1$. The roots are all simple except when $p$ is even, when $-1, -1$ is a double root, giving a linear (and not exponential) instability.

In summary, the $m = 1$ scheme is unstable unless $h = L/p$ and $p$ is odd, which is not always possible.

**Case $m = 2$:**

$$6Q_2(\xi) = r^3 \xi^{p+2} + P_1(r) \xi^{p+1} + P_2(r) \xi^p + 6 \sum_{n=2}^{p-1} \xi^n + 4 \xi + 2$$

with $P_1(r) = 1 + 3r + 3r^2 - 2r^3$ and $P_2(r) = 5 + 3r - 3r^2 + r^3$. When $r = 0$ the modulus of the product of the $p + 1$ roots is equal to 2, and it can be shown that there is a real root which is strictly less than $-3$, and so as above one or more of the roots has $|\xi_j| < 1$ which means that the scheme is unstable when $r = 0$ (and also when $r$ is close to 0 because the roots depend continuously on $r$). When $r > 0$ there are $p + 2$ roots and a similar argument shows that the scheme is also unstable in this case.

**Case $m = 3$:** similar arguments to those above can be used to show that the scheme is also unstable in this case.

5.3 Simple oscillatory kernel $K(t) = \cos(\omega t)$, transform $K(s) = s/(s^2 + \omega^2)$.

The Laplace transform of this kernel has two poles on the imaginary axes at $s = \pm i\omega$ and it is convenient to use the left contour in Figure 3.1 to evaluate $Q_m(\xi)$. The general formula is

$$Q_m(\xi) = \frac{B_m(\xi, i\omega h)}{2(1 - e^{-i\omega h})} + \frac{B_m(\xi, -i\omega h)}{2(1 - e^{i\omega h})}.$$

**Case $m = 0$:**

$$Q_0(\xi) = \left(\frac{\sin(\omega h)}{\omega h}\right) \left(\frac{1 - \xi}{\xi^2 - 2\xi \cos(\omega h) + 1}\right),$$
Clearly \( Q_0(\xi) = 0 \) for all choices of \( \xi \) when \( \omega h = k\pi \) for any \( k \in \mathbb{Z}/0 \) and so the scheme is unstable. In fact all of the weights \( q_j \) are zero for these values of \( \omega \) and the scheme is undefined. However if we avoid these values of \( \omega \) then the scheme makes sense and is stable since \( Q(\xi) \) has only one (simple) root at \( \xi = 1 \) when \( \omega h \in (0, \pi) \) and none when \( \omega h = 0 \). The scheme is thus stable for all frequencies \( \omega \) in the contiguous interval \( \omega h \in [0, \pi) \).

**Case \( m = 1 \):**

\[
Q_1(\xi) = \left( \frac{1 - \cos(\omega h)}{(\omega h)^2} \right) \left( \frac{1 - \xi^2}{\xi^2 - 2\xi \cos(\omega h) + 1} \right).
\]

This is similar to the previous case: the scheme is unstable when \( \omega h = 2k\pi \) for any \( k \in \mathbb{Z}/0 \), but away from these values it has a simple root at \( \xi = -1 \) and another at \( \xi = 1 \) when \( \omega h \in (0, 2\pi) \). Hence by Lemma 3.1 the scheme is stable in the contiguous interval \( \omega h \in [0, 2\pi) \).

**Case \( m = 2 \):** This is not so straightforward since the frequency \( \omega \) and \( \xi \) terms do not decouple cleanly as they do for \( m = 0, 1 \). Here

\[
Q_2(\xi) = \frac{2(1 - \xi)(G_1(\omega h)(\xi^2 + 1) + G_2(\omega h)\xi)}{(\xi^2 - 2\xi \cos(\omega h) + 1)}
\]

where

\[
G_1(\omega h) = \frac{\omega h - \sin(\omega h)}{(\omega h)^3}, \quad G_2(\omega h) = \frac{\sin(\omega h) + \sin(\omega h) \cos(\omega h) - 2\omega h \cos(\omega h)}{(\omega h)^3}.
\]

Note that both \( G_1(x) \) and \( G_2(x) \) are equal to \( 1/6 + O(x^2) \) as \( x \to 0 \).

Lemma 3.1 is only satisfied when \( G_1(\omega h)^2 - 4G_2(\omega h)^2 < 0 \), in which case there are two simple roots with unit modulus when \( \cos(\omega h) = 1 \), and three simple roots with unit modulus when \( \cos(\omega h) \neq 1 \). The condition \( G_2(\omega h)^2 - 4G_1(\omega h)^2 < 0 \) is guaranteed for \( 0 \leq \omega h < 1.9747 \ldots \), the smallest positive solution of the nonlinear equation \( G_2(\omega h) = 2G_1(\omega h) \). The scheme is stable in the contiguous interval \( \omega h \in [0, 1.9747 \ldots) \), it is unstable for all \( 1.9747 \ldots < \omega h < \pi \), and there are alternating intervals of stability and instability for larger values of \( \omega h \).

**Case \( m = 3 \):** This result is of course more complicated. There is a complex conjugate pair of simple roots with

\[
|\xi| = 1 - \frac{(\omega h)^2}{15} + O((\omega h)^4)
\]

which satisfy Lemma 3.1 when \( \omega \) is a constant independent of \( h \). However, when \( \omega h \) is fixed the scheme is unstable in the sense of Definition 3.1. So this case is different from the \( m = 0 : 2 \) cases above since there is no interval of stability.

### 5.4 Bessel function kernel \( K(t) = J_0(\omega t) \), transform \( \tilde{K}(s) = 1/\sqrt{s^2 + \omega^2} \)

As noted in the Introduction, this is the kernel function that arises when we consider TDBIE scattering from the flat surface \( \mathbb{R}^2 \). Its Laplace transform has a branch cut between the values \( s = \pm i\omega \), and the \( Z \)-transform \( Q_m(\xi) \) of the weights is not a rational polynomial. We can still establish stability directly for the impulse response sequence \( \{p_n\} \) defined in (3.4) using a change of variable in the \( Z \)-transform inversion formula [14, eq. 37.7] to get

\[
p_n = \frac{e^{nh\sigma}}{2\pi} \int_{-\pi}^{\pi} e^{iny} \frac{Q_m(e^{-x-iy})}{Q_m(e^{-x+iy})} \, dy,
\]

where we have set \( \xi = e^{-sh} \) with \( s = \sigma + i\eta \) and \( \sigma > 0 \) and then changed to scaled variables \( x = \sigma h \) and \( y = \eta h \). This yields the bound

\[
|p_n| \leq \frac{e^{\sigma T}}{2\pi} \int_{-\pi}^{\pi} \frac{|q_n| \, dy}{|Q_m(e^{-x+iy})|},
\]

when \( t_n \leq T \), which holds for any fixed \( \sigma > 0 \) when the singularities of the integrand are to the left of \( x \). Note that this bound is independent of \( n \), and the scheme is stable at a given frequency \( \omega \) if the integral term in (5.2) remains bounded as \( h \to 0 \). This can be demonstrated using the right contour in Figure 3.1 to calculate \( Q_m(e^{-sh}) \) but it is more straightforward to work directly with (3.7).
It follows from standard properties of B-splines that
\[ q_0 = \int_0^h \left(1 - \frac{t}{h}\right)^m J_0(\omega t) \, dt = K^{(-m-1)}(h) \]
when \( \phi_j = b_j^{m-m} \), where we define functions \( K^{(-k)}(t) \) by
\[ K^{(0)}(t) = J_0(\omega t), \quad K^{(-k-1)}(t) = \int_0^t K^{(-k)}(t') \, dt' \quad \text{for } k = 0, 1, \ldots \]
Note that
\[ K^{(-m-1)}(t) \leq \frac{m+1}{m+1}! \]
for all \( t \geq 0 \).

Properties of the B-spline basis functions can also be exploited to write (3.7) as
\[ Q_m(\xi) = \frac{(1 - \xi)^{m+1}}{\xi h^m} \mathcal{Z}\{K^{(-m-1)}\}(\xi) + C_m(\xi) \] (5.5)
where the “correction” terms are \( C_0 = 0, \quad C_1 = 0, \quad C_2(\xi) = (1 - z) \frac{K^{(-3)}(h)}{h^2}, \quad C_3(\xi) = (1 - \xi)(5 - 3\xi) \frac{K^{(-4)}(h)}{h^3} + \xi(1 - \xi) \frac{K^{(-4)}(2h)}{2 h^3} \).

The presence of these terms is because for \( m = 0 \) the basis functions are pure translates, while for \( m \geq 2 \), there are different shaped basis functions at the start. The function \( K^{(-k)} \) has Laplace transform
\[ \mathcal{L}\{K^{(-k)}\}(s) = \frac{1}{s^k \sqrt{s^2 + \omega^2}} \]
and it follows from the Poisson sum formula relating \( Z \) and Laplace transforms that
\[ \mathcal{Z}\{K^{(-k)}\}(e^{-sh}) = \frac{1}{h} \sum_{j \in \mathbb{Z}} \frac{1}{s_j^2 + s_j^2 + \omega^2}, \]
where \( s_j = s + i2\pi j/h \). Thus
\[ \mathcal{Z}\{K^{(-m-1)}\}(e^{-sh}) = f^{m+1} \sum_{j \in \mathbb{Z}} f^m_j \]
where
\[ f^m(\xi) = \frac{1}{\xi^{m+1} \sqrt{\xi^2 + (\omega h)^2}} \quad \text{and} \quad f^m_j = f^m(s_j h), \]
and we use this expression in (5.5) in order to bound the integral term in (5.3).

When \( m = 0 \) it is possible to obtain an analytic bound when \( \omega \leq \pi/h \), and a careful numerical approximation of the integral (5.3) indicates that the \( p_n \) are bounded for \( \omega \) up to (at least) 20 \( \pi/h \). The situation is more complicated for \( m = 1 \) and obtaining an analytic bound does not appear to be tractable, and in these cases we give numerical bounds.

**Case m = 0**: From (5.4) and (5.5)

\[ \frac{|q_0|}{|Q_0(e^{-x+iy})|} = \frac{1}{|e^{x+iy} - 1|} \left| \sum_{k \in \mathbb{Z}} f^0_k \right| \]
and
\[ \left| \sum_{k \in \mathbb{Z}} f^0_k \right| = |f^0_0| \left( 1 + \sum_{k \in \mathbb{Z}/0} \mathfrak{R}(f^0_k/f^0_0) \right)^2 + \left( \sum_{k \in \mathbb{Z}/0} \mathfrak{I}(f^0_k/f^0_0) \right)^2 \right)^{1/2} \geq |f^0_0| \left( 1 + \sum_{k \in \mathbb{Z}/0} \mathfrak{R}(f^0_k/f^0_0) \right) \]

It can be shown that

\[ 1 + \min_y \sum_{k \in \mathbb{Z}/0} \mathfrak{R}(f^0_k/f^0_0) = 1 + \sum_{k \in \mathbb{Z}/0} \mathfrak{R}(f^0_k/f^0_0) |y=0 > \frac{2}{3} \]
when \(0 \leq x \leq 1\), \(0 \leq \omega h \leq \pi\) and \(|y| \leq \pi\). In this case
\[
\left| \frac{q_0}{Q_0(e^{-x-iy})} \right| \leq \frac{3 \sqrt{x^2 + 2\pi^2} \sqrt{x^2 + y^2}}{|e^{x+iy} - 1|} \leq \frac{3 \sqrt{x^2 + 2\pi^2}}{2} \pi e^{-x/2}
\]
using Jordan’s inequality. Together with (5.3) this proves that the scheme is stable in the sense of Definition 3.1 for frequency \(\omega\) in the contiguous interval \(0 \leq \omega h \leq \pi\). Numerical evaluation of the right hand side of (5.3) indicates that the bound is
\[
|p_n| \leq 1.3 e^{\sigma T}
\]
when \(h\) is sufficiently small (so that \(x < 0.1\)) and \(0 \leq \omega h \leq 20\pi\). Further numerical tests computing \(p_n\) directly from (3.4) for a finite number of steps \(n \leq 2500\) and the same range of values of \(\omega h\) indicate that \(|p_n| \leq 1\), consistent with the estimate above. There is no indication of instability at any value of \(\omega h\) tested and we speculate that this scheme is stable for all \(\omega\).

**Case \(m = 1\):**

Finding an explicit bound for the integral in (5.3) is significantly more complicated and perhaps even intractable here so we only consider its direct numerical evaluation over a range of frequencies and values of \(z = h\sigma\) close to 0. However there is an extra complication because \(1/Q_1(z)\) has a pole at \(z = -1\). This is most obvious when we set \(\omega = 0\) and get \(1/Q_1(z) = (1 - z)/(1 + z)\) from (5.1). When \(\omega \neq 0\) there is no simple formula, but it is still possible to show by direct evaluation of the summation formula for \(Q_1(e^{-x-iy})\) that the pole remains when \(0 < \omega h < \pi\). The pole renders the bound in (5.3) less useful since
\[
\int_{-\pi}^{\pi} \frac{dy}{|Q_1(e^{-x-iy})|} = O(\log(1/x)) \rightarrow \infty \quad \text{as} \quad x \rightarrow 0
\]
(where \(x = h\sigma\) and hence
\[
|p_n| \leq (C_0 + C_1 \log(1/h)) e^{\sigma T}
\]
as \(h \rightarrow 0\), which does not satisfy the stability requirement of Definition 3.1. It could still be useful in convergence analysis for TDBIE approximations, but would give a suboptimal convergence rate reduced by the amount \(\log h\).

Fortunately the singularity can be removed by writing
\[
\frac{1}{Q_1(\xi)} = \frac{a}{(1 + \xi)} + \Delta P(\xi), \quad \text{where} \quad a = \lim_{\xi \rightarrow -1} Q_1(\xi)(1 + \xi)
\]
so that \(\Delta P(\xi)\) is bounded as \(\xi \rightarrow -1\). The sequence \(\{p_n\}\) can then be written as
\[
p_n = aq_0 (-1)^n + \Delta p_n, \quad n = 0, 1, \ldots
\]
where \(\Delta p_n\) is bounded in the same way as (5.3)
\[
|\Delta p_n| \leq \frac{q_0}{2\pi} \int_{-\pi}^{\pi} |\Delta P(x + iy)| dy.
\]
Numerical evaluation of the integral over frequencies \(0 \leq \omega h < \pi\) indicates that
\[
|\Delta p_n| \leq 1.1 \quad \text{and} \quad 0 < aq_0 \leq 2
\]
for \(nh \leq T\) and \(0 < x \leq 1/10\). Combining this with direct evaluation of (5.3) when \(\omega h \in [\pi, 20\pi]\) and there is not a pole indicates that
\[
|p_n| \leq Ce^{\sigma T} \quad \text{where} \quad C = \left\{\begin{array}{ll}
3.1, & \omega h \in [0, \pi) \\
1.1, & \omega h \in [\pi, 20\pi]
\end{array}\right.
\]
for \(nh \leq T\) and \(0 < x \leq 1/10\) satisfying stability Definition 3.1. Further numerical tests computing \(p_n\) directly from (3.4) for a finite number of steps \(n \leq 2500\) and the same range of values of \(\omega h\) indicate that \(|p_n| \leq 2\) for \(\omega h \in [0, 0.7\pi]\) and \(|p_n| \leq 1\) for \(\omega h \in [0.7\pi, 20\pi]\), consistent with the estimate above. Again we speculate that this scheme is stable for all \(\omega\).

**Case \(m = 2\):** The function \(1/Q_2(\xi)\) appears to have two poles on the unit circle when \(\omega h \in [0, L]\) where \(L \approx 2.55\), symmetrically located at \(\xi = e^{\pm i\mu(\omega h)}\). In the simple case \(\omega = 0\), (5.1) gives \(Q_2(\xi) = (1 + \xi + \xi^2)/(1 - \xi)/3\), and so \(\mu(0) = 2\pi/3\). Numerical evidence indicates that \(\mu(\omega h) > \omega h\) and that \(\mu\)
increases until the two poles meet where \( \mu(L) = \pi \). At that point stability in the sense of Definition 3.1 breaks down since there does not appear to be any compensating factor in the numerator to reduce the order of this double singularity.

We locate the poles numerically, and remove them from the integrand \( 1/Q_2(\xi) \) in a similar way to the previous case. The simplest form that captures the main features of the behaviour is

\[
1/Q_2(\xi) = \frac{a(1 - \xi)}{\xi^2 - 2\xi \cos \mu + 1} + \Delta P(\xi),
\]

so that by direct inversion of the Z transform

\[
p_n = aq_0(\cos(n\mu) - \sin(n\mu) \tan(\mu/2)) + \Delta p_n.
\]

For \( 0 \leq \omega h \leq L \approx 2.55 \) we find that \( 0 < aq_0 \leq 1 \) and from (5.6) that \( |\Delta p_n| \leq 0.8 \), giving

\[
|p_n| \leq 0.8 + \sec(\mu(\omega h)/2)
\]

for \( nh \leq T \) when \( 0 < x \leq 1/10 \). This satisfies the stability Definition 3.1 since \( 2\pi/3 \leq \mu(\omega h) < \pi \), but since \( \sec(\mu/2) \to \infty \) as \( \mu \to \pi \), the possibility for instability is clear. Further numerical tests computing \( p_n \) directly from (3.4) for a finite number of steps show very close and consistent agreement with this bound on \( |p_n| \), with instability appearing as predicted at \( \omega h = L \approx 2.55 \).

In summary, we have a contiguous interval of stability \( \omega h \in [0, L) \) with \( L \approx 2.55 \).

**Case \( m = 3 \):** Not surprisingly this case is more complicated still. It appears that \( 1/Q_3(\xi) \) has three poles. When \( \omega = 0 \) the explicit formula (5.1) gives \( Q_3(\xi) = (1 + \xi)(1 + \xi^2)/(1 - \xi)/4 \), and so the three poles of \( 1/Q_3(\xi) \) are on the unit circle at \( \xi = -1, e^{\pm i\pi/2} \). However, when \( \omega h > 0 \) increases, the real-valued pole at \( \xi = -1 \) moves (harmlessly) outside the unit circle while the other complex conjugate pair moves inside causing instability in the same way as for the \( \cos \omega t \) kernel. Further numerical tests computing \( p_n \) directly from (3.4) for fixed values of \( \omega h \) show behaviour consistent with this; we see apparent stability for larger values of \( h \) which disappears as \( h \to 0 \). This scheme is stable only when \( \omega \) is fixed (so that \( \omega h \to 0 \)), and not when \( \omega h \) is fixed as required in TDBIE analysis and calculations.

## 6 Conclusions

We have derived a new “convolution spline” approximation for (1.4). Key properties of the method are the backward time aspect (2.9), that the basis functions are (mainly) translates, and satisfy the sum to unity property (2.12). We have shown that this method based on B-splines of degree \( m \geq 1 \) is second order accurate when applied to a sufficiently smooth Volterra integral (the \( m = 0 \) case has previously been shown to first order).

The analysis of section 5.4 gives a stability bound for the \( p_n \) coefficients for this approximation of the Bessel function kernel VIE. This means that the convergence proof in [11] for the TDBIE (1.1) on \( \Gamma = \mathbb{R}^2 \) can be applied to an approximation which is a Fourier interpolant in space, without the need to impose an additional (essentially uncheckable) stability assumption. This proof, together with the compact support of the B-spline basis functions makes this type of approximation an attractive prospect for time-stepping the TDBIE (1.1). However, the result in Section 5.2 that only the \( m = 0 \) approximation is stable for a discontinuous kernel is potentially problematic, because the TDBIE kernel is not smooth (it is likely to be continuous but may have infinite derivatives). As an illustration consider the ‘cartoon’ TDBIE problem for which \( \Gamma = (0, 1)^2 \) which is approximated in space by a piecewise constant function on a single spatial element. This can be written as the VIE (1.4) where the kernel is \( K_C \) for a collocation scheme (at the midpoint of \( \Gamma \)) and \( K_G \) for a Galerkin scheme, where

\[
2\pi K_C(t) = \begin{cases} 
\pi, & 0 < 2t \leq 1 \\
\pi - 4 \cos^{-1} \left( \frac{1}{2t} \right), & 1 < 2t \leq \sqrt{2} \\
0, & 2t > \sqrt{2}
\end{cases}
\]

\[
2\pi K_G(t) = \begin{cases} 
\pi + t^2 - 4t, & 0 < t \leq 1 \\
\pi - 4 \cos^{-1} \left( \frac{1}{2t} \right) - 2t + 4\sqrt{t^2 - 1}, & 1 < t \leq \sqrt{2} \\
0, & t > \sqrt{2}
\end{cases}
\]

Numerical tests indicate that the kernel \( K_C \) is smooth enough for second order convergence of the schemes with \( m = 1 : 3 \), but only the \( m = 0 \) scheme is stable for the kernel \( K_G \). Whilst this should make it possible to use these methods to time-step (1.1) in the Galerkin case, this is not ideal because it is a very complicated method to implement for all but the simplest surfaces \( \Gamma \) (it involves approximating
integrals on subregions of $\Gamma \times \Gamma$. However, the framework of Section 3 applies to any approximation of the form (2.9) for which the basis functions are (essentially) translates, and there are other choices of basis functions which seem to give very stable approximations of (1.1), even in the collocation case. Development and testing of these methods is work in progress.

References