Overlapping Grids for Welltest Analysis

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Abstract. The use of approximations based on overlapping grids is examined for the solution of time dependent diffusion problems arising in welltest analysis, which is used to determine large scale properties of oil and gas reservoirs. Results are shown for a 2D model reservoir. Analysis of an overlapping grid scheme for a 1D model problem is carried out and the scheme is shown to satisfy a discrete maximum principle, conserve mass, be stable and to converge. More general results and future work are outlined.

1. Introduction

Welltest analysis is used to determine large scale properties of oil and gas reservoirs, and in particular to determine the likely productivity and the best way to achieve it. In its simplest form a welltest consists of shutting-in the reservoir (that is closing the valve on the wellbore) long enough for the reservoir to reach a uniform steady pressure distribution. Then the valve is opened and the flow rate and pressure at the output from the reservoir are recorded to give a time history of the pressure drop at the output as oil and/or gas flow out. The data are usually processed using Laplace transforms or some similar deconvolution method to normalise the flow rate and produce a pressure transient signature for the reservoir at constant outflow rate. Real tests can be more complicated, but the general idea is to obtain a pressure transient signature for a constant flow rate. General introductions to welltest analysis are given in [13, 19, 21].

Remarkably, experienced engineers can often deduce a great deal of information about the reservoir from this pressure transient signature, particularly when it is combined with other information like seismic surveys. Different types of formations can produce distinctive signatures, and some bulk properties can be deduced from various presentations of the pressure transient graph [2, 21]. This may also include examination of the rate of change of pressure with time or its logarithmic derivative as shown in Figure 2. Further quantitative information can be obtained by fitting the solution of a mathematical model of the pressure behaviour to the measured data.

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data from the real reservoir, where the model parameters to be fitted represent quantities such as diffusivity and formation thickness.

Almost all models used in welltest analysis involve the linear diffusion equation for pressure, derived using Darcy’s law relating pressure gradient to fluid velocity. Even models of pressure in gas reservoirs, which one would expect to be nonlinear, are transformed via some approximations to a linear diffusion equation for a nonlinear function of pressure. There are of course many factors to be accounted for in welltest analysis. For example the reservoir may be heterogeneous with layers and faults in the rock and the permeability of the rocks may be anisotropic because of way the layers have formed. The model may be modified to deal with drill damage close to the wellbore by use of a “skin factor” [2, 21], and to deal with the flow through the wellbore itself (often called “wellbore storage”) [10].

A standard welltest modelling problem in a heterogeneous, isotropic reservoir with an undamaged wellbore is then to find the pressure \( p(x,t) \) that solves

\[
\frac{\partial p}{\partial t} = k \Delta p, \quad x \in \Omega \\
p_n = 0, \quad x \in \partial\Omega_0, \quad p_n = F, \quad x \in \partial\Omega_1 \\
p(x,0) = p_0
\]

where \( p_0 \) is the initial uniform pressure, \( k \) is the diffusivity (constant in this case), \( \Omega \) is the region occupied by the reservoir, \( \partial\Omega_0 \) is the outer boundary of the reservoir (assumed impermeable), \( n \) is the inward normal direction and there is a constant flux \( F \) through the wellbore surface \( \partial\Omega_1 \). The wellbore is a relatively thin (radius \( O(10) \) cm) tubular hole cut into the reservoir rock and \( \partial\Omega_1 \) is the surface of that tube. The main quantity of interest is the pressure and its time derivative in the vicinity of the gauge down the wellbore as a function of time. Knowledge of the pressure field elsewhere is usually not required. Results for a model problem are shown in Figures 1 and 2.

When the geometry is suitable, then analytical techniques such as Green’s functions and the method of images can produce the required solutions of the linear diffusion equation [5]. Recently boundary integral techniques have also been used [10]. Despite the obvious geometric limitations, specialised commercial software packages based on these techniques and tailored to welltest analysis have been developed, and are used throughout the oil industry to great effect. These analytical models are used in conjunction with parameter fitting (optimization) algorithms to obtain estimates of the parameters of interest. The whole fitting process typically takes only a few minutes to run on a 200 MHz Pentium PC.

When the geometry is not suitable for analytical techniques, then solution of the diffusion equation by various numerical methods is also used in commercial welltest analysis packages. So far this appears to be restricted to one-off direct calculations with known parameters and geometry, since the computational costs of combining this with parameter fitting are currently too high for the target market. Demand for the analysis of more complicated reservoir structures and the availability of increasingly more cost effective computing power will almost certainly lead to parameter fitting full numerical models for general reservoir geometries becoming a standard feature in commercial packages. One of our aims here is to develop numerical approximation techniques that are suited to this purpose.

In the rest of this paper we outline why overlapping grids are particularly suited to this problem. We then illustrate how properties of overlapping grid schemes for
the diffusion equation can be obtained by analysis of a simple model problem and indicate how this can be generalised to more realistic situations.

2. Overlapping grids

In order to solve the partial differential equation (1) within the complicated geometry of a typical reservoir, we propose to use uniform grid PDE approximations on overlapping grids. See \[4, 8, 11, 20\] for similar examples involving diffusion. Overlapping or overset grids are perhaps the most general form of domain decomposition. Domain decomposition is commonly used to tear apart a non-uniform finite element mesh into regions that join together without overlapping. The overlapping grid form of domain decomposition is much less common, but it has been developed for a variety of very good reasons. The most important reasons in the welltest analysis context are listed below:

- When features like wellbores are introduced or repositioned the rest of the grids do not change, unlike a global finite element grid which would have to be built again from scratch to fit the new geometry (see for example \[12\]). This is particularly important when well or fault positions are parameters to be found by matching the model solution to the measured pressure time history.
- It allows the modular construction of the computational domain from a simple library of components, rather than using a complicated global finite element meshing procedure (see for example \[6, 7, 9, 22\]).
- When the component grids are uniform or quasi-uniform (that is they can be mapped onto a uniform grid by a change of variables), then extremely efficient solution procedures can be used on each independently. For diffusion problems, alternating direction implicit (ADI) methods may well give the best performance on these grids because the linear algebra required to step forward in time is very simple (see for example \[16\] for some recent work involving coordinate transformation and ADI methods and \[15\] for work in the context of welltesting).
- Different time steps and space mesh sizes can be used on each of the component grids. This allows regions of very different types of rocks to be treated more efficiently since long timesteps can be used where things change slowly, with higher resolution provided where required (for example, near the wellbore). This has been used in \[24\] for parabolic problems and in \[3\] for seismic problems.

Figure 1 gives a schematic illustration of overlapping grids for a 2D model reservoir containing a vertical well and a fault. A uniform grid is used in the reservoir and the outer boundary is modelled following the Cartesian grid approach of \[14\]. A uniform polar grid with a logarithmic scaling in the radial direction is used around the well and a uniform grid is used around the fault. Information is passed between the grids using interpolation at their edges. The pressure time history at the wellbore for problem (1) in this reservoir is shown in Figure 2.

3. Analysis of a 1D overlapping grid approximation

To illustrate the main points, we examine a simple 1D model problem approximated on two overlapping sections of uniform space grid, using linear interpolation
to pass information between the grids. The problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, 1) \text{ and } 0 \leq t \leq T,$$

with Neumann boundary conditions $u_x(0, t) = u_x(1, t) = 0$, and given initial data $u(x, 0) = u^0(x)$. We only consider semi-discrete approximations here. Analogous results are obtained for appropriate fully discrete approximations.
The two overlapping sections of the space grid have nodes \( \{x_j^{[\alpha]}, j = 0: N^{[\alpha]}, \alpha = 1, 2\} \) where

\[
x_j^{[1]} = jh, \quad j = 0: N^{[1]} \quad \text{and} \quad x_j^{[2]} = 1 + (j - N^{[2]})h, \quad j = 0: N^{[2]}.
\]

The grid sections overlap in the sense that \( x_0^{[2]} < x_0^{[1]} \) with

\[
x_{N^{[1]}}^{[1]} - x_0^{[2]} = (1 + \varepsilon)h, \quad \varepsilon \in (0, 1),
\]

where \( \varepsilon \) measures the offset between the grids. The approximate solution at the nodes is

\[
u_j^{[\alpha]}(t) \approx u(x_j^{[\alpha]}, t)
\]

for \( j = 0: N^{[\alpha]}, \alpha = 1, 2 \). Figure 3 shows a schematic representation of locations of the mesh points and solution components.

The standard finite difference approximation of the problem on this composite grid is given by

\[
\dot{u}_j^{[\alpha]} = \frac{1}{h^2} \left( u_{j-1}^{[\alpha]} - 2u_j^{[\alpha]} + u_{j+1}^{[\alpha]} \right)
\]

for \( j = 1: N^{[\alpha]} - 1, \alpha = 1, 2 \) where the dot is shorthand for the time derivative. The boundary conditions are approximated by the standard second order formulae

\[
u_0^{[1]} = \frac{1}{h^2} \left( 2u_1^{[1]} - 2u_0^{[1]} \right) \quad \text{and} \quad \dot{u}_{N^{[2]}}^{[2]} = \frac{1}{h^2} \left( 2u_{N^{[2]}-1}^{[2]} - 2u_{N^{[2]}}^{[2]} \right).
\]

Information is passed between the approximate solutions on the two grids by linear interpolation

\[
u_{N^{[1]}}^{[1]} = (1 - \varepsilon)u_0^{[2]} + \varepsilon u_0^{[1]} \quad \text{and} \quad u_{N^{[1]}}^{[2]} = (1 - \varepsilon)u_{N^{[1]}-1}^{[1]} + \varepsilon u_{N^{[1]}-2}^{[1]}.
\]

Note that the coefficients \( \varepsilon \) and \( 1 - \varepsilon \) are positive from (2).

The solution components \( u_{N^{[1]}}^{[1]}, u_0^{[2]} \) can of course be eliminated from the calculations. This leaves the system of ODEs

\[
\dot{u} = Bu
\]
where

\[ B = \frac{1}{h^2} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & -2 & 1 & 0 & 0 \\ 0 & \varepsilon & 0 & \varepsilon & 0 & 0 \\ -2 & 0 & -2 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & \varepsilon & 0 & \varepsilon \\ 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \]

and \( \mathbf{u} = (u_0^1, u_1^1, \ldots, u_{N-1}^1, u_1^2, u_2^2, \ldots, u_{N-1}^2)^T \).

**Discrete maximum principle.** It is relatively easy to show that this linear system of ODEs has the property that if all components of \( \mathbf{u}(t) \) are nonnegative at \( t = 0 \), then they remain nonnegative for all \( t > 0 \). The fact that the coefficients in the interpolation (3) are positive and sum to 1 is important in proving this. Further, since (4) is invariant under the change of dependent variables \( \mathbf{u} \mapsto \pm \mathbf{u} + C \mathbf{1} \) where \( \mathbf{1} \) is the vector of 1’s, the non-negativity property can be used twice to show that the elements of \( \mathbf{u}(t) \) are bounded above and below by the maximum and minimum elements of \( \mathbf{u}(0) \), and hence

\[ \| \mathbf{u}(t) \|_{\infty} \leq \| \mathbf{u}(0) \|_{\infty} \]

for all \( t \geq 0 \). Thus the approximation satisfies a maximum principle analogous to that for the exact problem [17, Ch. 2.11].

**Mass conservation.** Locally in each subgrid the approximation is mass conserving, but mass may be gained or lost through the interpolation conditions (3) (see [18]). The situation is complicated in the overlapped regions where the approximate solution has two alternative values. However, it is possible to show that the approximation above satisfies a generalised mass conservation condition

\[ \frac{d}{dt} (\mathbf{w}^T \mathbf{u}) = 0 \]

where vector \( \mathbf{w} \) is independent of the mesh size \( h \) and is a left null-vector of \( B \) (that is \( \mathbf{w}^T B = 0^T \)). Thus \( \mathbf{w}^T \mathbf{u} \) can be taken to be proportional to the mass in this approximation.

It is easy to see that \( B \) has left and right null-vectors since by inspection \( B \mathbf{1} = 0 \) implying that \( B \) is singular. In fact the vector \( \mathbf{w} \) can be found explicitly (up to a multiplicative constant) for this example by solving the difference equations defined by \( B^T \mathbf{w} = 0 \). This gives

\[ \mathbf{w} = \begin{pmatrix} \frac{1}{2}, 1, \ldots, 1, \frac{1}{2}, 1, \ldots, 1, 1 \end{pmatrix}^T \]

\[ = \begin{pmatrix} w_0^1, w_1^1, \ldots, w_{N-1}^1, w_1^2, w_2^2, \ldots, w_{N-1}^2 \end{pmatrix}^T \]

using the same labelling convention for elements of \( \mathbf{w} \) as for \( \mathbf{u} \). Note that the uniform single grid case has \( \mathbf{w} = (1/2, 1, \ldots, 1, 1/2)^T \) and is obtained by setting \( \varepsilon = 0 \) above.

**\( l_2 \) stability.** We can use the properties of \( B \) and \( \mathbf{w} \) to obtain \( l_2 \) stability results for the semi-discrete approximation. Similar methods are used in [1]. First we define the vector norm

\[ \| \mathbf{u} \|_{W, l_2} = \sqrt{h \mathbf{u}^T W \mathbf{u}} \]
where $W = \text{diag}(w)$. Premultiplying the ODE (4) by $u^T W$ we get

$$u^T W \dot{u} = u^T W B u$$

and then rearranging and taking the symmetric part of $WB$,

$$\frac{d}{dt} \|u\|_{W,l_2}^2 = h u^T (WB + B^T W) u.$$ 

The symmetric matrix $M = WB + B^T W$ has negative diagonal elements $m_{j,j} < 0$ and off-diagonal elements $m_{j,k} \geq 0$, $k \neq j$. It also has the property $M_1 = 0$ since

$$M_1 = (WB + B^T W)_1 = WB_1 + B^T w = 0$$

which follows from $B_1 = 0$ and the definition $W = \text{diag}(w)$. It is then easy to show that $M$ is negative semidefinite and so $WB$ is negative semidefinite. Hence

$$\frac{d}{dt} \|u(t)\|_{W,l_2} \leq \|u(0)\|_{W,l_2},$$

which is equivalent to stability in the discrete $l_2$ norm.

$l_2$ convergence. Now consider the behaviour of errors. We define the vector of nodal errors $\varepsilon$ componentwise by

$$e_j^{[0]}(t) = u_j^{[0]}(t) - u(x_j^{[0]}, t), \quad j = 0: N^{[0]} , \alpha = 1, 2.$$ 

Then it is straightforward to show that

$$\dot{\varepsilon} = B\varepsilon + \tau$$

where $\tau$ is the local truncation error vector defined by

$$\tau_j^{[0]} = \frac{h^2}{12} \frac{1}{u_{xxx}(x_j^{[0]}, t)} + \begin{cases} O(h^4), & j = 0: N^{[0]} - 2, \alpha = 1 \\
\left(\varepsilon - \varepsilon^2\right) u_{xx}(x_j^{[0]}, t) + O(h), & j = N^{[0]} - 1, \alpha = 1 \\
\left(\varepsilon - \varepsilon^2\right) u_{xx}(x_j^{[0]}, t) + O(h), & j = 1, \alpha = 2 \\
O(h^4), & j = 2: N^{[0]}, \alpha = 2 \end{cases}.$$ 

In other words we have the standard $O(h^2)$ uniform grid truncation error at the node points away from the overlap region, and an $O(1)$ truncation error at the two overlap points. This apparently catastrophic error comes about because the error in the linear interpolation formulae (3) is $O(h^2)$ and is divided by $h^2$ in the second central difference approximation, resulting in an error of size $O(1)$. However, using the fact that $WB$ is negative semidefinite and the definition of $\|\cdot\|_{W,l_2}$,

$$h \varepsilon^T W \dot{\varepsilon} = \frac{1}{2} \frac{d}{dt} \|\varepsilon\|_{W,l_2}^2 = h \varepsilon^T W B \varepsilon + h \varepsilon^T W \tau \leq \|\varepsilon\|_{W,l_2} \|\tau\|_{W,l_2} \leq \|\varepsilon\|_{W,l_2} (C_\varepsilon h^{1/2} + O(h))$$

where $C_\varepsilon$ depends only on $\varepsilon$. Finally

$$\frac{d}{dt} \|\varepsilon\|_{W,l_2} \leq C_\varepsilon h^{1/2} + O(h)$$

giving $\|\varepsilon\|_{W,l_2} = O(h^{1/2})$ as $h \to 0$ on the finite time interval $t \in [0, T]$. 

4. Generalisations and further work

We are exploring generalisations of the analyses outlined in the previous section to more than one space dimension and to more general interpolation formulae. For example, the $l^2$ convergence result for the 1D example above is $||e||_{W,l^2} = O(h^{\min((2s-1)/2,2)})$ where $s = 1$ for linear interpolation, $s = 2$ for quadratic etc.. However, our analysis is not optimal since numerical experiments (see Figure 4) indicate that the order attained is actually $O(h^{\min(s,2)})$, which is $O(h^{1/2})$ better than we have obtained above. Where it is possible, a more detailed analysis like that in [23, Ch. 3.10] resolves this problem.

We are currently using 2D overlapping grids for welltest simulation (see Figures 1 and 2) and are developing the analysis of the 2D situation. One experimental measurement of the errors introduced by interpolation between pieces of grid in 2D is described below. We consider the problem

$$u_t - \Delta u, x \in [0,1]^2$$

with Neumann boundary conditions and given initial data. The exact solution is obtained by separation of variables. The grid for approximation is in two pieces, each locally uniform with the same sized square mesh. The larger grid is a standard uniform square mesh that fits the unit square, but with a hole cut in its interior. The smaller grid piece covers the hole, but is aligned at a different angle to the larger grid. The grids are connected together by interpolation at their edges. Results are shown in Figure 4, and indicate that the 1D and 2D convergence rates are the same.

We have not yet explored the use of linear algebra solvers designed specifically for domain decomposition problems in the fully discrete approximation of (1). In the 2D problems described above, the direct sparse solvers in Matlab were fast enough. However, we are moving onto 3D problems where the choice of linear solver will be much more important.

5. Conclusions

We have shown how overlapping grid schemes appear well-suited to use in welltest analysis problems and demonstrated that, at least in simple cases, they have the desirable properties of mass conservation, a maximum principle, stability and convergence. There is further work to do to extend the results to more general cases and to establish if there is a real advantage over more standard finite element methods for typical welltest problems.

References

Figure 4. Comparison of results for the 1D problem of Section 3 and the 2D problem of Section 4, showing $O(h^s)$ convergence where $s = 1$ for linear interpolation and $s = 2$ for quadratic.


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