Logic and Proof course
Solutions to exercises from chapters 10 and 11

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10.1 (a) $\{m \in \mathbb{R} | -10 \leq m \leq 10 \wedge m \neq 0\}$.
(Set $m : m \in \mathbb{R} \wedge -10 \leq m \leq 10 \wedge m \neq 0 : m$).

(b) $\{m \in \mathbb{N} | x \bmod 2 \neq 0\}$.
(Set $m : m \in \mathbb{N} \wedge x \bmod 2 \neq 0 : m$).

(c) $\{m \in \mathbb{Z} | x \bmod 6 = 0\}$.
(Set $m : m \in \mathbb{N} \wedge x \bmod 6 = 0 : m$).

(d) $\{k \in \mathbb{N} | \exists_{m,n}[(m,n) \in \mathbb{N} \times \mathbb{N} : k = m^2 + n^2]\}$ or $\{m^2 + n^2 | (m,n) \in \mathbb{N} \times \mathbb{N}\}$
(Set $k : \exists_{m,n}[(m,n) \in \mathbb{N} \times \mathbb{N} : k = m^2 + n^2] : k$) or
(Set $m, n : (m,n) \in \mathbb{N} \times \mathbb{N} : m^2 + n^2$).

10.2 Ignore.

10.3 (a) 4950.

(b) 1

(c) Left to reader.

(d) Left to reader.

10.4 (a) The scope of $\forall_m$ is everything behind it.
The scope of $\forall_n$ is the part $[n \in \mathbb{N} : 3m + n > 3]$.

(b) a) Scope of $\forall_m$ is everything behind it.
b) Scope of $\exists_m$ is everything behind it.
c) Scope of $\forall_x$ is everything behind it. Scope of $\exists_y$ is $[y \in \mathbf{R} : y > x^2]$.
d) Scope of $\forall_m$ is $[m \in \mathbf{N} \Rightarrow m^2 > m]$.

11.1 (a) Let g be the weather is good, l be Charles will go to the lake, a be Charles will go to his aunt, s be Charles will go to the sea, and f be Charles will go fishing. Then we have: $g \Rightarrow l \wedge \neg a$ and $l \vee s \Rightarrow f$.

(b) We prove (1) as follows:

{ Hypothesis}
(1) $g \Rightarrow l \wedge \neg a$
{ Hypothesis}
(2) $l \vee s \Rightarrow f$
{Assume }
(3) g
{By (1) and (3)}
(4) $l \wedge \neg a$
{By (4)}
(5) $\neg a$

We prove (1) as follows:
{ Hypothesis}
(1) $g \Rightarrow l \wedge \neg a$
{ Hypothesis}
(2) $l \vee s \Rightarrow f$
{Assume }
(3) g
{By (1) and (3)}
(4) $l \wedge \neg a$
{By (4)}
(5) l
{By (5)}
(6) $l \vee s$
{By (6) and (2)}
(7) f

11.2 Ignore.

11.3 (a) We prove this as follows:
{ Assume}
(1) m and n are uneven
{By definition}
(2) $m = 2k + 1$ and $n = 2k' + 1$
{By arithmetic}
(3) $mn = (2k + 1)(2k' + 1) = 2(2kk' + k + k') + 1$
{By definition}
(4) mn is uneven

(b) Left as an exercise.

(c) We prove this as follows:
{ Assume}
(1) m is a multiple of 6
{By definition and arithmetic}
(2) $m = 6k = 3(2k)$
{By definition}
(3) m is a multiple of 3

11.4 (a) True. We prove this as follows:

{ Assume}

(1) p is even and $p \neq 2$

{ By hypothesis, }

(2) $p = 2k$ and $k \neq 1$.

{ By arithmetic}

(3) $1, 2$ and p are divisors of p

{By definition}

(4) p is not prime.

(b) False. 2, 3, 5 are all prime but 2+3+5 = 10 is not prime.

(c) True. We prove this as follows:

{ By Arithmetic}

(1) 29 = 1+4x7

{By Arithmetic}

(2) 29 is prime

{ Logic}

(3) $\exists_x[x$ is prime $\wedge\, x$ is 1 plus a multiple of seven $]$

11.5 (a) Ignore.

(b) We prove one side as follows:

{ Assume}

(1) $x = 1$

{By Arithmetic}

(2) $x^3 + x^2 - x - 1 = 0$

We prove the other side as follows:
$x = -1$ is a solution to $x^3 + x^2 - x - 1 = 0$.

11.6 We prove as follows:

{By Arithmetic}

(1) $(2n + 1)^2 = 4n(n + 1) + 1$

{ By Arithmetic}

(2) Case $n = 2k$ is even:
$(2n + 1)^2 = 4n(n + 1) + 1 = 8k(n + 1) + 1$ is 1 plus a multiple of 8.

{By Arithmetic}

(3) Case $n = 2k + 1$ is uneven:
$(2n + 1)^2 = 4n(2k + 2) + 1 = 8n(k + 1) + 1$ is 1 plus a multiple of 8.

(4) So, in all cases, $(2n + 1)^2$ is 1 plus a multiple of 8.

11.7 (a) We prove this as follows:

{ Assume}

(1) $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded

{By definition}

(2) there are M, M' such that $|a_n| < M$ and $|b_n| < M$ for all $n \in \mathbb{N}$

{ By Arithmetic}

(3) $|a_n + b_n| < |a_n| + |b_n| < M + M'$

{By definition}

(4) $(a_n + b_n)_{n\in\mathbb{N}}$ is bounded.

(b) Ignore.