Logic and Proof course
Solutions to exercises from chapters 8 and 9

Fairouz Kamareddine

8.4 (a) $\forall_x[x \in \mathbf{D} : x \neq 0]$

(b) $\forall_x[x \in \mathbf{D} : x > 10 \wedge x < 25]$

(c) $\forall_x[x \in \mathbf{D} : x > 10] \vee \forall_x[x \in \mathbf{D} : x < 25]$

(d) $\forall_{x,y}[(x,y) \in \mathbf{D} : x \neq y \Rightarrow |x-1| > 1]$

8.5 (a) $\exists_m[m \in \mathbb{Z} : m > 1000]$

(b) $\exists_m[m \in \mathbb{Z} : m = m^2]$

(c) $\exists_m[m \in \mathbb{R} : m > 0] \wedge \exists_m[m \in \mathbb{R} : m < 0]$

(d) $\neg\exists_m[m \in \mathbb{R} : m > 0 \wedge m < 0]$

8.6 (a) $\forall_n[n \in \mathbb{N} : \exists_m[m \in \mathbb{R} : m - n = 1]]$

(b) $\neg\exists_n[n \in \mathbb{N} : \forall_m[m \in \mathbb{R} : n > m]]$

(c) $\forall_n[n \in \mathbb{N} : \forall_m[m \in \mathbb{R} : (m + n \geq m) \wedge (m + n \geq n)]]$

(d) $\exists_n[n \in \mathbb{N} : \exists_m[m \in \mathbb{R} : m^2 + n^2 = 58]]$
Alternatively: $\exists_{m,n}[(m,n) \in \mathbb{R} \times \mathbb{N} : m^2 + n^2 = 58]$

9.3 (a) As follows

$$
\begin{array}{ll}
 & \forall_x[P : \mathtt{True}] \\
\stackrel{val}{=\!=} & \quad \{\text{ 'exchange trick'}\} \\
 & \forall_x[\neg\mathtt{True} : \neg P] \\
\stackrel{val}{=\!=} & \quad \{\text{ Inversion}\} \\
 & \forall_x[\mathtt{False} : \neg P] \\
\stackrel{val}{=\!=} & \quad \{\text{ Empty Domain}\} \\
 & \mathtt{True}
\end{array}
$$

(b) As follows

$$
\begin{array}{ll}
 & \forall_x[P : \mathtt{False}] \\
\stackrel{val}{=\!=} & \quad \{\text{ 'exchange trick'}\} \\
 & \forall_x[\neg\mathtt{False} : \neg P] \\
\stackrel{val}{=\!=} & \quad \{\text{ Inversion}\} \\
 & \forall_x[\mathtt{True} : \neg P] \\
\stackrel{val}{=\!=} & \quad \{\text{ Remark 8.4.1}\} \\
 & \forall_x[\neg P]
\end{array}
$$

(c) As follows

$$
\begin{array}{ll}
 & \exists_x[P : \texttt{True}] \\
\overset{val}{=\!=} & \quad \{ \text{'exchange trick'}\} \\
 & \exists_x[\texttt{True} : P] \\
\overset{val}{=\!=} & \quad \{ \text{Remark 8.4.1}\} \\
 & \exists_x[P]
\end{array}
$$

(d) As follows

$$
\begin{array}{ll}
 & \exists_x[P : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{'exchange trick'}\} \\
 & \exists_x[\texttt{False} : P] \\
\overset{val}{=\!=} & \quad \{ \text{Empty Domain}\} \\
 & \texttt{False}
\end{array}
$$

9.4 (a) As follows

$$
\begin{array}{ll}
 & \forall_x[P \vee Q : \neg P] \\
\overset{val}{=\!=} & \quad \{ \text{Negation } \} \\
 & \forall_x[P \vee Q : P \Rightarrow \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{Domain Weakening}\} \\
 & \forall_x[(P \vee Q) \wedge P : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{True/False elimination}\} \\
 & \forall_x[(P \vee Q) \wedge (P \vee \texttt{False}) : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{Distributivity}\} \\
 & \forall_x[P \vee (Q \wedge \texttt{False}) : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{True/False elimination}\} \\
 & \forall_x[P \vee \texttt{False} : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{True/False elimination}\} \\
 & \forall_x[P : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{\text{9.3 b)}\} \\
 & \forall_x[\neg P]
\end{array}
$$

(b) As follows

$$
\begin{array}{ll}
 & \forall_x[P \wedge Q : \neg P] \\
\overset{val}{=\!=} & \quad \{ \text{Negation } \} \\
 & \forall_x[P \wedge Q : P \Rightarrow \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{Domain Weakening}\} \\
 & \forall_x[P \wedge Q \wedge P : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{ \text{Idempotence}\} \\
 & \forall_x[P \wedge Q : \texttt{False}] \\
\overset{val}{=\!=} & \quad \{\text{9.3 b), De Morgan }\} \\
 & \forall_x[\neg P \vee \neg Q]]
\end{array}
$$

(c) Left as an exercise.

(d) Left as an exercise.

9.5 (a) As follows

$$
\begin{aligned}
& \exists_k[P : Q] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Double negation } \} \\
& \exists_k[P : \neg\neg Q] \\
\stackrel{val}{=\!=} & \quad \{ \text{ De Morgan} \} \\
& \neg\forall_k[P : \neg Q] \\
\stackrel{val}{=\!=} & \quad \{ \text{ 'Exchange trick'} \} \\
& \neg\forall_k[\neg\neg Q : \neg P] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Double negation } \} \\
& \neg\forall_k[Q : \neg P]
\end{aligned}
$$

(b) As follows

$$
\begin{aligned}
& \forall_k[P : Q \vee R] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Double negation } \} \\
& \forall_k[P : \neg\neg(Q \vee R)] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Negation } \} \\
& \forall_k[P : \neg(Q \vee R) \Rightarrow \texttt{False}] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Domain Weakening} \} \\
& \forall_k[P \wedge \neg(Q \vee R) : \texttt{False}] \\
\stackrel{val}{=\!=} & \quad \{ \text{ De Morgan} \} \\
& \forall_k[P \wedge \neg Q \wedge \neg R : \texttt{False}] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Domain Weakening} \} \\
& \forall_k[P \wedge \neg Q : \neg R \Rightarrow \texttt{False}] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Negation } \} \\
& \forall_k[P \wedge \neg Q : \neg\neg R] \\
\stackrel{val}{=\!=} & \quad \{ \text{ Double negation } \} \\
& \forall_k[P \wedge \neg Q : R]
\end{aligned}
$$

9.6 (a) Assume k is a natural number. Take P to be $k > 3$ and Q to be $k^2 > 3$.

(b) Take P to be $k \in \mathbb{N}$, Q to be $k = 4$ and R to be $k^2 = 4$.