Coinduction and Corecursion in Coq; Inductive and Coinductive Components of Corecursive Functions.

Ekaterina Komendantskaya, joint work with Yves Bertot

INRIA Sophia Antipolis

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Outline

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Filter Function, [Bertot05]

Definition

(Filter for streams).
For a given predicate P,

$$\text{filter (SCons x tl)} = \begin{cases} \text{SCons x (filter tl)} & \text{if } P(x) \\ \text{filter tl} & \text{otherwise.} \end{cases}$$

Inductive Types and Recursive Functions

Coq = COC [Coquand,Huet'88] + CIC [Coquand,Paulin'93]

```
Inductive nat :  Set :=
| O : nat
| S : nat -> nat.
```

```
Fixpoint div2 n :  nat :=
 match n with
| O => 0
| S O => 0
| S (S n') => S (div2 n')
end.
```

Coinductive Types and Corecursive Functions

Coq = COC ['88] + CIC ['93] + CCC [Gimenez'96]

```
CoInductive str (A: Set) :  Set :=
SCons:  A -> str A -> str A.
```

```
CoFixpoint repeat (a:  A): str A :=
SCons a (repeat a).
```

Termination

We require all computations to terminate, because of:

- Curry-Howard Isomorphism (propositions $\rightarrow$ types; proofs $\rightarrow$ programs): non-terminating proofs can lead to inconsistency.

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Example

The function `div2` is terminative.

Productive Values

Values in co-inductive types are productive when all observations of fragments made using recursive functions are guaranteed to be computable in finite time.

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The element of the stream at position *n* can be found by:

$$\begin{cases} \texttt{nth 0 (SCons a tl)} = \texttt{a} \\ \texttt{nth (S n) (SCons a tl)} = \texttt{nth n tl} \end{cases}$$

A given stream `s` is productive if we can be sure that the computation of the list `nth n s` is guaranteed to terminate, whatever the value of `n` is.

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Example

For any `n`, the value `repeat n` is productive.

Productive Functions

We call a function *productive at the input value i*, if it outputs a productive value at i.

Example

Filter is productive only on certain inputs.

A more general example

Definition

Let A, B be of type `Set`. For a predicate $P : B \to \texttt{bool}$ and functions $h : B \to A$, g, $g' : B \to B$, we define the function *dyn*:

$$\text{dyn (x)} = \begin{cases} \text{SCons } h(x) \text{ (dyn } (g(x))) & \text{if } P(x) \\ \text{dyn } (g'(x)) & \text{otherwise.} \end{cases}$$

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Example

Suppose B is the set of natural numbers, $h = id$, $g = +1$; $g' = *2$; $P =$ "even". If we take $x = 1$, `dyn` will compute the infinite list: 2, 6, 14, 30, 62, 126, ...

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If B is a set of streams, we can have `dyn` = filter.

Totally-, Partially-, Non- Productive Functions

- Totally Productive
  (Function `repeat`)
- Partially Productive
  (Filters on streams and trees; `dyn`).
- Non-Productive
  Computing `nth 0 (filter even (repeat 1))` provokes the following computation:
  `filter even (repeat 1) repeat 1` $\rightsquigarrow$ `filter even (1::repeat 1)` $\rightsquigarrow$ `filter even (repeat 1)`...

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Our method makes it possible to formalise totally and partially productive functions in Coq, using inductive and coinductive predicates to characterise the arguments on which these functions output productive values.

Structural Recursion

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 end.
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General Recursion

Definitions where the recursive calls are not required to be on structurally smaller arguments, that is, where the recursive calls can be performed on any argument, are called *general recursive* arguments.

Example

$$\begin{cases} \log(\mathrm{S}\ 0) = 0 \\ \log(\mathrm{S}(\mathrm{S}\ n)) = \mathrm{S}(\log\ \mathrm{S}(\mathrm{div2}\ n)). \end{cases}$$

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Method of Ad-hoc Predicates [Aczel77], [Bove02].

```
Fixpoint log (x:nat)(h:log_domain x){struct h} :
nat :=
```

```
match x as y return x = y -> nat with
| 0 => fun h' => False_rec nat (log_domain_non_0 x h
h')
| S 0 => fun h' => 0
| S (S p) =>
fun h' => S (log (S (div2 p)) (log_domain_inv x p h
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end (refl_equal x).
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Guardedness [Gimenez96: Calculus of Coinductive Constructions]

The guardedness condition insures that

* each corecursive call is made under at least one constructor;
** if the recursive call is under a constructor, it does not appear as an argument of any function.

Violation of any of these two conditions makes a function rejected by the guardedness test in Coq.

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Example

```
CoFixpoint repeat (a:A): str A := SCons a (repeat a).
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Non-guarded functions:

[$*$] is not satisfied:
Filters, `dyn`;
[$**$] is not satisfied:
Consider the following function computing lists of ordered natural numbers:
`nats = (SCons 1 (map (+ 1) nats)).`
where the function `map` above is defined as follows:

```
map f (s: str): str := Cons (f (hd s)) (map f (tl s)).
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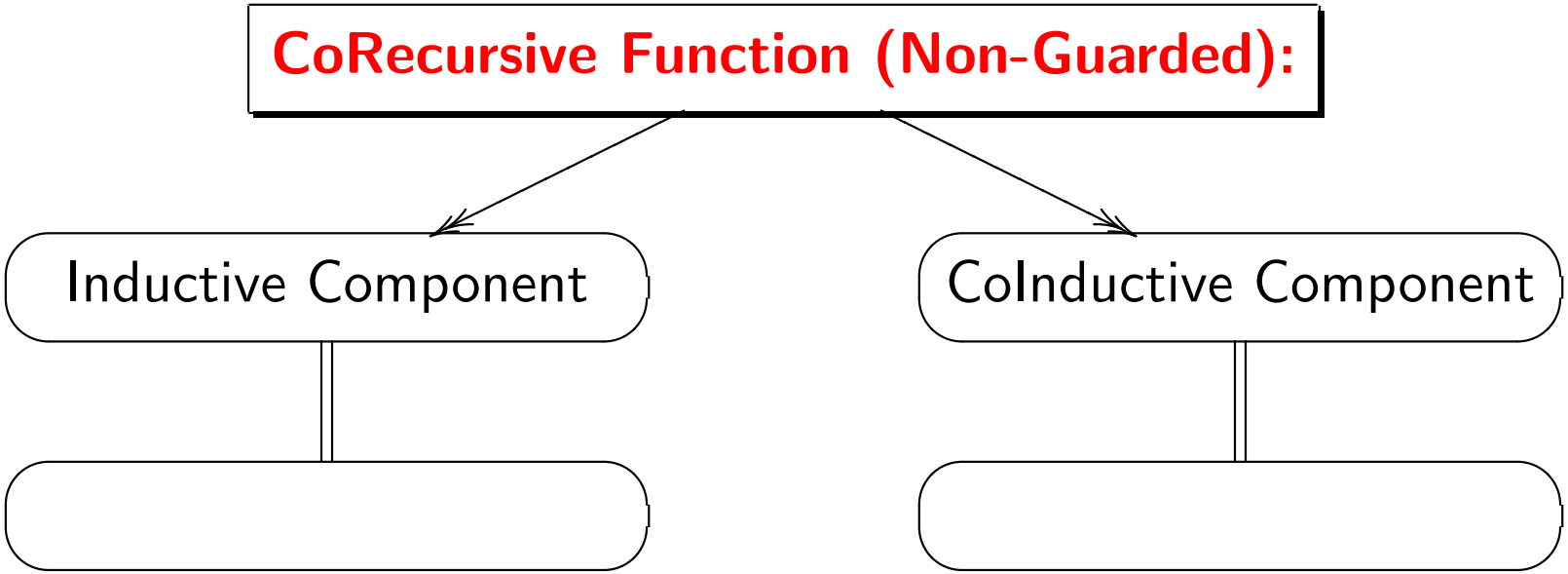
Our Method: Inductive and Coinductive Components



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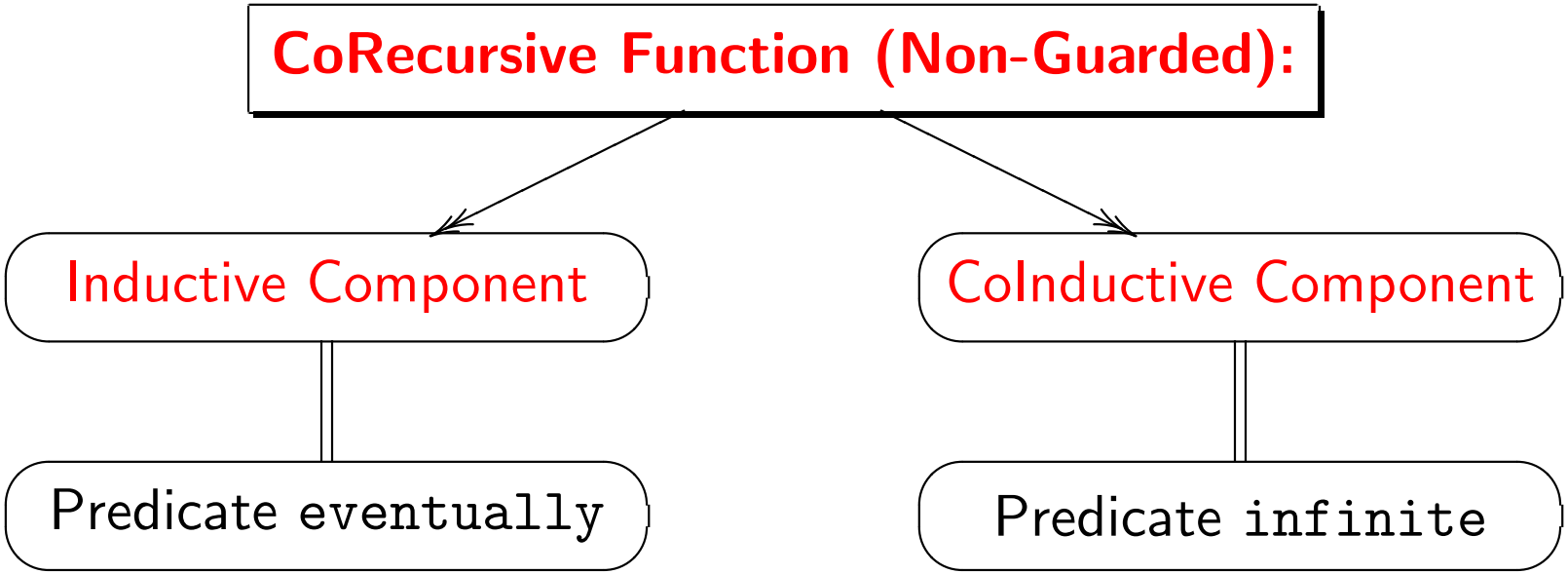

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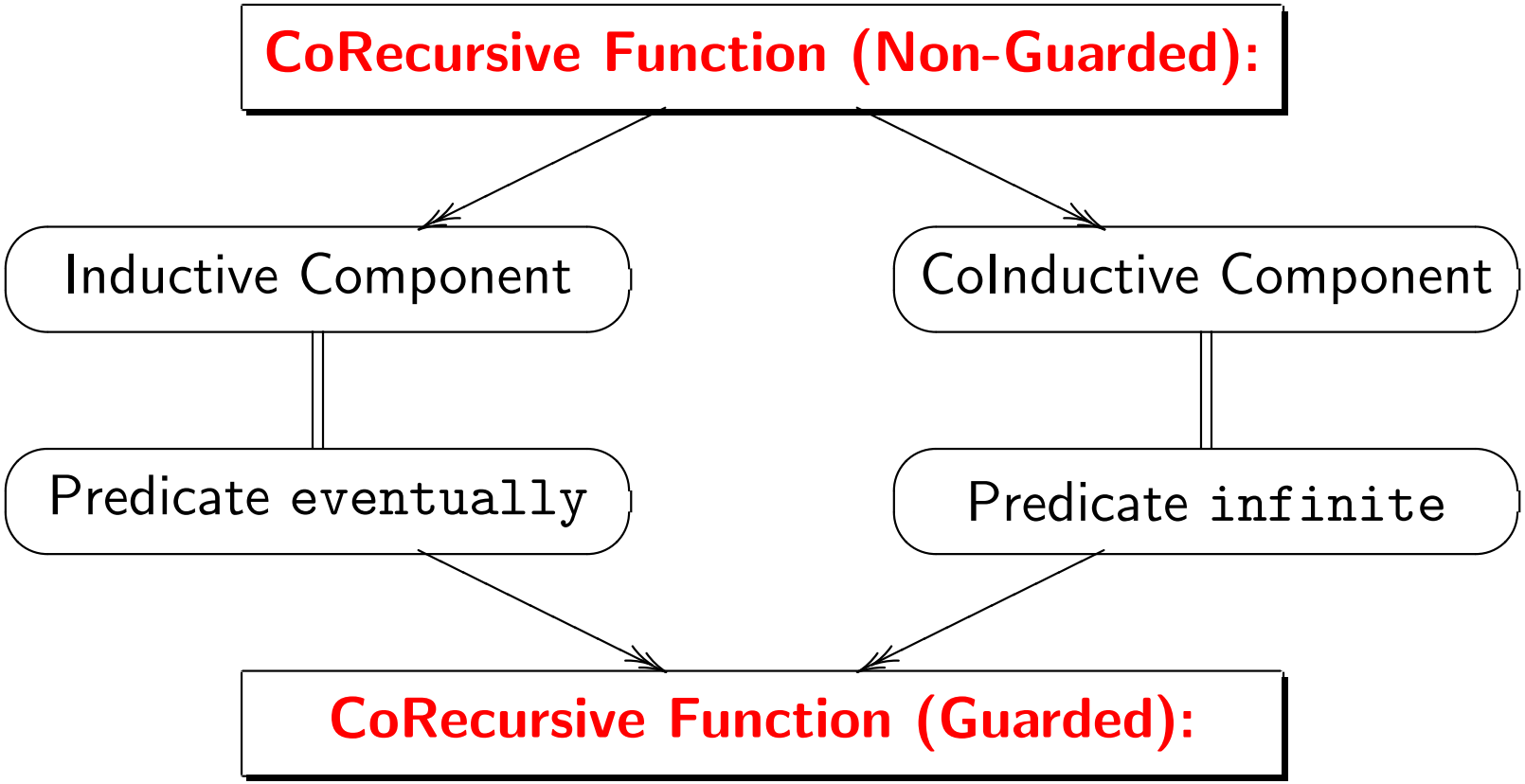

Predicate eventually: First-Step Productivity

Using eventually, we can describe the inductive component of a corecursive function. This component is a recursive function that performs all the computations and tests that lead to the first guarded corecursive call.

```
Inductive eventually_s:  str A -> Prop :=
| ev_b:  forall x s, P x -> eventually_s (SCons A x s)
| ev_r:  forall x s, not P x
-> eventually_s s -> eventually_s (SCons A x s).
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```

eventually for dyn

```
Inductive eventually_dyn (x: B) : Prop :=
| ev_dyn1 : P x = true -> eventually_dyn x
| ev_dyn2 : P x = false -> eventually_dyn (g' x) ->
eventually_dyn x.
```

Compare `eventually` with $\Diamond$ in [Pnueli81, Jacobs02].

Inversion Lemmas

```
Lemma eventually_s_inv:
forall (s :  str A),
eventually_s s -> forall x s', s = SCons A x s' ->
not P x -> eventually_s s'.
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```

```
Lemma eventually_dyn_inv :
```

```
forall x, eventually_dyn x -> P x = false ->
eventually_dyn (g' x).
```

Inductive Component of Filter

```
Fixpoint pre_filter_s (s : str A) (h :
eventually_s s) struct h : A * str A :=
```

```
match s as b return s = b -> A*str A with
SCons x s' =>
fun heq =>
match P_dec x with
| left _ => (x, s')
| right hn =>
pre_filter_s s' (eventually_s_inv s h x s' heq hn)
end
end (refl_equal s).
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```

Inductive Component of dyn

```
Fixpoint pre_dyn(x:B)(d:eventually_dyn x){struct
d}:  A*B:=
```

```
 match P x as b return P x = b -> A*B with
| true => fun t => (h x, g x)
| false => fun t =>
pre_dyn (g' x) (eventually_dyn_inv x d t)
end (refl_equal (P x)).
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Inductive Component of dyn

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```

Coinductive Predicate `infinite`

Corecursive computations are introduced by repeating computations performed by the inductive component. This can happen only if recursive calls satisfy the `eventually` predicate repeatedly. We need the predicate `infinite` to express this.

```
CoInductive infinite_s :  str -> Prop :=

 al_cons: forall (s:str A) (h: eventually s),
  infinite_s (snd(pre_filter_s s h)) ->  infinite_s s.
```

The same predicate for dyn

```
CoInductive infinite_dyn (x :  B): Prop :=

  di : forall (d: eventually_dyn x),
   infinite_dyn (snd (pre_dyn x d)) -> infinite_dyn x.
```

The `infinite` predicate describes exactly those arguments to the function for which the function is guaranteed to be productive.

Relating eventually and infinite

```
Lemma infinite_eventually_dyn :
forall x, infinite_dyn x -> eventually_dyn x.
```

```
Lemma infinite_always_dyn :
forall x, infinite_dyn x ->
forall e:  eventually_dyn x,
infinite_dyn (snd (pre_dyn x e)).
```

Guarded Representation of a filter

```
CoFixpoint filter (s : str A) : forall (h:
infinite_s s), str A :=
```

```
match s return infinite_s s -> str A with
| SCons x s' =>
fun h' : infinite_s (SCons A x s') =>
SCons A (fst
(pre_filter_s _ infinite_eventually_s (SCons A x s')
h')))
(filter _ (infinite_always_s (SCons A x s') h'))
end.
```

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```

Guarded Representation of dyn

```
CoFixpoint dyn (x:B)(h:infinite_dyn x) :  str :=
SCons (fst (pre_dyn x (infinite_eventually_dyn ev x
h)))
(dyn _ (infinite_always_dyn x h
(infinite_eventually_dyn x h))).
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Guarded Representation of dyn

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```

Recursive Equation Lemma for dyn

```
Theorem dyn_equation :

forall x i: infinite_dyn x, bisimilar_s (dyn x i)
(match Px as b return Px = b -> infinite_dyn x -> str
with
|true => fun t i =>
SCons(h x)(dyn(g x)(dyn_step1 x t i))
|false => fun t i => dyn (g' x) (dyn_step2 x t i)
end (refl_equal (P x)) i).
```

More Complicated Applications of Our Method:

Expression trees and dynamic filtering on expression trees.

The dynamic filter function on expression trees was used to establish a normalisation algorithm for an admissible representation of a closed interval of real numbers in [Geuvers1993], [Niqui2004].

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Expression trees and dynamic filtering on expression trees.

The dynamic filter function on expression trees was used to establish a normalisation algorithm for an admissible representation of a closed interval of real numbers in [Geuvers1993], [Niqui2004]. The function was not guarded.

We applied our method to give a Coq formalisation of the function.

Other Productive Non-Guarded Functions

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Every terminative function gives rise to a non-guarded totally productive function.

Example

Terminative function $x - 1$ gives rise to the totally productive function `f: stream nat -> stream nat:`

$$\texttt{f (x::y::tl)} = \begin{cases} \texttt{x::f(y::tl)} & \text{if } x \leq y \\ \texttt{f((x-1)::y::tl)} & \text{otherwise.} \end{cases}$$

Conclusions

1. We generalised the method of [Bertot05] to a wider class of functions:
   - various output data types including streams, expression and binary trees;
   - included dynamically changing functions;

   and thereby gave a general analysis of the method.
2. We work with partial productivity, and not just total productivity.
3. We establish the uniform Recursive Equation Lemmas for the functions we formalise, this was not achieved in [Bertot05].

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Future work $\rightarrow$ further automatisation.

Thank you!