Coalgebraic Semantics for Parallel Derivation Strategies in Logic Programming.

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Outline

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1. Logic programs...

2. Coalgebraic Semantics for them...

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1. Logic programs...
2. Coalgebraic Semantics for them...
3. Lawvere theory in this picture

First-order syntax

We fix the *alphabet* **A** to consist of

- Signature Σ:
  - constant symbols a, b, c, possibly with finite subscripts;
  - function symbols $f, g, h, f_1 \ldots$, with arities;
- variables $x, y, z, x_1 \ldots$;
- predicate symbols $P, Q, R, P_1, P_2 \ldots$, with arities;

Term:

$$t = a \mid x \mid f(t)$$

Atomic Formula: $At = P(t_1, \ldots, t_n)$, where P is a predicate symbol of arity n and t_i is a term.

Example

Signature: $\{O, S\}$ - for natural numbers; {a,b,c} for a graph with 3 nodes.
Atoms: nat(x), edge(a,b), etc...

Logic programs

A first-order logic program consists of a finite set of clauses of the form

$$A \leftarrow A_1, \ldots, A_n$$

where A and the A_i's are atomic formulae, typically containing free variables; and $A_1, \ldots, A_n$ is to mean the conjunction of the A_i's.

Definition

Let a goal G be $\leftarrow A_1, \ldots, A_m, \ldots, A_k$ and a clause C be $A \leftarrow B_1, \ldots, B_q$. Then G' is *derived* from G and C using mgu θ if the following conditions hold:

- A_m is an atom, called the *selected* atom, in G.
- θ is an *mgu* of A_m and A.
- G' is the goal $\leftarrow (A_1, \ldots, A_{m-1}, B_1, \ldots, B_q, A_{m+1}, \ldots, A_k)\theta$.

Example: Goal $\leftarrow$ p(a).

```
q(b,a)  ←  s(a,b)
q(b,a)  ←
s(a,b)  ←
  p(a)  ←  q(b,a),s(a,b)
```

$$\leftarrow \mathtt{p(a)}$$
$$\Big|$$
$$\leftarrow \mathtt{q(b,a)}, \mathtt{s(a,b)}$$

Example: Goal $\leftarrow$ p(a).

$$
\begin{array}{rcl}
\texttt{q(b,a)} & \leftarrow & \texttt{s(a,b)} \\
\texttt{q(b,a)} & \leftarrow & \\
\texttt{s(a,b)} & \leftarrow & \\
\texttt{p(a)} & \leftarrow & \texttt{q(b,a)}, \texttt{s(a,b)}
\end{array}
$$

$$
\begin{array}{c}
\leftarrow \texttt{p(a)} \\
| \\
\leftarrow \texttt{q(b,a)}, \texttt{s(a,b)} \\
| \\
\leftarrow \texttt{s(a,b)}, \texttt{s(a,b)}
\end{array}
$$

Example: Goal $\leftarrow$ p(a).

```
q(b,a)  ←  s(a,b)
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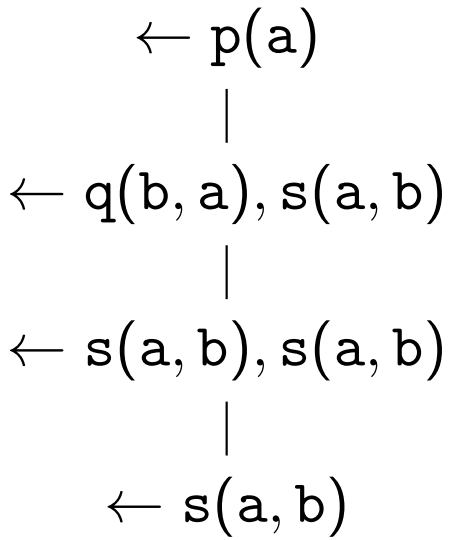
```



Example: Goal $\leftarrow$ p(a).

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\begin{array}{rcl}
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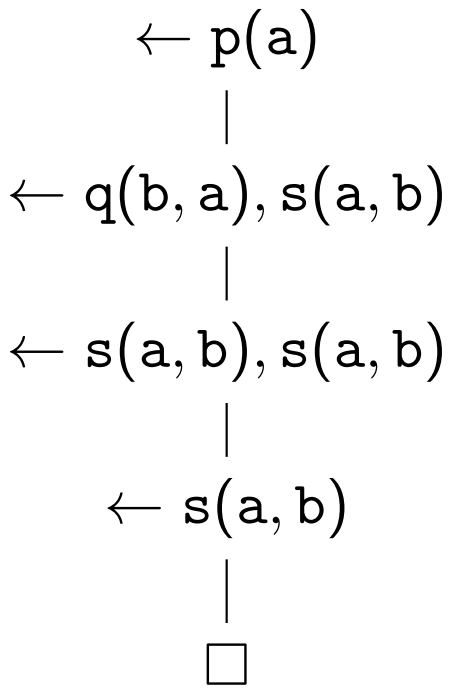
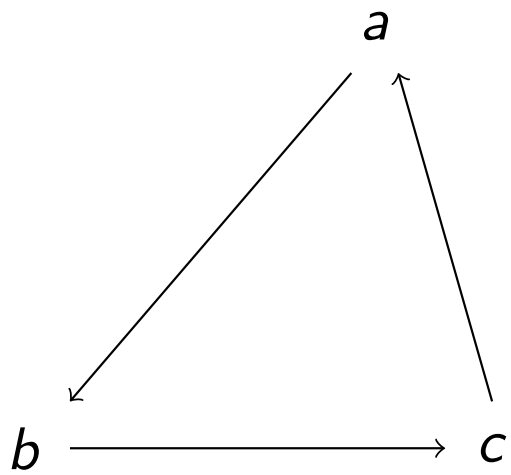
\texttt{s(a,b)} & \leftarrow & \\
\texttt{p(a)} & \leftarrow & \texttt{q(b,a)}, \texttt{s(a,b)}
\end{array}
$$



Logic program: Graph connectivity

Example

```
connected(x,x)  ←
connected(x,y)  ←  edge(x,z),
connected(z,y).
     edge(a,b)  ←
     edge(b,c)  ←
     edge(c,a)  ←
```



Example: Goal $\leftarrow$ connected(a, c).

Example

```
connected(x,x)  ←
connected(x,y)  ←  edge(x,z),
connected(z,y)
     edge(a,b)  ←
     edge(b,c)  ←
     edge(c,a)  ←
```

$$\leftarrow \texttt{connected}(\texttt{a}, \texttt{c})$$

$$\Big|$$

$$\leftarrow \texttt{edge}(\texttt{a}, \texttt{x}), \texttt{connected}(\texttt{x}, \texttt{c})$$

Example: Goal $\leftarrow$ connected(a, c).

Example

```
connected(x,x)  ←
connected(x,y)  ←  edge(x,z),
connected(z,y)
     edge(a,b)  ←
     edge(b,c)  ←
     edge(c,a)  ←
```

$$\leftarrow \texttt{connected}(\texttt{a},\texttt{c})$$
$$\Big|$$
$$\leftarrow \texttt{edge}(\texttt{a},\texttt{x}),\texttt{connected}(\texttt{x},\texttt{c})$$
$$\Big|$$
$$\leftarrow \texttt{connected}(\texttt{b},\texttt{c})$$

Example: Goal $\leftarrow$ connected(a, c).

Example

```
connected(x,x)  ←
connected(x,y)  ←  edge(x,z),
connected(z,y)
     edge(a,b)  ←
     edge(b,c)  ←
     edge(c,a)  ←
```

$\leftarrow$ connected(a, c)

|

$\leftarrow$ edge(a, b), connected(b, c)

|

$\leftarrow$ connected(b, c)

|

$\leftarrow$ edge(b, x), connected(x, c)

Example: Goal $\leftarrow$ connected(a, c).

Example

```
connected(x,x)  ←
connected(x,y)  ←  edge(x,z),
connected(z,y).
     edge(a,b)  ←
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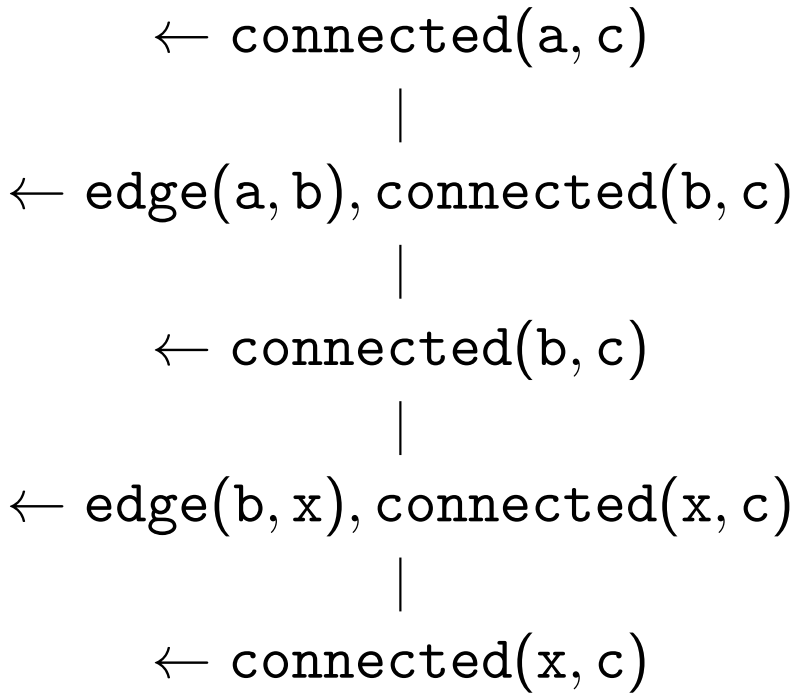
```



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Example

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connected(x,x)   ←
connected(x,y)   ←   edge(x,z),
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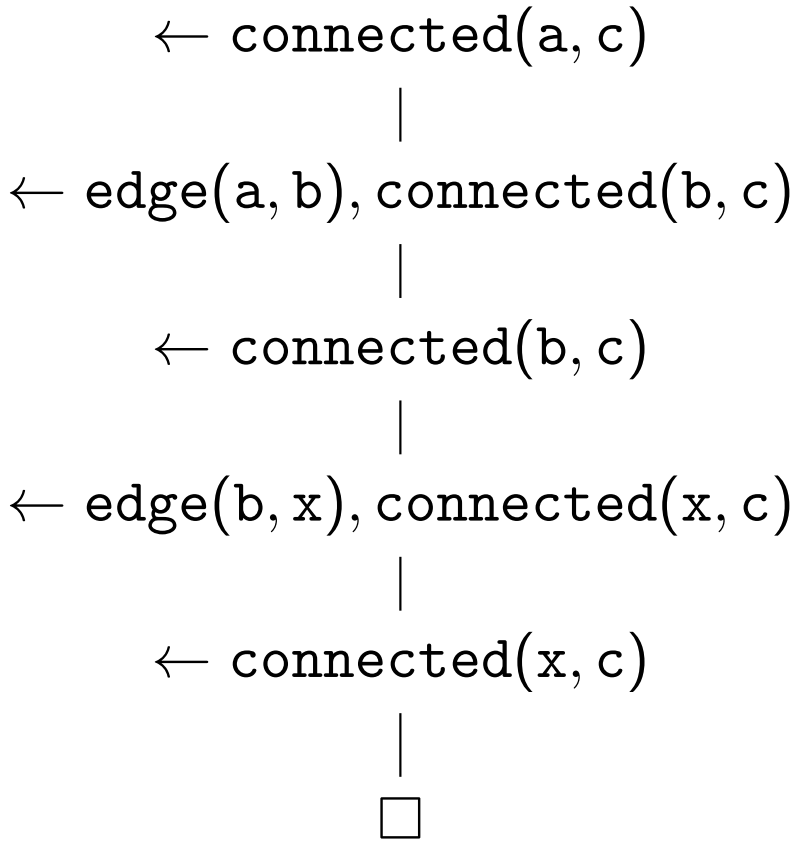
     edge(a,b)   ←
     edge(b,c)   ←
     edge(c,a)   ←
```



Recursion and Corecursion in Logic Programming

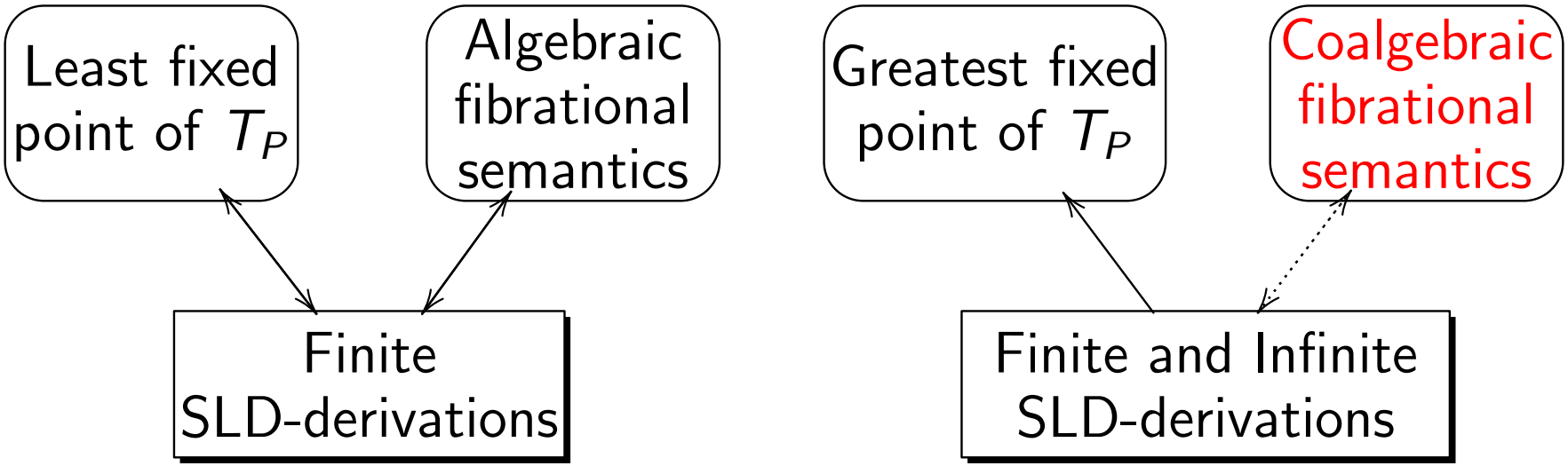
Example

$$\begin{aligned}
\texttt{nat(0)} &\leftarrow \\
\texttt{nat(s(x))} &\leftarrow \texttt{nat(x)} \\
\texttt{list(nil)} &\leftarrow \\
\texttt{list(cons x y)} &\leftarrow \texttt{nat(x), list(y)}
\end{aligned}$$

Example

$$\begin{aligned}
\texttt{bit(0)} &\leftarrow \\
\texttt{bit(1)} &\leftarrow \\
\texttt{stream(cons (x,y))} &\leftarrow \texttt{bit(x),stream(y)}
\end{aligned}$$

Algebraic and coalgebraic semantics for LP



Coalgebraic Analysis of Logic Programs

Generally, given a functor F, an F-coalgebra is a pair (S, α) consisting of a set S and a function $\alpha : S \longrightarrow F(S)$. We will take a powerset functor P_f.

Proposition

For any set At, there is a bijection between the set of variable-free logic programs over the set of atoms At and the set of $P_f P_f$-coalgebra structures on At.

Proof.

Given a variable-free logic program P, let At be the set of all atoms appearing in P. Then P can be identified with a $P_f P_f$-coalgebra (At, p), where $p : At \longrightarrow P_f(P_f(\mathrm{At}))$ sends an atom A to the set of bodies of those clauses in P with head A, each body being viewed as the set of atoms that appear in it. □

Example

Example

Consider the logic program from the previous Example.

$$\begin{aligned} \texttt{q(b,a)} &\leftarrow \texttt{s(a,b)} \\ \texttt{q(b,a)} &\leftarrow \\ \texttt{s(a,b)} &\leftarrow \\ \texttt{p(a)} &\leftarrow \texttt{q(b,a)}, \texttt{s(a,b)} \end{aligned}$$

The program has three atoms, namely $\texttt{q(b,a)}$, $\texttt{s(a,b)}$ and $\texttt{p(a)}$. So $At = \{\texttt{q(b,a)}, \texttt{s(a,b)}, \texttt{p(a)}\}$. And the program can be identified with the $P_f P_f$-coalgebra structure on At given by
$p(\texttt{q(b,a)}) = \{\{\}, \{\texttt{s(a,b)}\}\}$, where $\{\}$ is the empty set.
$p(\texttt{s(a,b)}) = \{\{\}\}$, i.e., the one element set consisting of the empty set.
$p(\texttt{p(a)}) = \{\{\texttt{q(b,a)}, \texttt{s(a,b)}\}\}$.

Coalgebraic Analysis of derivations in Logic Programs

Theorem

Given an endofunctor $H : Set \longrightarrow Set$ with a rank, the forgetful functor $U : H\text{-}Coalg \longrightarrow Set$ has a right adjoint R.

R is constructed as follows. Given $Y \in Set$, we define a transfinite sequence of objects as follows. Put $Y_0 = Y$, and $Y_{\alpha+1} = Y \times H(Y_\alpha)$. We define $\delta_\alpha : Y_{n+1} \longrightarrow Y_n$ inductively by

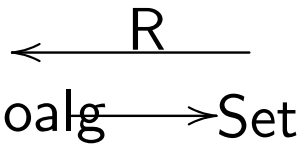
$$Y_{\alpha+1} = Y \times HY_\alpha \overset{Y \times H\delta_{\alpha-1}}{\longrightarrow} Y \times HY_{\alpha-1} = Y_\alpha,$$

with the case of $\alpha = 0$ given by the map $Y_1 = Y \times HY \overset{\pi_1}{\longrightarrow} Y$. For a limit ordinal, let $Y_\alpha = \lim_{\beta<\alpha}(Y_\beta)$, determined by the sequence

$$Y_{\beta+1} \overset{\delta_\beta}{\longrightarrow} Y_\beta.$$

If H has a rank, there exists α such that Y_α is isomorphic to $Y \times HY_\alpha$. This Y_α forms the cofree coalgebra on Y.

Coalgebraic Analysis of derivations in Logic Programs



Corollary

If H has a rank, U has a right adjoint R and putting $G = RU$, G possesses a canonical comonad structure and there is a coherent isomorphism of categories

$$G\text{-}Coalg \cong H\text{-}Coalg,$$

where G-$Coalg$ is the category of G-coalgebras for the comonad G.

Given an H-coalgebra $p : Y \longrightarrow HY$, we construct maps $p_\alpha : Y \longrightarrow Y_\alpha$ for each ordinal α as follows. The map $p_0 : Y \longrightarrow Y$ is the identity, and for a successor ordinal, $p_{\alpha+1} = \langle id, Hp_\alpha \circ p \rangle : Y \longrightarrow Y \times HY_\alpha$. For limit ordinals, p_α is given by the appropriate limit. By definition, the object GY is given by Y_α for some α, and the corresponding p_α is the required G-coalgebra.

Coalgebraic Analysis of derivations in Logic Programs

Taking $p : \mathrm{At} \longrightarrow P_f P_f(\mathrm{At})$, by the proof of Theorem 1, the corresponding $C(P_f P_f)$-coalgebra where $C(P_f P_f)$ is the cofree comonad on $P_f P_f$ is given as follows: $C(P_f P_f)(\mathrm{At})$ is given by a limit of the form

$$\ldots \longrightarrow \mathrm{At} \times P_f P_f(\mathrm{At} \times P_f P_f(\mathrm{At})) \longrightarrow \mathrm{At} \times P_f P_f(\mathrm{At}) \longrightarrow \mathrm{At}.$$

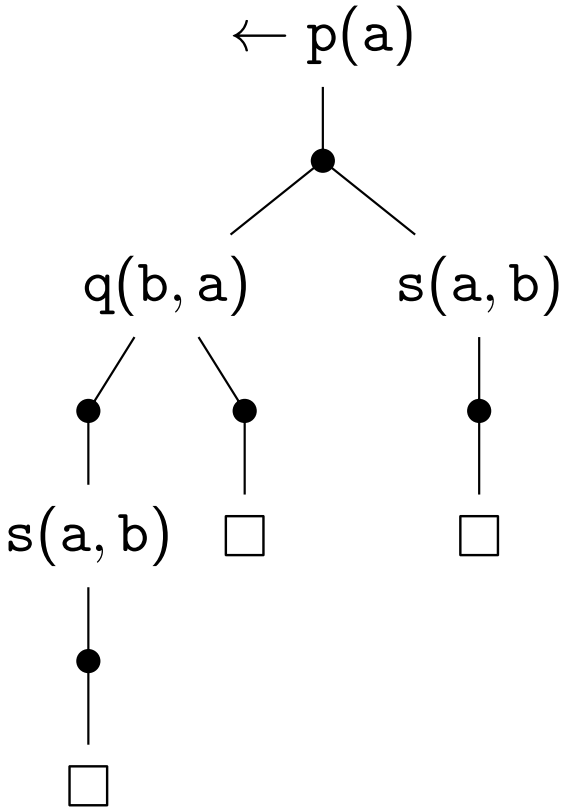
This chain has length ω. As above, we inductively define the objects $\mathrm{At}_0 = \mathrm{At}$ and $\mathrm{At}_{n+1} = \mathrm{At} \times P_f P_f \mathrm{At}_n$, and the cone

$$\begin{aligned} p_0 &= id : \mathrm{At} \longrightarrow \mathrm{At}(= \mathrm{At}_0) \\ p_{n+1} &= \langle id, P_f P_f(p_n) \circ p \rangle : \mathrm{At} \longrightarrow \mathrm{At} \times P_f P_f \mathrm{At}_n (= \mathrm{At}_{n+1}) \end{aligned}$$

and the limit determines the required coalgebra $\overline{p} : \mathrm{At} \longrightarrow C(P_f P_f)(\mathrm{At})$.

Examples of a derivations

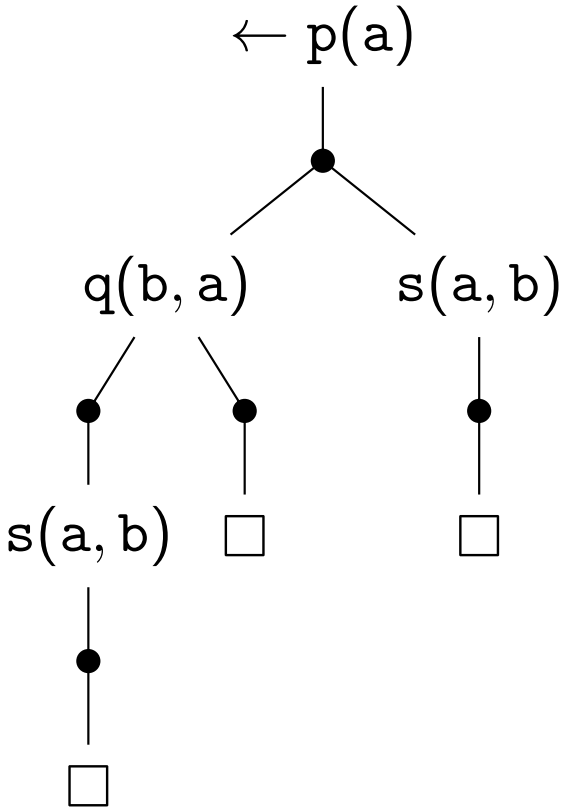
The action of
$\overline{p} : \mathrm{At} \longrightarrow C(P_f P_f)(\mathrm{At})$ on
$\mathtt{p(a)}$

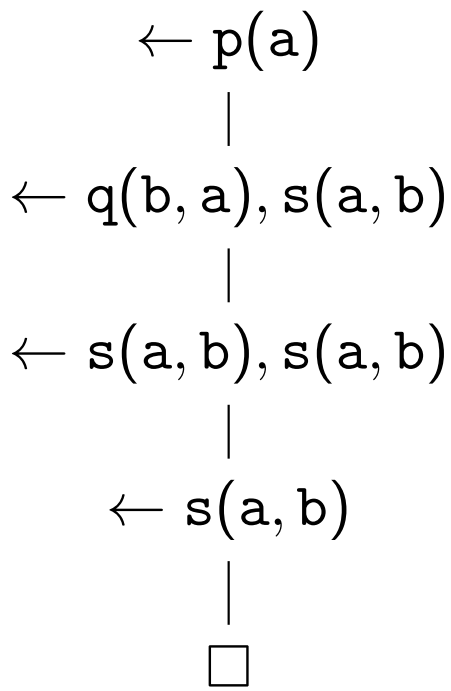


Examples of a derivations

The action of $\overline{p} : \mathrm{At} \longrightarrow C(P_f P_f)(\mathrm{At})$ on $\mathtt{p(a)}$

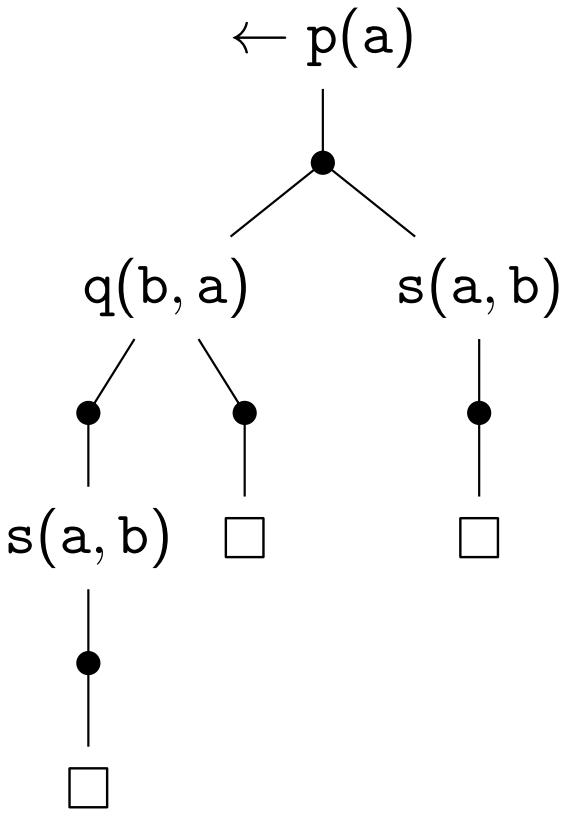
The SLD derivation from a previous example



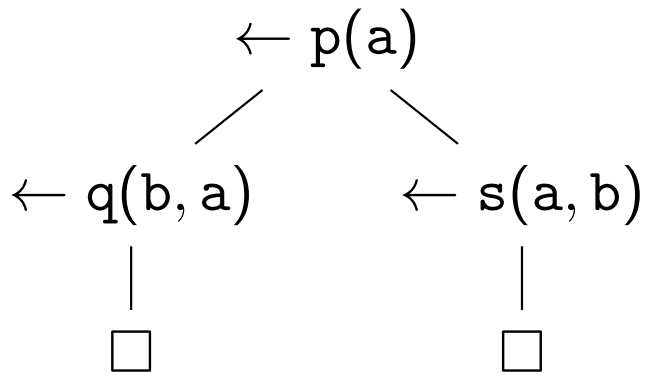


Examples of a derivations

The action of $\overline{p} : \mathrm{At} \longrightarrow C(P_f P_f)(\mathrm{At})$ on $\mathtt{p(a)}$



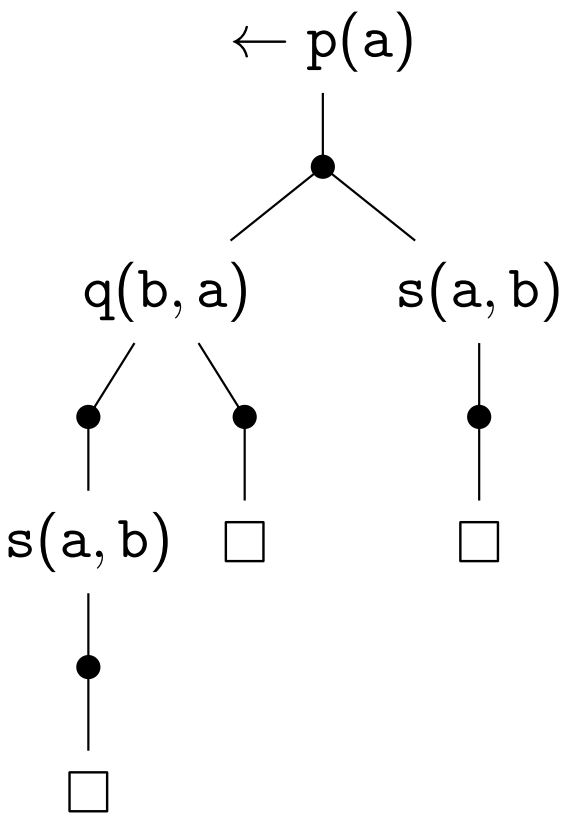
The proof tree

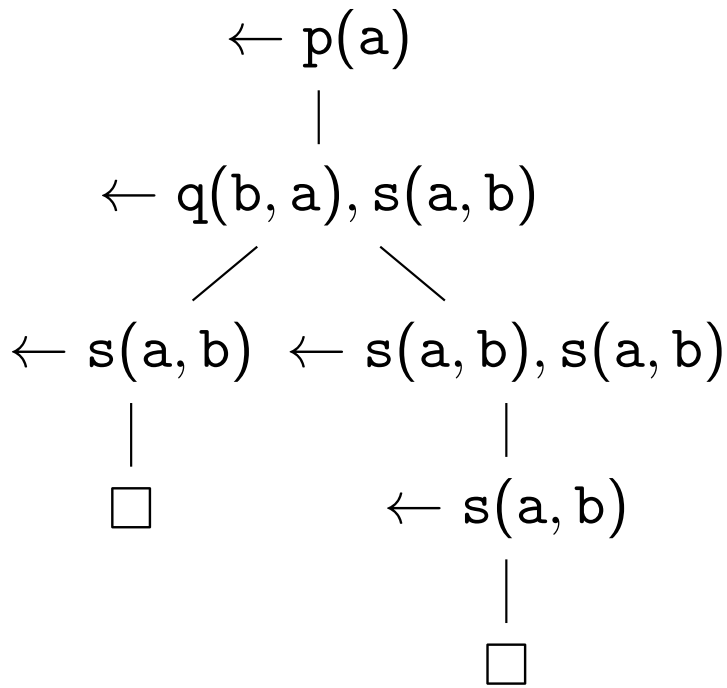


Examples of a derivations

The action of $\overline{p} : \mathrm{At} \longrightarrow C(P_f P_f)(\mathrm{At})$ on $\mathtt{p(a)}$

The SLD tree





Is there anything at all in practice of Logic Programming that corresponds to the action of $C(P_f P_f)$-comonad?

From the examples above, it's clear that:

Sequential SLD-derivation

is the least suitable for the model given by $C(P_f P_f)$-comonad.

Proof trees

exhibit an and-parallelism in derivations - that is, parallel proof search over conjuncts in a goal, but the choices of different clauses to use in the process are not reflected - except for - one can use a sequence of proof-trees for this purpose.

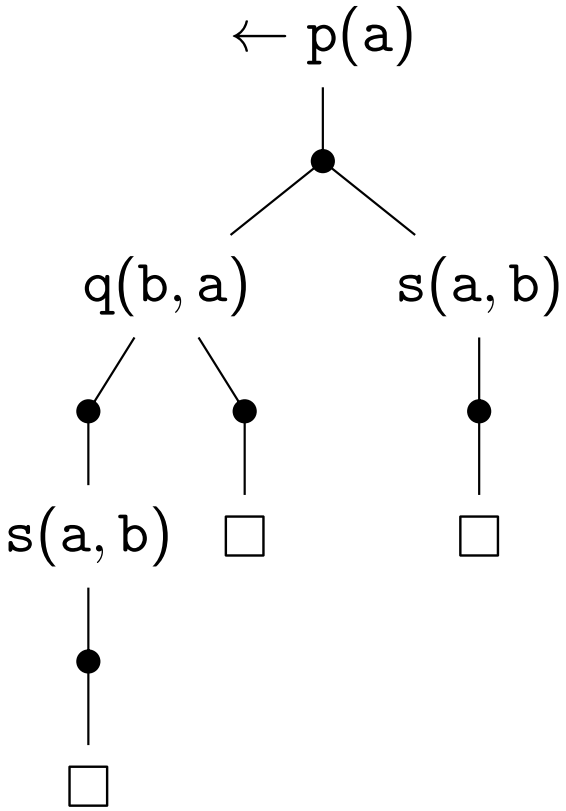
SLD-trees

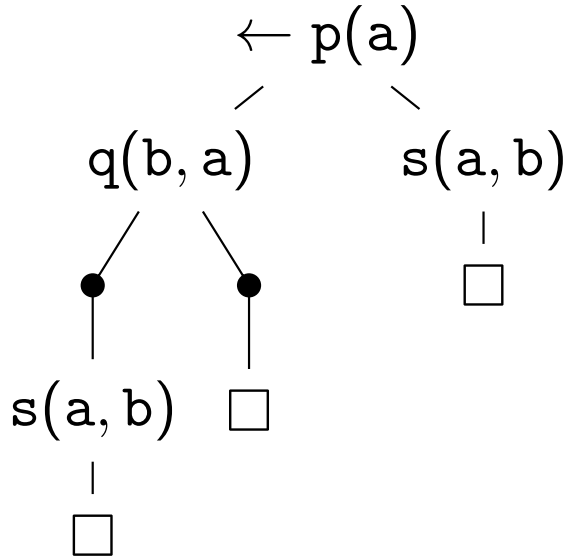
exhibit an or-parallelism in derivations - that is, they show different possibilities of derivations if there are multiple clauses that unify with a goal; but they process conjuncts in a goal sequentially.

It turns out that the answer lies in the combination of the two kinds of parallelism:

$\overline{p} : \mathrm{At} \longrightarrow C(P_f P_f)(\mathrm{At})$ on p(a)

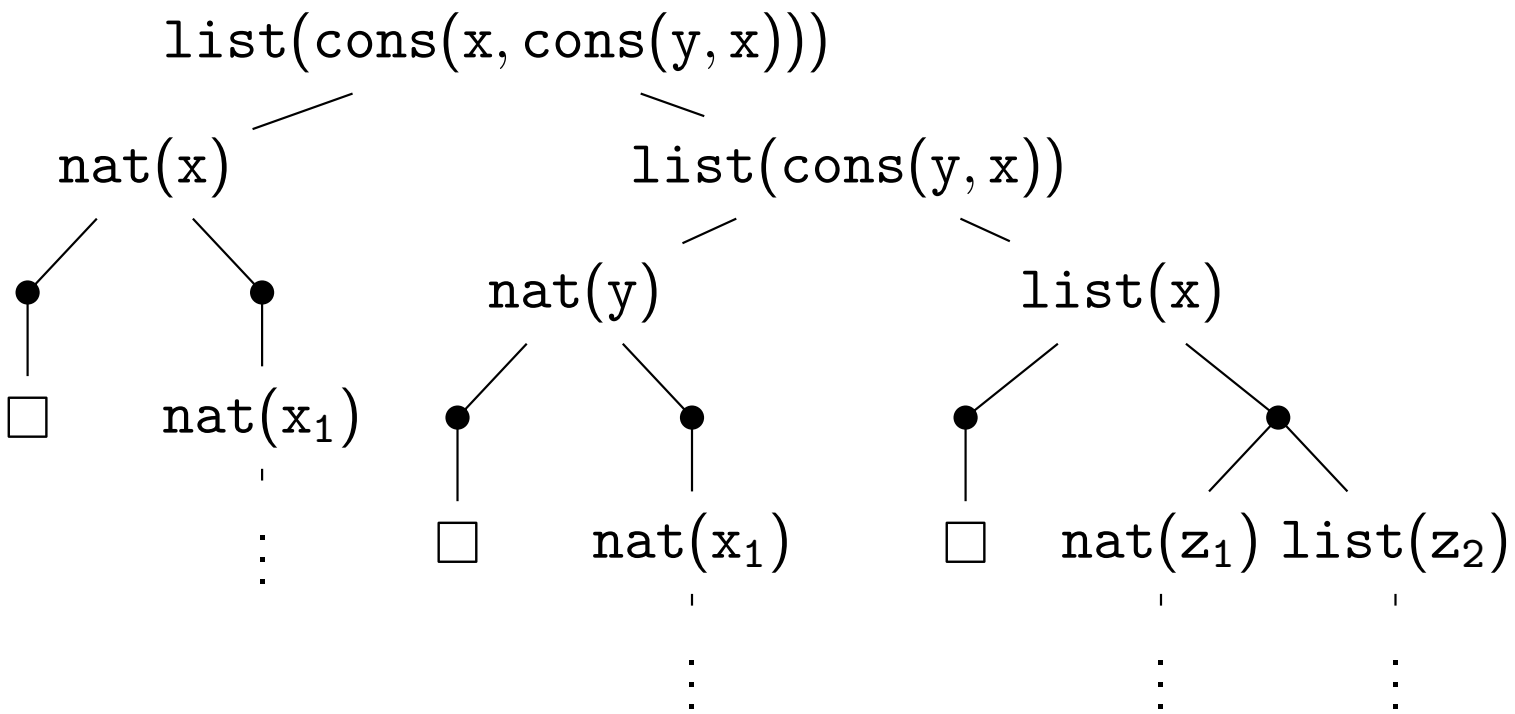
The and-or parallel tree (as defined in the literature)





Except for... they are unsound for first-order programs.

Why unsound?


list(cons(x, cons(y, x)))
nat(x)
list(cons(y, x))
nat(y)
list(x)
□
nat(x1)
□
nat(x1)
□
nat(z1)
list(z2)

Lawvere theories and the first-order signature

- We use *Lawvere theories*,
- model most general unifiers (mgu's) by *equalisers*,

Given a signature Σ, the Lawvere theory $\mathcal{L}_\Sigma$:

Define the set $\mathrm{ob}(\mathcal{L}_\Sigma)$ to be the set of natural numbers.
For each natural number n, let $x_1, \ldots, x_n$ be a specified list of distinct variables. Define $\mathrm{ob}(\mathcal{L}_\Sigma)(n, m)$ to be the set of m-tuples $(t_1, \ldots, t_m)$ of terms generated by the function symbols in Σ and variables $x_1, \ldots, x_n$. Define composition in $\mathcal{L}_\Sigma$ by substitution. The interpretation $\| \ \|_\Sigma$ sends an n-ary function symbol f to $f(x_1, \ldots, x_n)$.

Example

The constants `0` and `nil` are modelled by maps from 0 to 1 in $\mathcal{L}_\Sigma$, `s` is modelled by a map from 1 to 1, and `cons` is modelled by a map from 2 to 1. The term `s(0)` is therefore modelled by the map from 0 to 1 given by the composite of the maps modelling `s` and `0`; similarly for the term `s(nil)`, although the latter does not make semantic sense.

Coalgebraic semantics for the first-order case

Assume we have a signature Σ of function symbols and, for each natural number n, a specified list of variables $x_1, \ldots, x_n$. Then, given an arbitrary logic program with signature Σ,

we can consider the functor $At : \mathcal{L}_\Sigma^{op} \to Set$ that sends a natural number n to the set of all atomic formulae with variables among $x_1, \ldots, x_n$ generated by the function symbols in Σ and the predicate symbols appearing in the logic program. A map $f : n \to m$ in $\mathcal{L}_\Sigma$ is sent to the function $At(f) : At(m) \to At(n)$ that sends an atomic formula $A(x_1, \ldots, x_m)$ to $A(f_1(x_1, \ldots, x_n)/x_1, \ldots, f_m(x_1, \ldots, x_n)/x_m)$, i.e., $At(f)$ is defined by substitution.

Attempt to model P by $[\mathcal{L}_\Sigma^{op}, P_f P_f]$-coalgebra

$p : At \longrightarrow P_f P_f At$

Naturality fails: ListNat

There is a map in $\mathcal{L}_\Sigma$ of the form $0 \to 1$ that models the nullary function symbol 0. So, naturality of the map $p : At \longrightarrow P_f P_f At$ in $[\mathcal{L}_\Sigma^{op}, Set]$ would yield commutativity of the diagram

$$\begin{array}{ccc} At(1) & \longrightarrow & P_f P_f At(1) \\ \downarrow & & \downarrow \\ At(0) & \longrightarrow & P_f P_f At(0) \end{array}$$

No clause "$\mathtt{nat(x)} \leftarrow$", but the commutativity of the diagram would in turn imply that there cannot be a clause in ListNat of the form $\mathtt{nat(0)} \leftarrow$. In fact, there is!

The solution: Lax naturality, Posets and and P_c

The diagram need not commute, but rather the composite via $P_f P_f At(m)$ need only yield a subset of that via $At(n)$.

$$\begin{array}{ccc} At(m) & \longrightarrow & P_f P_f At(m) \\ \downarrow & & \downarrow \geq \\ At(n) & \longrightarrow & P_f P_f At(n) \end{array}$$

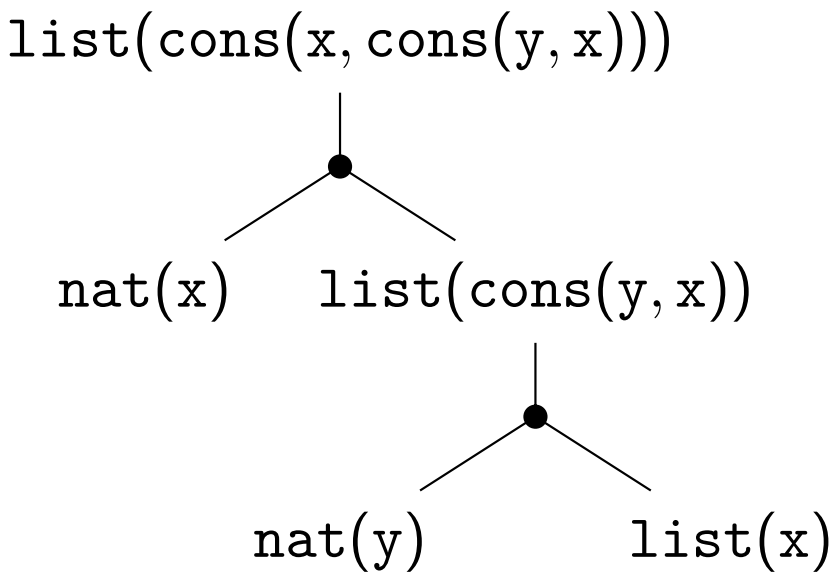
$p : At \longrightarrow P_c P_f At$ gives a $Lax(\mathcal{L}_\Sigma^{op}, P_c P_f)$-coalgebra structure on At; and p determines a $Lax(\mathcal{L}_\Sigma^{op}, C(P_c P_f))$-coalgebra structure $\bar{p} : At \longrightarrow C(P_c P_f)(At)$.

Note that we extend *Set* to *Poset* in $At : \mathcal{L}_\Sigma^{op} \to Set$, and change P_f to P_c for the endofunctor $P_c P_f$.

Examples of first-order coinductive trees determined by the semantics:

$A(x, y) \in At(2)$

Then apply At to the map $(s, s) : 1 \to 2$ in $\mathcal{L}_\Sigma$



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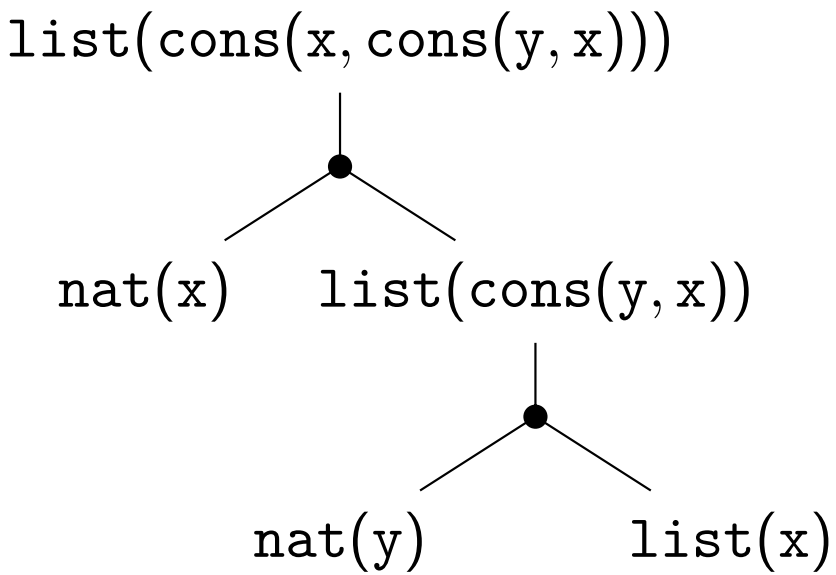
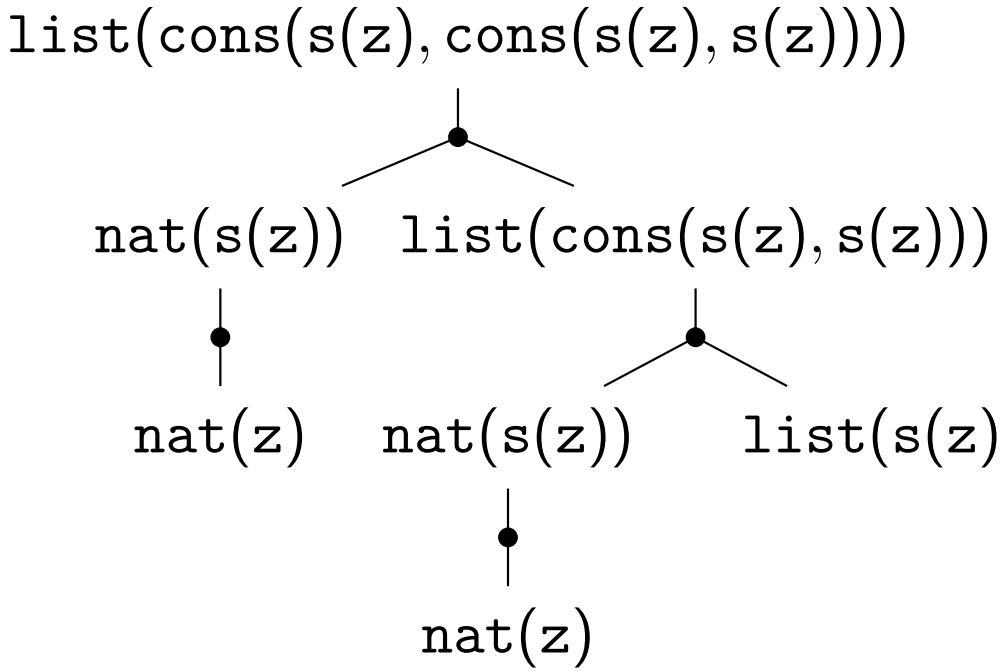
$A(z) \in At(1)$

$At((s, s))(A(x, y))$ is an element of $P_c P_f At(1)$.

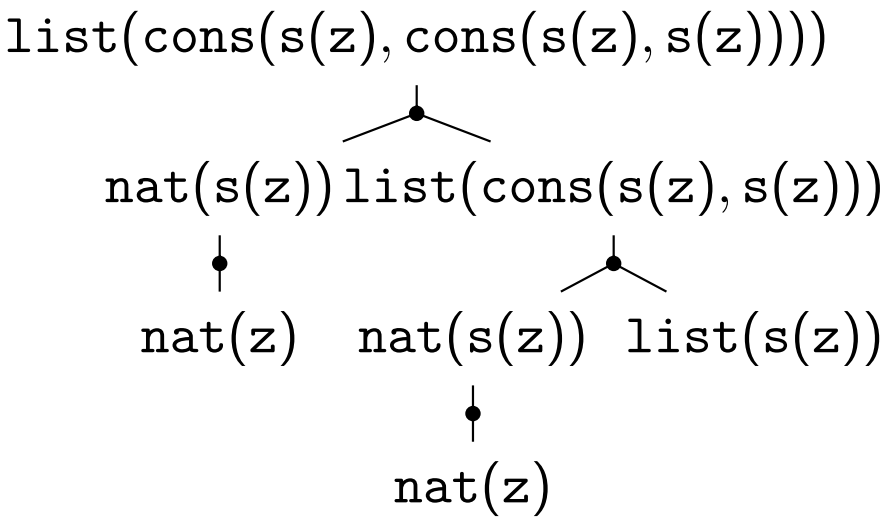




Examples of first-order coinductive trees determined by the semantics:

$A(z) \in At(1)$

Then apply At to the map $O : 0 \to 1$ in $\mathcal{L}_\Sigma$.



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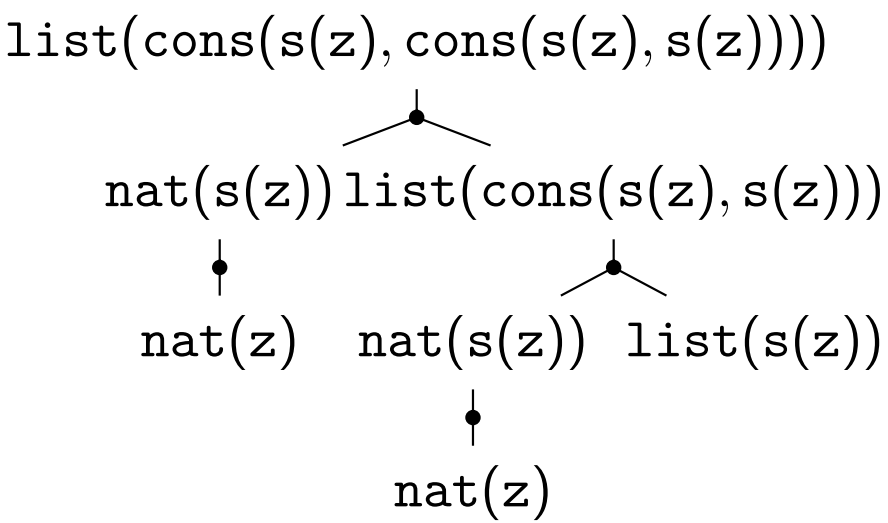
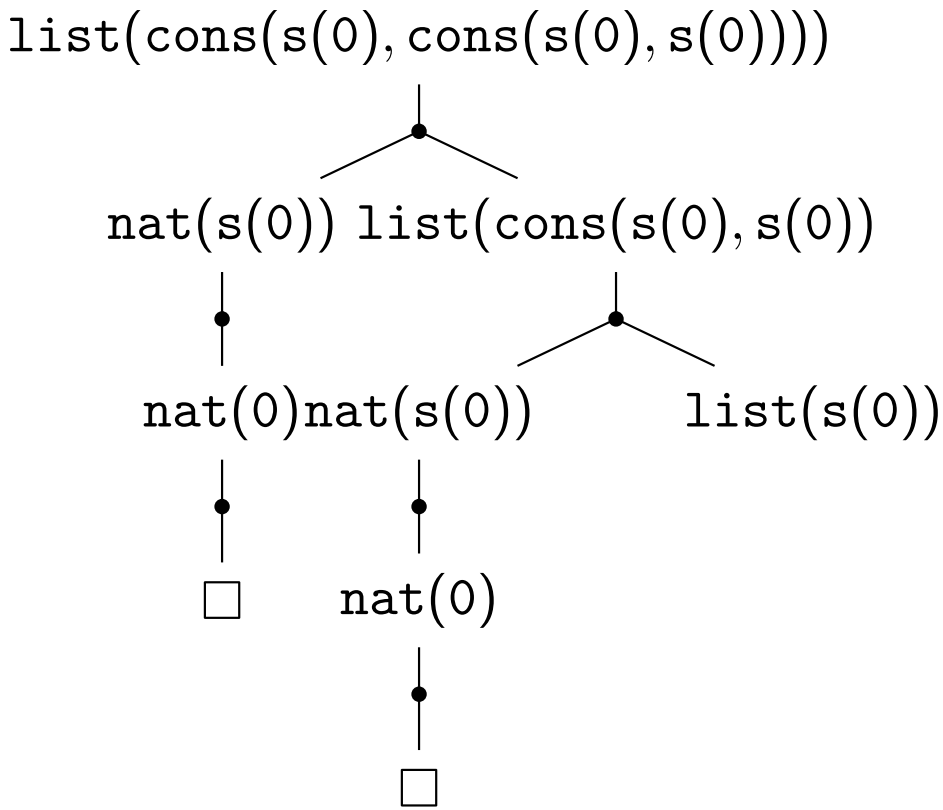
$A(z) \in At(1)$

Then apply At to the map $O : 0 \to 1$ in $\mathcal{L}_\Sigma$.



$pAt(0)At((s,s))(A(x,y))$



Thank you!