Structural Resolution

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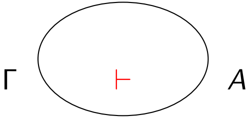
Outline

Proof methods: structural, unstructured, and?

Abstracting from the details, all proof-search and proof-inference methods can be classified as

more or less Structural...



Proof inference methods: structural

Constructive Type theory

is more Structural...

$$\Gamma \quad \vdash p : \quad A$$

To prove $\Gamma \vdash A$, we need to show that type A has inhabitant p; namely, we have to conSTRUCT it.

Proof inference methods

Resolution-based first-order automated theorem provers (ATPs)

are less Structural...

$$\Gamma \qquad \vdash \qquad A$$

To prove $\Gamma \vdash A$, we need to assume A is false, and derive a contradiction from $\Gamma \cup \neg A$.

It only matters if resolution <u>finitely succeeds</u>; the proof structure is irrelevant.

Logic Programming...

SLD resolution $=$ Unification $+$ Search

SLD-resolution + unification in LP derivations.

Program **NatList**:

Example

1.`nat(0)` $\leftarrow$
2.`nat(s(x))` $\leftarrow$ `nat(x)`
3.`list(nil)` $\leftarrow$
4.`list(cons(x,y))` $\leftarrow$

`nat(x), list(y)`

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$$\leftarrow \texttt{list}(\texttt{cons}(\texttt{x},\texttt{y}))$$
$$|$$
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$$\begin{array}{c} \leftarrow \texttt{list(cons(x,y))} \\ | \\ \leftarrow \texttt{nat(x)},\texttt{list(y)} \\ | \\ \leftarrow \texttt{list(y)} \\ | \\ \Box \end{array}$$

The answer is "Yes", $\texttt{NatList} \vdash \mathit{list}(\mathit{cons}(x, y))$ if $x/0$, y/nil, but we can get more substitutions by backtracking.

SLD-refutation = finite successful SLD-derivation. SLD-refutations are sound and complete.

Problem

LP has never received a coherent, uniform theory of *Universal Termination*.

the program P is terminating, if, given any term A, a derivation for $P \vdash A$ returns an answer in a finite number of derivation steps.

- The survey [deSchreye, 1994] lists some 119 approaches to termination in LP, neither using universal termination.
- The consensus has not been reached to this day.

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Reasons? – The lack of structural theory, namely:

Reason-1. *Non-determinism of proof-search in LP: –* termination depends on the searching strategy and order of clauses.

NatList2:

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$$\dots$$

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We have no means to analyse the structure of computations but run a search... which may be deceiving.

Reason 2. *Termination and (deciding) entailment are closely connected in LP.*

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No answer, as derivation never terminates. Neverthless, the program could be given a coindutive meaning...

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| \\
\leftarrow \mathtt{bit(x), stream(y)} \\
| \\
\leftarrow \mathtt{stream(y)} \\
| \\
\leftarrow \mathtt{bit(x_1), stream(y_1)} \\
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No distinction between type, function definition, and proof that could help to separate the issues...

Reason 3. *"Lack of directionality" in LP:*

Structurally recursive addition:

$$
\begin{array}{lll}
1.\mathtt{add}(\mathtt{0}, \mathtt{Y}, \mathtt{Y}) & \leftarrow & \\
2.\mathtt{add}(\mathtt{s}(\mathtt{X}), \mathtt{Y}, \mathtt{s}(\mathtt{Z})) & \leftarrow & \mathtt{add}(\mathtt{X}, \mathtt{Y}, \mathtt{Z})
\end{array}
$$

If the third argument in add is "thought of" as the "output", and the other arguments – as "inputs", then, giving queries with variable-free "inputs" will guarantee termination by structural recursion on the first argument. But otherwise, there will be non-terminating derivations for queries to add.
As a consequence, in LP, it is common to talk about existential termination (only for some derivations, for queries of certain kinds, or satisfying certain conditions/measures), not programs in general.

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This unstructured approach to $\vdash$ gives us too little formal support to analyse termination

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- Or may be it is a recursive program with coinductive interpretation? (again, **NatList2**)

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- Or may be it is just some bad loop without particular computational meaning:

$$badstream(scons(x, y)) \leftarrow badstream(scons(x, y))$$

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We are missing a theory, a language, to talk about such things...

Problems with LP termination and static program analysis

From its conception in 1960's, LP/ATP has not formulated a theory of universal termination!

All below programs do not terminate, and fail to produce any answer in PROLOG.

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$\mathtt{nat(s(x))} \leftarrow \mathtt{nat(x)}$ $\mathtt{nat(0)} \leftarrow$	$\mathtt{stream(scons(x,y))} \leftarrow \mathtt{nat(x), stream(y)}$	$\mathtt{bad(x)} \leftarrow \mathtt{bad(x)}$

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No termination – no program analysis

New methods. In search of a missing link

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Is there a mysterious Missing link theory?

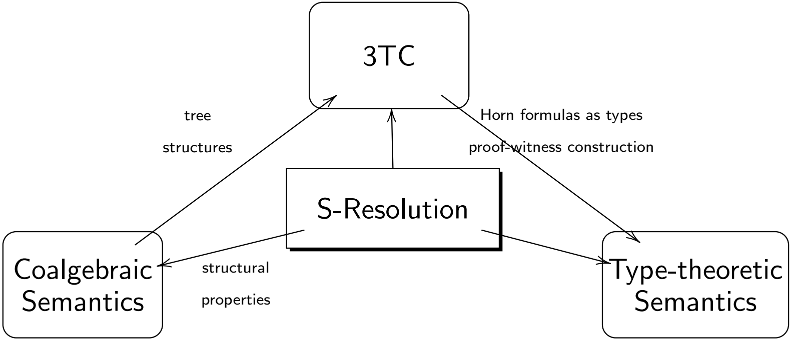
– Structural Resolution (also S-Resolution)

Is there place for a DISCOVERY here, which could expose A BETTER STRUCTURED resolution?

What IS

S-Resolution?

Structural Resolution:



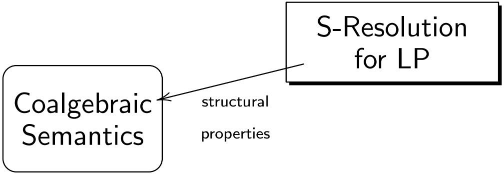
Outline

Structural Resolution:

[2010-2014, K, Power]

Discovery A:

(A) Structural Properties of Programs Uniquely determine Structural Properties of Computations



Outline

Motivation

Coalgebraic Semantics for Structural Resolution

The Three Tier Tree calculus for Structural Resolution

Type-Theoretic view of Structural Resolution

Conclusions and Future work

Our running example

Example

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2. $\mathtt{nat}(\mathtt{0}) \leftarrow$
3. $\mathtt{stream}(\mathtt{scons}(\mathtt{x}, \mathtt{y})) \leftarrow \mathtt{nat}(\mathtt{x}), \mathtt{stream}(\mathtt{y})$

Note: double-hopeless for SLD-resolution-based ATP!

Defining structural resolution from first principles...

Main credo: we do not impose types or extra annotations, but look deep for “sub-atomic” structures innate in first-order proofs.

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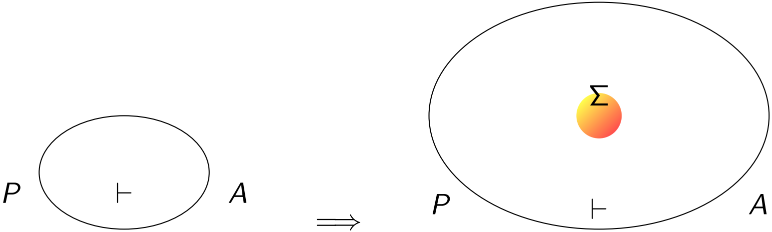
Given a logic program P there is a first-order signature Σ in P...



Example

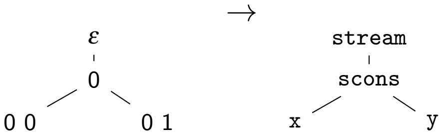
For our example, $\Sigma = \{0, \mathtt{s}, \mathtt{scons}, \mathtt{nat}, \mathtt{stream}\}$ + Variables.

Tier-1: Term-trees, given Σ:

Let $\mathbb{N}^*$ denote the set of all finite words over $\mathbb{N}$.

A set $L \subseteq \mathbb{N}^*$ is a *(finitely branching) tree language*, satisfying prefix closedness conditions.

A term tree is a map $L \rightarrow \Sigma \cup Var$, satisfying term **arity** restrictions.



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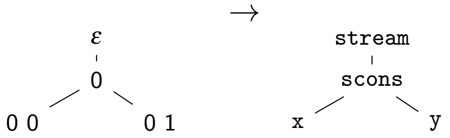

Calculus: first-order unification

Given two terms t_1, t_2, and a substitution θ, θ is a unifier if $\theta(t_1) = \theta(t_2)$, and matcher if $t_1 = \theta(t_2)$.

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$$\begin{array}{ccc} & \varepsilon & \\ & | & \\ & 0 & \\ / & & \backslash \\ 0\,0 & & 0\,1 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} & \texttt{stream} & \\ & | & \\ & \texttt{scons} & \\ / & & \backslash \\ \texttt{x} & & \texttt{y} \end{array}$$

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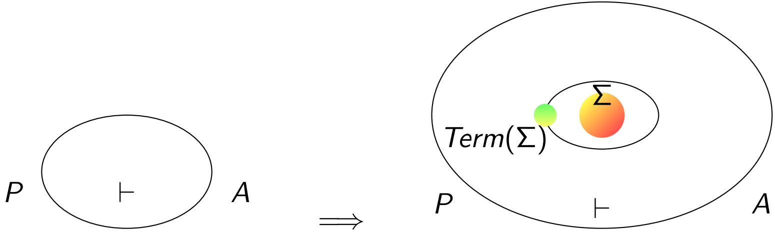
Notation:

Term(Σ)	Set of *finite* term trees over Σ
Term$^{\infty}(\Sigma)$	Set of *infinite* term trees over Σ
Term$^{\omega}(\Sigma)$	Set of *finite and infinite* term trees over Σ

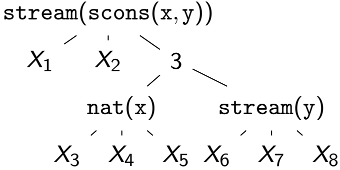
Constructing the structural resolution from first principles...

- Given a logic program P there is a first-order signature Σ...
- First tier of Terms builds on it...



Tier-2: rewriting trees

A rewriting tree is a map $L \to \textbf{Term}(\Sigma) \cup \textbf{Clause}(\Sigma) \cup Var_R$, subject to arity conditions.



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Interesting: Variables of Tier 2 and finiteness of rewriting trees for our “difficult” example!

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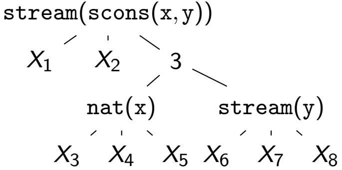

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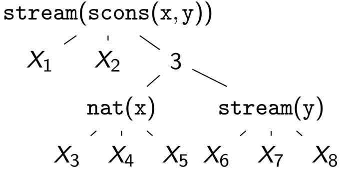

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Calculus: tree transition by Tier-2 variable substitution

Notation:

$\textbf{Rew}(P)$	all *finite* rewriting trees over P and $\textbf{Term}(\Sigma)$
$\textbf{Rew}^{\infty}(P)$	all *infinite* rewriting trees over P and $\textbf{Term}(\Sigma)$
$\textbf{Rew}^{\omega}(P)$	all *finite and infinite* rewriting trees over P and $\textbf{Term}(\Sigma)$

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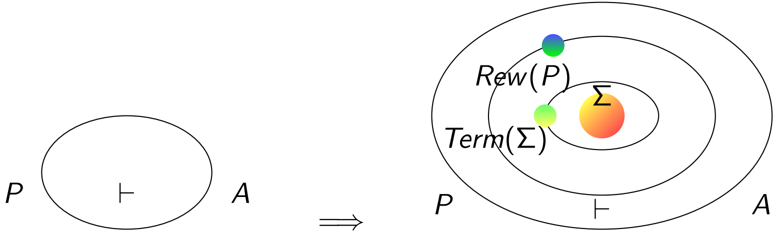
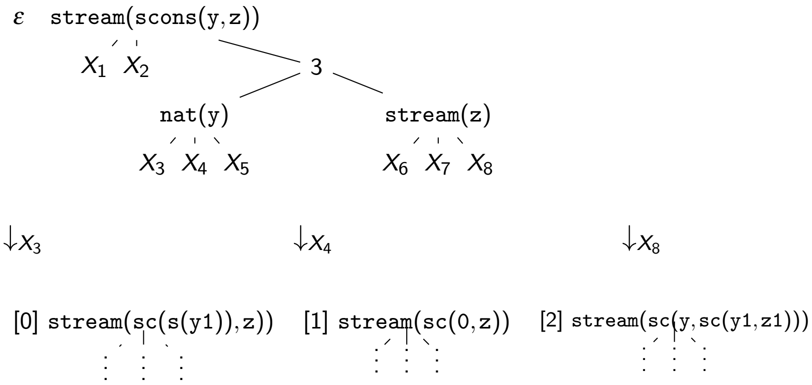
- Given a logic program P there is a first-order signature Σ...
- First tier of Terms builds on it...
- Term-trees give rise to a new tier of rewriting trees...



Tier-3: Derivation trees

A derivation tree is a map $L \rightarrow \mathbf{Rew}(P)$, subject to Arity condition (given by the number of clauses in P).



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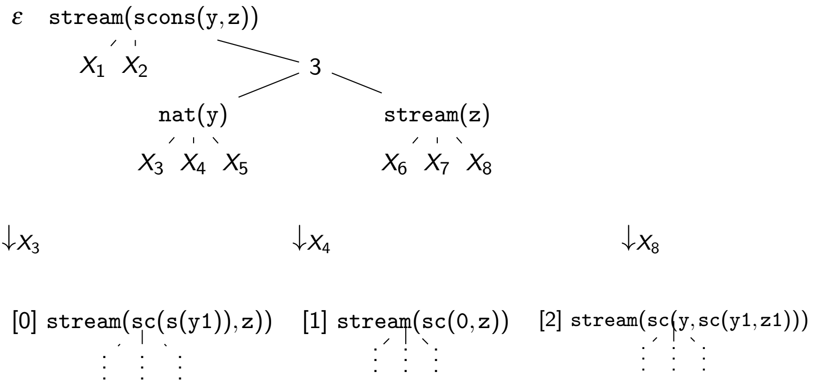

Note: this derivation tree is infinite.

Tier-3 laws and notation

Notation:

$\mathbf{Der}(P)$	all *finite* derivation trees over $\mathbf{Rew}(P)$
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$\mathbf{Der}^{\omega}(P)$	all *finite and infinite* derivation trees over $\mathbf{Rew}(P)$

Tier-3 laws and notation

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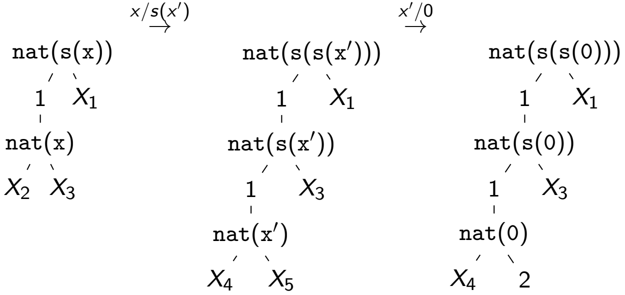
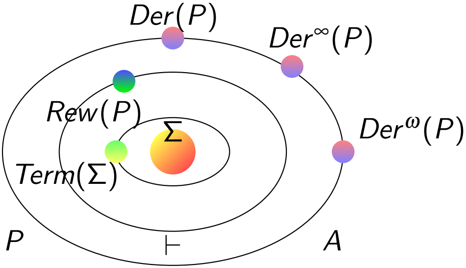
An SLD-derivation for a program P and goal A corresponds to a branch in a derivation tree for P and A.



Constructing the structural resolution from first principles...

- Given a logic program P there is a first-order signature Σ...
- First tier of Terms builds on it...
- Term-trees give rise to a new tier of rewriting trees.
- And then, derivations by Structural resolution emerge!

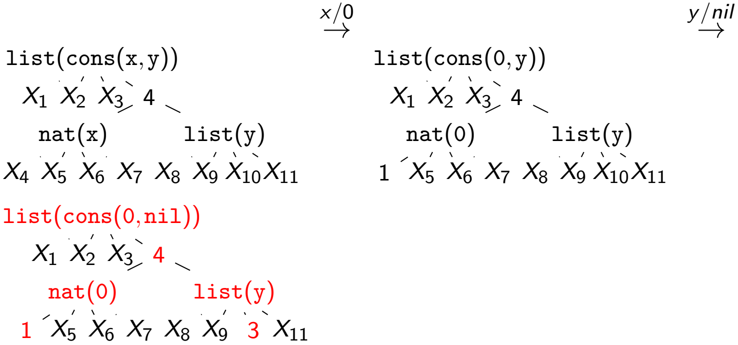


Gains:

- We found a missing theory of constructive resolution!
- Now to prove $P \vdash A$, we need to construct a rewriting tree $rew \in Rew(P)$ that proves A:

$$P \vdash rew : A$$

To prove $ListNat \vdash list(cons(x, y))$, we need to construct a rewriting tree that proves it:



New theory of universal productivity for resolution

A program P is productive, if it gives rise to rewriting trees only in **Rew**(P).

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In the class of Productive LPs, we can further distinguish:

- finite LP that give rise to derivations in $\mathbf{Der}(P)$,
- inductive LPs all derivations for which are in $\mathbf{Der}^{\omega}(P)$;
- coinductive LPs all derivations for which are in $\mathbf{Der}^{\infty}(P)$

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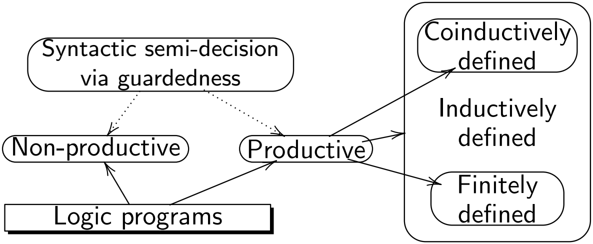
Productive inductive program	Productive coinductive program	Non-productive program
rewriting trees in $Rew(P)$, derivation trees $Der^{\omega}(P)$	rewriting trees in $Rew(P)$, derivation trees in $Der^{\infty}(P)$	rewriting trees do not belong to $Rew(P)$

Theory of universal Productivity in LP!



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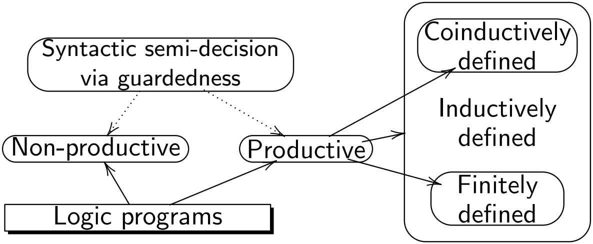
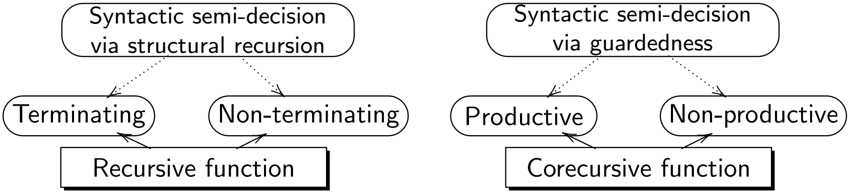
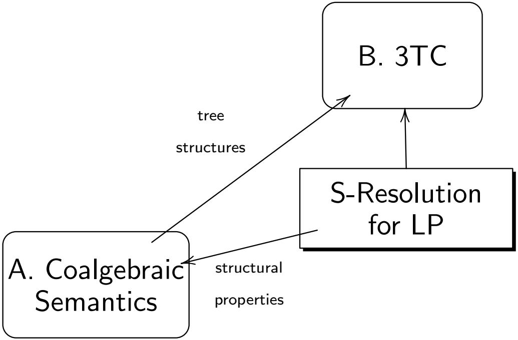




Structural Resolution:

Discovery B:

(B) Structures suggested by (A) can give a sound calculus, and solve problems known to be hard for LP: universal productivity and coinductive proof inference.



More questions still:

- What is the proof-theoretic meaning of S-Resolution?
- What is the constructive content of proofs by resolution?
- How do the rewriting trees relate to term rewriting systems?
- Does the informal analogy of 3TC

  $$P \vdash \mathit{rew} : A$$

  really have any relation to type theory?
- How exactly does the intuition that rewriting trees may serve as proof-witnesses in S-derivations relate to the type theory setting?

Outline

Horn formula view of LP

$$\kappa_1 : \Rightarrow \mathrm{Nat}(0)$$
$$\kappa_2 : \ \mathrm{Nat}(x) \Rightarrow \mathrm{Nat}(s(x))$$
$$\kappa_3 : \Rightarrow \mathrm{List}(\mathrm{nil})$$
$$\kappa_4 : \mathrm{Nat}(x), \mathrm{List}(y) \Rightarrow \mathrm{List}(\mathrm{cons}(x, y))$$

Formalism: LP-Unif, LP-TM and LP-Struct

- **Term-matching reduction:**
  $\Phi \vdash \{A_1, ..., A_i, ..., A_n\} \rightarrow_{\kappa,\sigma} \{A_1, ..., \sigma B_1, ..., \sigma B_m, ..., A_n\}$, if there exists $\kappa : \forall \underline{x}. B_1, ..., B_n \Rightarrow C \in \Phi$ such that $C \mapsto_\sigma A_i$.

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- **Unification reduction:**
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- **LP-TM:** $(\Phi, \to)$
  LP-Unif: $(\Phi, \rightsquigarrow)$
  LP-Struct: $(\Phi, \to^\mu \cdot \hookrightarrow^1)$

Execution behavior of LP-TM

- Consider query $\text{List}(\text{cons}(x,y))$:
  $\{\text{List}(\text{cons}(x,y))\} \rightarrow_{\kappa_4,[x/x_1,y/y_1]} \{\text{Nat}(x), \text{List}(y)\}$
  Note Partial nature

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- Consider following Stream predicate:
  $\kappa : \text{Stream}(y) \Rightarrow \text{Stream}(\text{cons}(x, y))$
- In LP-TM:
  $\{\text{Stream}(\text{cons}(x, y))\} \rightarrow_{\kappa, [x/x_1, y/y_1]} \{\text{Stream}(y)\}$

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Note finiteness

LP-Struct: BList

For query $\mathrm{List}(\mathrm{cons}(x, y))$, in LP-Struct:

- $\{\mathrm{List}(\mathrm{cons}(x, y))\} \to \{\mathrm{Nat}(\mathrm{x}), \mathrm{List}(y)\}$

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For query $\mathrm{List}(\mathrm{cons}(x, y))$, in LP-Struct:

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- $\hookrightarrow_{[0/x]} \{\mathrm{Nat}(0), \mathrm{List}(y)\} \to \{\mathrm{List}(y)\}$

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- ...
- Partial answer: $\mathrm{cons}(x_1, \mathrm{cons}(x_2, \mathrm{cons}(x_3, y_3)))/y$

Formalization of a Type System

- Term $t ::= x \mid f(t_1, ..., t_n)$
  Atomic Formula $A, B, C, D ::= P(t_1, ..., t_n)$
  (Horn) Formula $F ::= A_1, ..., A_n \Rightarrow A$
  Proof Term $p, e ::= \kappa \mid a \mid \lambda a.e \mid e\ e'$

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- Girard's observation on intuitionistic sequent calculus with atomic formulas

$$\frac{}{\underline{B} \vdash A}\ axiom \qquad \frac{\underline{B} \vdash C}{\sigma \underline{B} \vdash \sigma C}\ subst \qquad \frac{\underline{A} \vdash D \quad \underline{B}, D \vdash C}{\underline{A}, \underline{B} \vdash C}\ cut$$

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- Is $\vdash Q$ provable?

Formalization of a Type System

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- Is $\vdash Q$ provable?
- We internalized "$\vdash$" as "$\Rightarrow$" and add proof term annotations

$$\frac{}{\kappa : \forall \underline{x}.F}\ \mathit{axiom} \qquad \frac{e : F}{e : \forall \underline{x}.F}\ \mathit{gen}$$

$$\frac{e : \forall \underline{x}.F}{e : [\underline{t}/\underline{x}]F}\ \mathit{inst} \qquad \frac{e_1 : \underline{A} \Rightarrow D \quad e_2 : \underline{B}, D \Rightarrow C}{\lambda \underline{a}.\lambda \underline{b}.(e_2\ \underline{b})\ (e_1\ \underline{a}) : \underline{A}, \underline{B} \Rightarrow C}\ \mathit{cut}$$

Soundness of LP-TM and LP-Unif

- *Soundness of LP-Unif*
  If $\Phi \vdash \{A\} \rightsquigarrow^*_\gamma \emptyset$, then there exists a proof $e : \forall \underline{x}. \Rightarrow \gamma A$ given axioms Φ.
- *Soundness of LP-TM*
  If $\Phi \vdash \{A\} \rightarrow^* \emptyset$, then there exists a proof $e : \forall \underline{x}. \Rightarrow A$ given axioms Φ.
- For example:
  $\{\mathrm{BList}(\mathrm{cons}(x, y))\} \rightsquigarrow \{\mathrm{Bit}(\mathrm{x}), \mathrm{BList}(y)\} \rightsquigarrow_{[0/x]} \{\mathrm{BList}(y)\}$
  $\rightsquigarrow_{[0/x, \mathrm{nil}/y]} \rightsquigarrow \emptyset$
- yields a proof $(\lambda a.(\kappa_4\ a)\ \kappa_1)\ \kappa_3 : \mathrm{BList}(\mathrm{cons}(0, nil))$
  (β-reducible to $(\kappa_4 \kappa_3)\kappa_1$).

Soundness of LP-TM and LP-Unif

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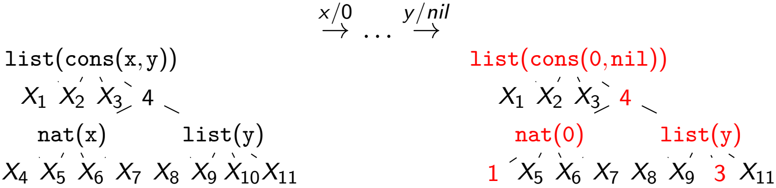
  $\leadsto_{[0/x,\mathrm{nil}/y]} \leadsto \emptyset$
- yields a proof $(\lambda a.(\kappa_4\ a)\ \kappa_1)\ \kappa_3 : \mathrm{BList}(\mathrm{cons}(0, nil))$ (β-reducible to $(\kappa_4 \kappa_3)\kappa_1$).
- Compare with the 3TC proof-witness:



LP-Struct is equivalent to LP-Unif

... for logic programs subject to realisability transformation

$\kappa_1 : \Rightarrow \mathrm{Nat}(0, c_{\kappa_1})$
$\kappa_2 : \ \mathrm{Nat}(x, u) \Rightarrow \mathrm{Nat}(s(x), f_{\kappa_2}(u))$
$\kappa_3 : \Rightarrow \mathrm{BList}(\mathrm{nil}, c_{\kappa_3})$
$\kappa_4 : \mathrm{Bit}(x, u_1), \mathrm{BList}(y, u_2) \Rightarrow \mathrm{BList}(\mathrm{cons}(x, y, f_{\kappa_4}(u_1, u_2)))$

- $\{\mathrm{BList}(\mathrm{cons}(x, y, u))\} \hookrightarrow_{[f_{\kappa_4}(u_1,u_2)/u]}$
  $\{\mathrm{BList}(\mathrm{cons}(x, y, f_{\kappa_4}(u_1, u_2)))\} \rightarrow \{\mathrm{Bit}(\mathrm{x}, \mathrm{u}_1), \mathrm{BList}(y, u_2)\}$

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- $\hookrightarrow_{[0/x, \mathrm{nil}/y, c_{\kappa_3}/u_2]} \{\mathrm{BList}(\mathrm{nil}, c_{\kappa_3})\} \to \emptyset$

Note the substitution for $u/f_{\kappa_4}(c_{\kappa_1}, c_{\kappa_3})$ matches the earlier computed proof term $(\kappa_4 \kappa_3)\kappa_1$.

Results about Realizability Transformation

- *Guarantees productivity = Termination of term-matching reduction*
  Directly inherited from 3TC
- *Preserves Provability*
- *Records Proof*
  in the extra argument substitutions
- *Preserves Computational behaviour of LP-Unif*
- *Helps to prove Operational Equivalence of LP-Unif and LP-Struct*
- *Helps to prove soundness of LP-Struct*

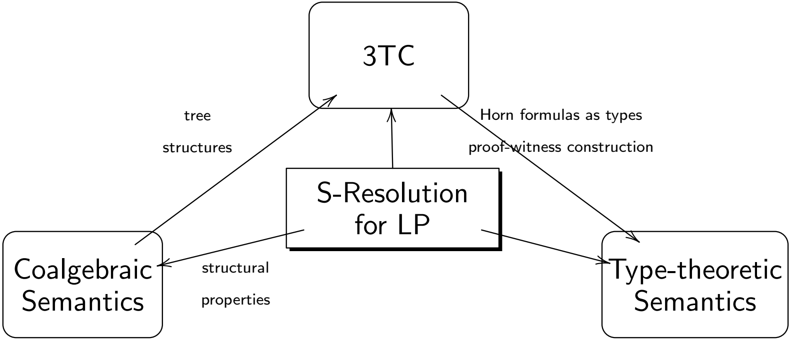
Gains from type-theoretic semantics for S-Resolution:

1. We established a direct relation to term-rewriting via LP-Struct;
2. We established a natural typed λ-calculus characterisation;
3. LP-Struct is sound wrt the type system;
4. Proof-witness is now formally defined as type inhabitant;
   directly inherited from 3TC
5. S-resolution is not equivalent to SLD-resolution, in general;
6. We exactly described the class of LPs that have structural properties (for which S-resolution and SLD-resolution are equivalent);
   directly inherited from 3TC
7. and gave an automated and static way to transform LPs to their constructive variants (via realisability transformation).

Structural Resolution:

Discovery C:

(C) The 3 Tier Tree calculus gives genuine insight into constructive nature of first-order automated proof: Horn-formulas as types and proof-witnesses as type inhabitants.



Outline

Structural Resolution ABC

S-resolution is Automated proof-search by resolution

in which:

(A) Structural Properties of Programs Uniquely determine Structural Properties of Computations

(B) These structures define a sound calculus, and solve problems known to be hard for LP: universal productivity and coinductive proof inference.

(C) The 3 Tier Tree calculus gives genuine insight into constructive nature of first-order automated proof

Current work

Applications of the above to Type Inference

... see the next talk by Fu Peng.

To construct equality class instance for datatype list and int:

$$\begin{array}{ll} \kappa_1 : & \mathrm{Eq}(x) \Rightarrow \mathrm{Eq}(\mathrm{list}(x)) \\ \kappa_2 : & \Rightarrow \mathrm{Eq}(\mathrm{int}) \end{array}$$

When we call a query $\mathrm{Eq}(\mathrm{list}(\mathrm{int}))$, we can use LP-TM to construct a proof for $\mathrm{Eq}(\mathrm{list}(\mathrm{int}))$, which is $\kappa_2\ \kappa_1$, and then the evaluation of $\kappa_2\ \kappa_1$ will correspond to the process of evidence construction, thus yielding computational meaning of the proof.

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Dreams for the Future

Structural resolution as a new —
better structured and more constructive —
foundation for Automated Proof Search, starting from LP and reaching as far as Resolution-based SAT and SMT solvers.

Thank you!

CoALP webpage:
http://staff.computing.dundee.ac.uk/katya/CoALP/

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