Statistical Machine Learning in Interactive Theorem Proving

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University of Dundee

8 November 2013

Outline

1. Introduction

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2. ML4PG: “Machine Learning for Proof General”

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3. Using ML4PG

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3. Using ML4PG
4. More Examples
   - Detecting patterns across mathematical libraries
   - Detecting irrelevant libraries

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1. Introduction
2. ML4PG: "Machine Learning for Proof General"
3. Using ML4PG
4. More Examples
   - Detecting patterns across mathematical libraries
   - Detecting irrelevant libraries
5. Conclusions and Further work

Interactive theorem proving:

- (typically) higher-order language (Agda,Coq,Isabelle/HOL)
- (often) dependently-typed (AGDA,Coq)
- Interactive proof development: tactic – prover response;
- Expressive enough to verify large areas of Maths, software, hardware.

Interactive theorem proving:

- (typically) higher-order language (Agda,Coq,Isabelle/HOL)
- (often) dependently-typed (AGDA,Coq)
- Interactive proof development: tactic – prover response;
- Expressive enough to verify large areas of Maths, software, hardware.
  - ...enriched with dependent types, (co)inductive types, type classes and provide rich programming environments;
  - ...applied in formal mathematical proofs: Four Colour Theorem (60,000 lines), Kepler conjecture (325,000 lines), Feit-Thompson Theorem (170,000 lines), etc.
  - ...applied in industrial proofs: seL4 microkernel (200,000 lines), verified C compiler (50,000 lines), ARM microprocessor (20,000 lines), etc.

Coq and SSReflect

- SSReflect is a dialect of Coq;
- The SSReflect library was developed as the infrastructure for formalisation of the Four Colour Theorem;
- played a key role in the formal proof of the Feit-Thompson theorem.

G. Gonthier. Formal proof - the four-color theorem. Notices of the American Mathematical Society, 55(11):13821393, 2008.

G. Gonthier et al. A Machine-Checked Proof of the Odd Order Theorem. In 4th Conference on Interactive Theorem Proving (ITP13), volume 7998 of Lecture Notes in Computer Science, pages 163179, 2013.

Challenges

- . . . size and sophistication of libraries stand on the way of efficient knowledge reuse;
- . . . manual handling of various proofs, strategies, libraries, becomes difficult;
- . . . team-development is hard, especially that ITPs are sensitive to notation;
- . . . comparison of proof similarities is hard.

An example: JVM

Java Virtual Machine (JVM) is a stack-based abstract machine which can execute Java bytecode.

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Goal

- Model a subset of the JVM in (e.g.) Coq, defining an interpreter for JVM programs,
- Verify the correctness of JVM programs within Coq.

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Java Virtual Machine (JVM) is a stack-based abstract machine which can execute Java bytecode.

Goal

- Model a subset of the JVM in (e.g.) Coq, defining an interpreter for JVM programs,
- Verify the correctness of JVM programs within Coq.

This work is inspired by:

H. Liu and J S. Moore. Executable JVM model for analytic reasoning: a study. Journal Science of Computer Programming - Special issue on advances in interpreters, virtual machines and emulators (IVME'03), 57(3):253–274, 2003.

Computing 5!

Java code:

```
static int factorial(int n)
{
  int a = 1;
  while (n != 0){
    a = a * n;
    n = n-1;
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Bytecode:

0	:	*iconst* 1
1	:	*istore* 1
2	:	*iload* 0
3	:	*ifeq* 13
4	:	*iload* 1
5	:	*iload* 0
6	:	*imul*
7	:	*istore* 1
8	:	*iload* 0
9	:	*iconst* 1
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11	:	*istore* 0
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JVM model:

counter:

0

stack:

				...

local variables:

5				...

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counter:

1

stack:

1				...

local variables:

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JVM model:

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2

stack:

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local variables:

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stack:

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JVM model:

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stack:

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local variables:

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JVM model:

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stack:

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JVM model:

counter:

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stack:

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local variables:

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JVM model:

counter:

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stack:

5				...

local variables:

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JVM model:

counter:

10

stack:

1	5			...

local variables:

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JVM model:

counter:

11

stack:

4				...

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counter:

12

stack:

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local variables:

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counter:

2

stack:

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local variables:

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JVM model:

counter:
13

stack:

0				...

local variables:

0	120			...

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JVM model:

counter:

14

stack:

120				...

local variables:

0	120			...

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counter:

15

stack:

120				...

local variables:

0	120			...

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JVM model:

counter:

15

stack:

120				...

local variables:

0	120			...

Goal (Factorial case)

$\forall n \in \mathbb{N}$, running the bytecode associated with the factorial program with n as input produces a state which contains $n!$ on top of the stack.

Formalisation of Java bytecode in Coq

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Methodology:

1. Write the specification of the function

```
Definition theta_fact (n : nat) := n`!.
```

Formalisation of Java bytecode in Coq

Goal (Factorial case)

$\forall n \in \mathbb{N}$, running the bytecode associated with the factorial program with n as input produces a state which contains $n!$ on top of the stack.

Methodology:

1. Write the specification of the function
2. Write the algorithm (tail recursive function)

```
Fixpoint helper_fact (n a : nat) :=
match n with
| 0 => a
| S p => helper_fact p (n * a)
end.

Definition fn_fact (n : nat) :=
   helper_fact n 1.
```

Formalisation of Java bytecode in Coq

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Methodology:

1. Write the specification of the function
2. Write the algorithm (tail recursive function)
3. Prove that the algorithm satisfies the specification

```
Lemma fn_fact_is_theta n :
    fn_fact n = theta_fact n.
```

Formalisation of Java bytecode in Coq

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Methodology:

1. Write the specification of the function
2. Write the algorithm (tail recursive function)
3. Prove that the algorithm satisfies the specification
4. Write the JVM program

```
Definition pi_fact :=
  [::(ICONST,1%Z);
     (ISTORE,1%Z);
     (ILOAD,0%Z);
     (IFEQ,10%Z);
     (ILOAD,1%Z);
     (ILOAD,0%Z);
     (IMUL, 0%Z);
     (ISTORE, 1%Z);
     (ILOAD, 0%Z);
     (ICONST, 1%Z);
     (ISUB, 0%Z);
     (ISTORE, 0%Z);
     (GOTO, (-10)%Z);
     (ILOAD, 1%Z);
     (HALT, 0%Z)].
```

Formalisation of Java bytecode in Coq

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$\forall n \in \mathbb{N}$, running the bytecode associated with the factorial program with n as input produces a state which contains $n!$ on top of the stack.

Methodology:

1. Write the specification of the function
2. Write the algorithm (tail recursive function)
3. Prove that the algorithm satisfies the specification
4. Write the JVM program
5. Define the function that schedules the program

```
Fixpoint loop_sched_fact (n : nat) :=
match n with
| 0 => nseq 3 0
| S p => nseq 11 0 ++ loop_sched_fact p
end.


Definition sched_fact (n : nat) :=
   nseq 2 0 ++ loop_sched_fact n.
```

Formalisation of Java bytecode in Coq

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Methodology:

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2. Write the algorithm (tail recursive function)
3. Prove that the algorithm satisfies the specification
4. Write the JVM program
5. Define the function that schedules the program
6. Prove that the code implements the algorithm

```
Lemma program_is_fn_fact n :
  run (sched_fact n)
    (make_state 0 [::n] [::] pi_fact) =
 (make_state 14 [::0;fn_fact n ]
    (push (fn_fact n ) [::]) pi_fact).
```

Formalisation of Java bytecode in Coq

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Methodology:

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2. Write the algorithm (tail recursive function)
3. Prove that the algorithm satisfies the specification
4. Write the JVM program
5. Define the function that schedules the program
6. Prove that the code implements the algorithm
7. Prove total correctness

```
Theorem total_correctness_fact n sf :
  sf = run (sched_fact n)
  (make_state 0 [::n] [::] pi_fact) ->
  next_inst sf = (HALT,0%Z) /\
  top (stack sf) = (n`!).
```

Formalisation of Java bytecode in Coq

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7. Prove total correctness

```
Lemma fn_fact_is_theta n :
   fn_fact n = theta_fact n.
```

Proof of lemma `fn_fact_is_theta`

emacs@joheras-HP-Compaq-6730b-GW687AV

File Edit Options Buffers Tools Coq Proof-General Holes Help

```
Lemma fn_fact_is_theta : forall (n : nat), fn_fact n = theta_fact n.
Proof.
```

```
-U:**-  lists.v       All L1     (Coq Script(0) Holes)----------------------------
```

```
1 subgoals, subgoal 1 (ID 13)

============================
 forall n : nat, fn_fact n = theta_fact n
```

```
-U:%%-  *response*     All L1     (Coq Response)------------------------------------
```

Proof of lemma `fn_fact_is_theta`

emacs@joheras-HP-Compaq-6730b-GW687AV

File Edit Options Buffers Tools Coq Proof-General Holes Help

```
Lemma fn_fact_is_theta : forall (n : nat), fn_fact n = theta_fact n.
Proof.
move => n.
```

-U:**- lists.v   All L1   (Coq Script(0) Holes)---------------------------

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1 subgoals, subgoal 1 (ID 14)

n : nat
============================
fn_fact n = theta_fact n
```

-U:%%- *response*   All L1   (Coq Response)-----------------------------------

Proof of lemma fn_fact_is_theta

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...and now?

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2. ML4PG: "Machine Learning for Proof General"
3. Using ML4PG
4. More Examples
   - Detecting patterns across mathematical libraries
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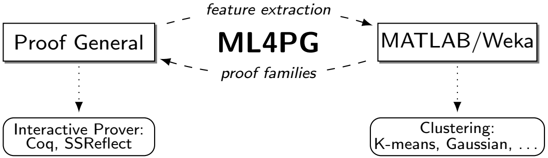
Machine Learning 4 Proof General: interfacing interfaces

...in [2013, Postproc. of UITP'12]



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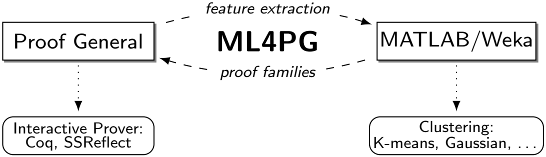

F.1. works on the background of Proof General extracting some low-level features from proofs in Coq/SSReflect.

F.2. automatically sends the gathered statistics to a chosen machine-learning interface and triggers execution of a clustering algorithm of user's choice;

F.3. does some post-processing of the results and displays families of related proofs to the user.

Features of this approach

1. Feature extraction:
   - features are extracted from higher-order propositions and proofs;
   - feature extraction is built on the method of proof-traces;
   - longer proofs are analysed by means of the proof-patch method.

What are the significant features of proofs?

1-2 names and the number of tactics used in one command line,

3 types of the tactic arguments;

4 relation of the tactic arguments to the (inductive) hypotheses or library lemmas,

5-7 three top symbols in the term-tree of the current subgoal, and

8 the number of subgoals each tactic command-line generates.

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5-7 three top symbols in the term-tree of the current subgoal, and

8 the number of subgoals each tactic command-line generates.

Taken within 5 proof steps;

...40 features for one proof patch.

Thus a proof fragment is given by a point in a 40-dimensional space.

Features of this approach

2. Machine-learning tools:
   - works with unsupervised learning (clustering) algorithms implemented in MATLAB and Weka;
   - uses algorithms such as Gaussian, K-means, and farthest-first.

ML4PG approach to proof-clustering

We have integrated Proof General with a variety of clustering algorithms:

ML4PG approach to proof-clustering

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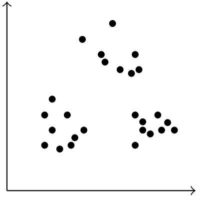

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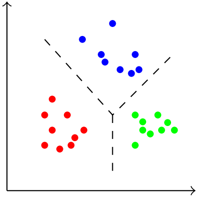

- Engines: Matlab, Weka, Octave, R, . . .

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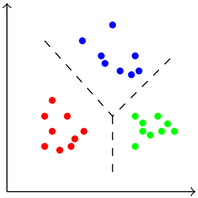

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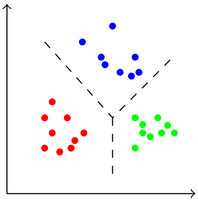

- Engines: Matlab, Weka, Octave, R, ...
- Algorithms: K-means, Gaussian Mixture models, simple Expectation Maximisation, ...

Order your own copy of Ml4PG!

- ML4PG is now a part of standard Proof General distribution
- Easy to find: just google "ML4PG" for our page with all software resources, libraries of examples, papers, etc.
  This talk:

  J. Heras and K. Komendantskaya. Recycling Proof-Patterns in Coq: Case Studies. 31 page. Submitted, available in ARXIV.
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Continuation of proof of lemma `fn_fact_is_theta`

emacs@joheras-HP-Compaq-6730b-GW687AV

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```
Lemma fn_fact_is_theta : forall (n : nat), fn_fact n = theta_fact n.
Proof.
move => n.
rewrite /fn_fact /theta_fact.
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1 subgoals, subgoal 1 (ID 14)

n : nat
============================
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 helper_fact n 1 = n`!
```

```
----------------------------------

Lemma fn_fact_is_theta is similar
to lemmas:
  - fn_expt_is_theta
  - fn_mult_is_theta
  - fn_power_is_theta

----------------------------------
```

-U:%%- *response* All L1 (Coq Response)

Proving lemma fn_fact_is_theta by analogy

Factorial	Exponentiation
`Lemma fn_fact_is_theta n : fn_fact n = n`!.` `Proof.` `move => n. rewrite /fn_fact.`	`Lemma fn_expt_is_theta n m : fn_expt n m = n^m.` `Proof.` `by move => n; rewrite /fn_expt helper_expt_is_theta` `      mul1n.` `Qed.` `Lemma helper_expt_is_theta n m a :` `   helper_expt n m a = a * (n ^ m).` `Proof.` `move : a; elim : n => [a| n IH a /=].` `      by rewrite /theta_expt expn0 muln1.` `by rewrite IH /theta_expt expnS` `      mulnA [a * _]mulnC.` `Qed.`

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Proof Strategy

Prove an auxiliary lemma about the helper considering the most general case. For example, if the helper function is defined with formal parameters n, m, and a, and the wrapper calls the helper initializing a at 0, the helper theorem must be about (helper n m a), not just about the special case (helper n m 0). Subsequently, instantiate the lemma for the concrete case.

Consistency of ML4PG clusters

Algorithm:	$g = 1$ $(n = 16)$	$g = 2$ $(n = 18)$	$g = 3$ $(n = 21)$	$g = 4$ $(n = 24)$	$g = 5$ $(n = 29)$
K-means	$30^{a,b,d}$	$\mathbf{4}^{a-d}$	$\mathbf{4}^{a-d}$	$2^{c,d}$	0
E.M.	21^{a-d}	7^{a-d}	7^{a-d}	0	0
FarthestFirst	28^{a-d}	25^{a-d}	0	0	0

a) Lemma about JVM multiplication program
b) Lemma about JVM power program
c) Lemma about JVM exponentiation program
d) Lemma about JVM factorial

Where else ML4PG can be applied?

Similarly, ML4PG can be used in:

1. Write the specification of the function
2. Write the algorithm (tail recursive function)
3. Prove that the algorithm satisfies the specification
4. Write the JVM program
5. Define the function that schedules the program
6. Prove that the code implements the algorithm
7. Prove total correctness

Proving lemma program_is_fn_fact by analogy

Factorial

```
Lemma program_is_fn_fact n :
run (sched_fact n)(make_state 0 [::n] [::] pi_fact)=
(make_state 14 [::0;fn_fact n ] (push (fn_fact n )[::])pi_fact).
Proof.
rewrite run_app.
```

Proving lemma program_is_fn_fact by analogy

Exponentiation (ML4PG suggestion)

```
Lemma program_is_fn_expt n m :
run (sched_expt n m)(make_state 0 [::n;m] [::] pi_expt)=
(make_state 14 [::0;fn_expt n m] (push (fn_expt n m)[::])pi_expt).
Proof.
rewrite run_app loop_is_helper_expt.
Qed.

Lemma loop_is_helper_expt n m a :
run (loop_sched_expt n)(make_state 2 [::n;m;a] [::] pi_expt)=
(make_state 14 [::0;(helper_expt n m a)] (push (helper_expt n m a)[::])pi_expt)
Proof.
move : n a; elim : m => [// | m IH n a].
by rewrite -IH subn1 -pred_Sn.
Qed.
```

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Prove that the loop implements the helper using an auxiliary lemma. Such a lemma about the loop must consider the general case as in the previous proof strategy. Subsequently, instantiate the result to the concrete case.

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ML4PG suggestions (for several parameters): Analogous theorems for multiplication, exponentiation and power.

Proving total correctness by analogy

Factorial

```
Theorem total_correctness_fact n sf :
sf = run (sched_fact n)(make_state 0 [::n] [::] pi_fact)->
next_inst sf = (HALT,0\%Z)/\ top (stack sf)= (n`!).
Proof.
move => H; split
```

Proving total correctness by analogy

Exponentiation (ML4PG suggestion)

```
Theorem total_correctness_expt n m sf :
sf = run (sched_expt m)(make_state 0 [::n;m] [::] pi_expt)->
next_inst sf = (HALT,0%Z)/\ top (stack sf)= (n^m).
Proof.
      by move => H; split; rewrite H program_is_fn_expt fn_expt_is_theta.
Qed.
```

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Combine lemmas of the two previous steps.

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The bigop library

- SSReflect library about indexed big “operations”

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- Examples:

$$\sum_{0 \leq i < 2n | odd\ i} i = n^2, \prod_{0 \leq i \leq n} i = n!, \bigcup_{i \in I} f(i), \ldots$$

The bigop library

- SSREFLECT library about indexed big “operations”
- Examples:

$$\sum_{0 \le i < 2n | odd\ i} i = n^2, \prod_{0 \le i \le n} i = n!, \bigcup_{i \in I} f(i), \ldots$$

- Applications:
  - Definition of matrix multiplication
  - Binomials
  - Union of sets
  - ...

Application of ML4PG: Inverse of nilpotent matrices

Definition

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Let M be a square matrix, M is nilpotent if it exists an n such that $M^n = 0$

Lemma

Let M be a nilpotent matrix, then

$$(1 - M) \times \sum_{0 \leq i < n} M^i = 1$$

where n is such that $M^n = 0$

```
Lemma inverse_I_minus_M_big (M : 'M_m) : (exists n, M^n = 0) ->
  (1 - M) *m (\sum_(0<=i<n) M^i) = 1.
```

Starting the proof

Goals and Subgoals	Proof-Steps (Tactics)
$\forall\ (M : M_n)(m : nat),\ M^m = 0 \implies (1 - M) \times \sum_{i=0}^{m-1} M^i = 1$	
	`move => M m nilpotent.`
$(1 - M) \times \sum_{i=0}^{m-1} M^i = 1$	
	`rewrite big_distrr mulmxBr mul1mx.`
$\sum_{i=0}^{m-1} M^i - M^{i+1}$	
	`case : n.`
$\forall\ (M : M_0)(m : nat),\ M^m = 0 \implies \sum_{i=0}^{m-1} M^i - M^{i+1}$	
	`by rewrite !thinmx0.`
$\forall\ (M : M_{n+1})(m : nat),\ M^m = 0 \implies \sum_{i=0}^{m-1} M^i - M^{i+1}$	

Suggestions provided by ML4PG

Theorem (Fundamental Lemma of Persistent Homology)

$\beta_i^{j,k} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$

$$\beta_n^{k,l} - \beta_n^{k,m} = \sum_{1 \leq i \leq k} \sum_{l < j \leq m} (\beta_n^{j,p-1} - \beta_n^{j,p}) - (\beta_n^{j-1,p-1} - \beta_n^{j-1,p})$$

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Proof

$$\sum_{1 \leq i \leq k} \sum\nolimits_{l < j \leq m} (\beta_n^{j,i-1} - \beta_n^{j,i}) - (\beta_n^{j-1,i-1} - \beta_n^{j-1,i}) \quad =$$

Suggestions provided by ML4PG

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Proof

$$\begin{aligned}
\sum_{1 \leq i \leq k} \sum_{l < j \leq m} (\beta_n^{j,i-1} - \beta_n^{j,i}) - (\beta_n^{j-1,i-1} - \beta_n^{j-1,i}) \quad &= \\
\sum_{1 \leq i \leq k} ((\beta_n^{l+1,i-1} - \beta_n^{l+1,i}) - (\beta_n^{l,i-1} - \beta_n^{l,i}) + \\
(\beta_n^{l+2,i-1} - \beta_n^{l+2,i}) - (\beta_n^{l+1,i-1} - \beta_n^{l+1,i}) + \\
\ldots \\
(\beta_n^{m-1,i-1} - \beta_n^{m-1,i}) - (\beta_n^{m-2,i-1} - \beta_n^{m-2,i}) + \\
(\beta_n^{m,i-1} - \beta_n^{m,i}) - (\beta_n^{m-1,i-1} - \beta_n^{m-1,i}))
\end{aligned}$$

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Proof

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\end{aligned}$$

Suggestions provided by ML4PG

Theorem (Fundamental Lemma of Persistent Homology)

$\beta_i^{j,k} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$

$$\beta_n^{k,l} - \beta_n^{k,m} = \sum_{1 \leq i \leq k} \sum_{l < j \leq m} (\beta_n^{j,p-1} - \beta_n^{j,p}) - (\beta_n^{j-1,p-1} - \beta_n^{j-1,p})$$

Proof

$$\begin{array}{rl} \sum_{1 \leq i \leq k} \sum_{l < j \leq m} (\beta_n^{j,i-1} - \beta_n^{j,i}) - (\beta_n^{j-1,i-1} - \beta_n^{j-1,i}) & = \\ \sum_{1 \leq i \leq k} ((\beta_n^{l+1,i-1} - \beta_n^{l+1,i}) - (\beta_n^{l,i-1} - \beta_n^{l,i}) + & \\ (\beta_n^{l+2,i-1} - \beta_n^{l+2,i}) - (\beta_n^{l+1,i-1} - \beta_n^{l+1,i}) + & \\ \ldots & \\ (\beta_n^{m-1,i-1} - \beta_n^{m-1,i}) - (\beta_n^{m-2,i-1} - \beta_n^{m-2,i}) + & \\ (\beta_n^{m,i-1} - \beta_n^{m,i}) - (\beta_n^{m-1,i-1} - \beta_n^{m-1,i})) & \end{array}$$

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Lemma

If $g : \mathbb{N} \to \mathbb{Z}$, then

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Proof Strategy

Apply case on n.

1. *Prove the base case (a simple task).*
2. *Prove the case $0 < n$:*
    1. *expand the summation,*
    2. *cancel the terms pairwise,*
    3. *the only terms remaining after the cancellation are the first and the last one.*

Lemma

Let M be a nilpotent matrix, then

$$(1 - M) \times \sum_{0 \leq i < n} M^i = 1$$

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An unusual discovery

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Goals and Subgoals	Proof-Steps (Tactics)
$\forall\ (M : M_n)(m : nat), M^m = 0 \implies \exists N, N \times (1 - M) = 1$	`move => M m nilpotent.`
$\exists N, N \times (1 - M) = 1$	`exists \sum_(0<=i<m.+1)(pot_matrix M i).`
$(\sum_{i=0}^{m-1} M^i) \times (1 - M)$	`rewrite big_distrl mulmxrB mulmx1.`
$\sum_{i=0}^{m-1} M^i - M^{i+1}$	`case : n.`
$\forall\ (M : M_0)(m : nat), M^m = 0 \implies \sum_{i=0}^{m-1} M^i - M^{i+1}$	`by rewrite !thinmx0.`
$\forall\ (M : M_{n+1})(m : nat), M^m = 0 \implies \sum_{i=0}^{m-1} M^i - M^{i+1}$	

Outline

1. Introduction
2. ML4PG: "Machine Learning for Proof General"
3. Using ML4PG
4. More Examples
   - Detecting patterns across mathematical libraries
   - Detecting irrelevant libraries
5. Conclusions and Further work

An example coming from Game Theory

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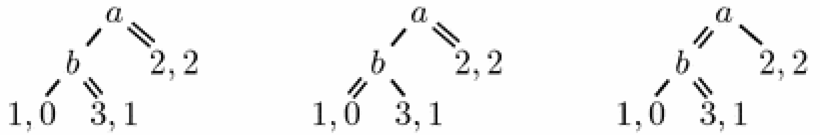

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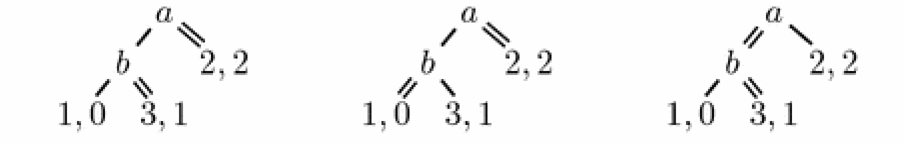

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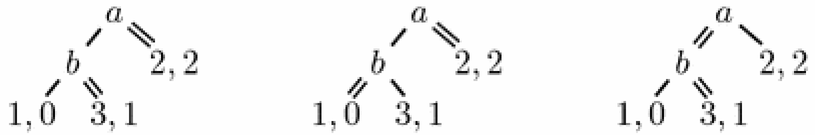

A *Nash equilibrium* is a strategy in which no agent can change one or more of his choices to obtain a better result.
A strategy is a *subgame perfect equilibrium* if it represents Nash equilibrium of every subgame of the original game.

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Is it possible to reuse patterns between these libraries? It is natural to think so, but . . .

Formalisations are just too different

Subgame Perfect Equilibrium implies Nash Equilibrium:

Binary case	General case

`Lemma SGP_is_NashEq :`
`forall s : Strategy, SGP s -> NashEq s.`
`Proof.`
`induction s.`
`unfold NashEq. intros _. induction s'.`
`intros. unfold stratPO. unfold agentConv in H.`
`rewrite (H a). trivial.`
`unfold agentConv. intros. contradiction.`
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`Qed.`
	`Lemma SPE_is_Eq :`
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ML4PG discovers the absence of patterns.

Comparison of the two examples

Orthogonal examples:

- Nilpotent matrices example:
  - Completely unrelated libraries, but common proof strategy.
- Nash example:
  - Similar results, but completely different proof strategies.

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it really works!!!!

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Related Work

- ACL2(ml) works as ML4PG in the ACL2 prover and also conceptualise new lemmas. Part of SICSA industrial grant.

Statistical Machine Learning in Interactive Theorem Proving

Katya Komendantskaya and Jonathan Heras
(Funded by EPSRC First Grant Scheme)

University of Dundee

8 November 2013