Proof-Pattern Recognition and Lemma Discovery in ACL2

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(joint work with K. Komendantskaya, M. Johansson and E. Maclean)

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31 January 2014
Scottish Theorem Proving seminar

Outline

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ACL2 (A Computational Logic for an Applicative Common Lisp) is the successor of the Boyer-Moore theorem prover.

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Applications of ACL2:

- Software and Hardware Verification (microprocessors, flash memories, JVM-like bytecode, ...)

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- Automatic: once a proof attempt is started, the user can no longer interact with ACL2.
- Interactive: the user has to supply a suitable collection of definitions and auxiliary lemmas to guide ACL2.

Challenges

- . . . size of ACL2 library stands on the way of efficient knowledge reuse;
- . . . manual handling of proofs, strategies, libraries becomes difficult;
- . . . team-development is hard;
- . . . comparison of proof similarities is hard;
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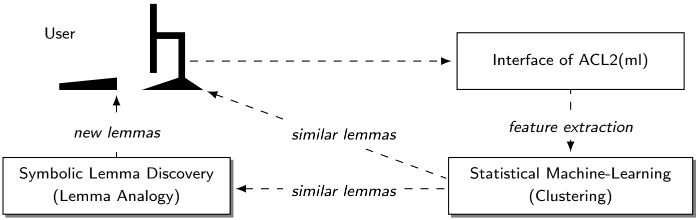
What can we do?

- Statistical methods can discover patterns in proofs but are weak for conceptualisation.
- Symbolic methods (Proof planning, lemma discovery) can conceptualise but have limitations.
- Combination of statistical and symbolic methods:
  - Statistical methods can take advantage of symbolic methods to conceptualise results.
  - Symbolic tools can use statistical results for efficient lemma discovery.

Outline

ACL2(ml)

. . . in [2013, Proceedings of LPAR'13]



F.1. works on the background of Emacs extracting some low-level features from ACL2 definitions and theorems.

F.2. automatically sends the gathered statistics to a chosen machine-learning interface and triggers execution of a clustering algorithm of user's choice;

F.3. does some post-processing of the results and

- **F.3.a** displays families of related proofs (or definitions) to the user.
- **F.3.b** uses the families of related proofs to discover new lemmas.

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Extracting features from ACL2

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   - We extract features directly from term trees of ACL2 terms.

Definition (Term tree)

A variable or a constant is represented by a tree consisting of one single node, labelled by the variable or the constant itself. A function application $f(t_1, \ldots, t_n)$ is represented by the tree with the root node labelled by f, and its immediate subtrees given by trees representing $t_1, \ldots, t_n$.

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(implies (natp n) (equal (fact-tail n) (fact n)))
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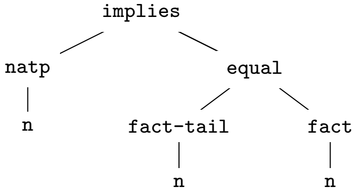

ACL2(ml) term tree matrices

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Definition (Term tree depth level)

Given a term tree T, the *depth* of the node t in T, denoted by *depth(t)*, is defined as follows:
$-$ $depth(t) = 0$, if t is a root node;
$-$ $depth(t) = n + 1$, where n is the depth of the parent node of t.

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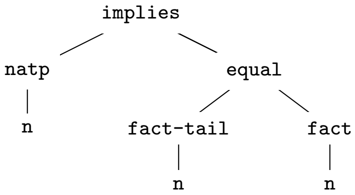
Definition (ACL2(ml) term tree matrices)

Given a term tree T for a term with signature Σ, and a function $[.] : \Sigma \rightarrow \mathbb{Q}$, the ACL2(ml) term tree matrix M_T is a 7×7 matrix that satisfies the following conditions:
− the $(0, j)$-th entry of M_T is a number $[t]$, such that t is a node in T, t is a variable and $depth(t) = j$.
− the (i, j)-th entry of M_T $(i \neq 0)$ is a number $[t]$, such that t is a node in T, t has arity $i + 1$ and $depth(t) = j$.

An example



	variables	arity 0	arity 1	arity 2
*td*0	0	0	0	[implies]
*td*1	0	0	[natp]	[equal]
*td*2	[n]	0	[fact-tail]::[fact]	0
*td*3	[n]::[n]	0	0	0

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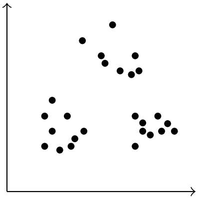

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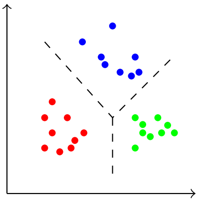

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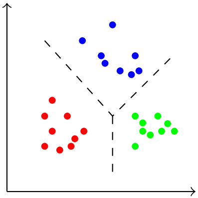

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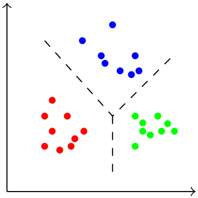

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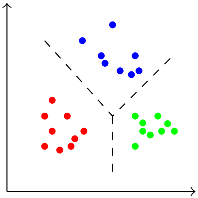

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Recurrent clustering

How is the function [.] defined?

Definition (Function [.])

Given the nth term definition of the library (call the term t), a function [.] is inductively defined for every symbol s in t as follows:
– $[s] = i$, if s is the ith distinct variable in t (formulas are implicitly universally quantified);
– $[s] = -[m]$, if t is a recursive definition defining the function s with measure function m;
– $[s] = k$, if s is a function imported from CLISP; and $[s] = k$ in the figure below;
– $[s] = 5 + 2 \times j + p$, where C_j is a cluster obtained as a result of definition clustering with granularity 3 for library definitions 1 to $n - 1$, $s \in C_j$ and p is the proximity value of s in C_j.

$*$ Type recognisers ($r = \{$symbolp, characterp, stringp, consp, acl2-numberp, integerp, rationalp, complex-rationalp$\}$): $[r_i] = 1 + \sum_{j=1}^{i} \frac{1}{10 \times 2^{j-1}}$ (where r_i is the ith element of r).

$*$ Constructors ($c = \{$cons, complex$\}$): $[c_i] = 2 + \sum_{j=1}^{i} \frac{1}{10 \times 2^{j-1}}$.

$*$ Accessors ($a^1 = \{$car, cdr$\}$, $a^2 = \{$denominator, numerator$\}$, $a^3 = \{$realpart, imagpart$\}$): $[a^j_i] = 3 + \frac{1}{10 \times j} + \frac{i-1}{100}$.

$*$ Operations on numbers ($o = \{$unary-/, unary–, binary-+, binary-*$\}$): $[o_i] = 4 + \sum_{j=1}^{i} \frac{1}{10 \times 2^{j-1}}$.

$*$ Integers and rational numbers: $[0] = 4.3$, $[n] = 4.3 + \frac{|n|}{10}$ (with $n \neq 0$ and $|n| < 1$) and $[n] = 4.3 + \frac{1}{100 * |n|}$ (with $n \neq 0$ and $|n| \geq 1$).

$*$ Boolean operations ($b = \{$equal, if, ¡$\}$): $[b_i] = 5 + \sum_{j=1}^{i} \frac{1}{10 \times 2^{j-1}}$.

Demo

- Finding similar theorems across libraries.
- Obtaining more precise clusters.
- Finding similar definitions across libraries.

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Terminology:

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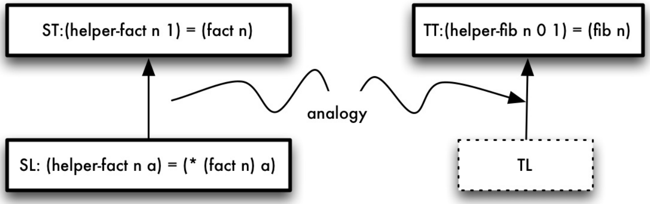
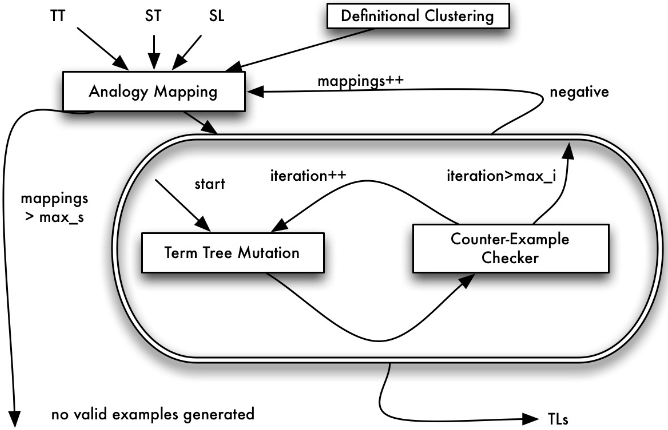


Overview of the process



Analogy mapping

Definition (Analogy Mapping $\mathcal{A}$)

For all symbols $s_1, \ldots, s_n$ occurring in the current ST, the set of admissible analogy mappings is the set of all mappings $\mathcal{A}$ such that
- $\mathcal{A}(s_i) = s_i$ for all shared background symbols; otherwise:
- $\mathcal{A}(s_i) = s_j$ for all combinations of $i, j \in 1 \ldots n$, such that s_i and s_j belong to the same cluster in the last iteration of definition clustering.

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Example

For our running example, the shared background theory includes symbols $\{$`+`, `*`,`-`, `1`, `0`$\}$. We thus get a mapping:
$\mathcal{A} = \{\texttt{fact} \mapsto \texttt{fib}, \texttt{helper-fact} \mapsto \texttt{helper-fib}, \texttt{+} \mapsto \texttt{+}, \texttt{1} \mapsto \texttt{1}, \ldots\}$

Term tree mutation

Term tree mutation consists of three iterations:

- Tree reconstruction.
- Node expansion.
- Term tree expansion.

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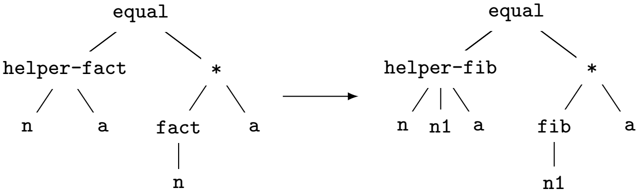

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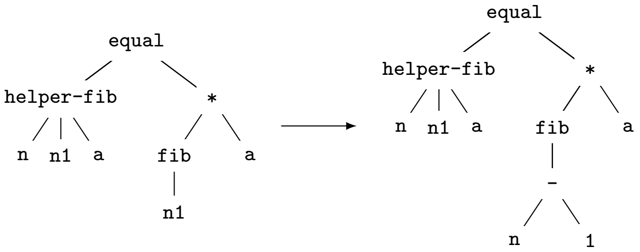

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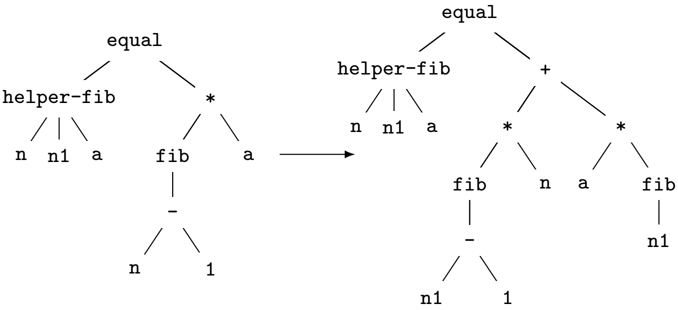

Using guards to generate preconditions

- Using the lemma analogy tool, ACL2(ml) generates the following suggestion:

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(equal (helper_fib n j k)
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- Solution: compute recursively the guards of a function f.

Using guards to generate preconditions

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(defun helper_fib (n j k)
     (if (zp n) j (if (equal n 1) k (helper_fib (- n 1) k (+ j k)))))

  * zp ->  (natp x)
  * equal -> t
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```
(defthm helper_fib_theta_fib
   (equal (helper_fib n j k) (+ (* (theta_fib (- n 1)) j) (* (theta_fib n) k))))
```

Guards:

```
(and (integerp n) (not (< n 0)) (acl2-numberp j) (acl2-numberp k)
     (not (< (+ -1 n) 0)))
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Demo

- Lemma discovery.
- Guard generation.

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Conclusions

- ACL2(ml) combines statistical machine learning (detection of patterns) with symbolic techniques (generation of lemmas).
- ACL2(ml) is different to other tools:
  - its methods of generating the proof-hints interactively and in real-time;
  - its flexible environment for integration of statistical and symbolic techniques.

Further work

- **Different patterns.** Statistical ACL2(ml) groups in the same clusters theorems whose lemmas cannot be mutated to generate any useful lemma.
- **Smaller lemmas.** The lemma analogy tool currently only adds term structure; therefore, it cannot generate smaller lemmas.
- **Conditional lemmas.** Discovering appropriate conditions for generated lemmas is a difficult problem for theory exploration systems.
- **New definitions.** Another big challenge in lemma discovery is invention of new concepts.

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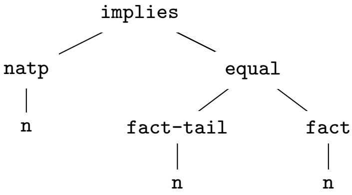

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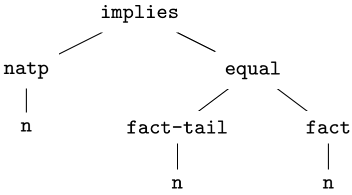

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n	0	0	0	0	0	0	0	0
n'	0	0	0	0	0	0	0	0
n''	0	0	0	0	0	0	0	0

Properties of these methods:

- different tree nodes are represented by distinct matrix entries;
- the matrix entries are binary;
- the size of the matrix depend on the tree size; and
- they can grow very large.

Evaluation

Scalability: ACL2(ml) works well with libraries of varied sizes and complexities.

(150 lemmas)	$g = 1$ ($n = 16$)	$g = 2$ ($n = 18$)	$g = 3$ ($n = 21$)	$g = 4$ ($n = 25$)	$g = 5$ ($n = 30$)
fib-fib-tail	$9^{a,b,c}$	$4^{a,b,c}$	$3^{a,c}$	2^{a}	2^{a}

(996 lemmas)	$g = 1$ ($n = 110$)	$g = 2$ ($n = 124$)	$g = 3$ ($n = 142$)	$g = 4$ ($n = 166$)	$g = 5$ ($n = 199$)
fib-fib-tail	$57^{a,b,c}$	$50^{a,b,c}$	$25^{a,b,c}$	$8^{a,b,c}$	$4^{a,b,c}$

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Usability of Statistical Suggestions (996 lemmas):

- 37% of clusters can be directly used by the Lemma Analogy tool of ACL2(ml) to mutate lemmas.
- 19% of cluster contain basic theorems whose proofs are similar (based on simplification).
- 15% of clusters contain theorems that use the same lemmas in their proofs.
- 15% of clusters consist of theorems that are used in the proofs of other theorems of the same cluster.
- 14% of clusters do not show a clear correlation that could be reused.

Evaluation II

Modularity: ACL2(ml) provides a flexible environment for integrating statistical and symbolic machine-learning methods.

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Lemma discovery: reduces the combinatorial explosion of theory exploration techniques. Comparison with QuickSpec:

		Target						
		fact	*power*	*expt*	*sum*	*sum_sq*	*mult*	*fib*
Source	*fact*	-	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	$\checkmark_2$	$\checkmark_1$
	power	$\checkmark_1$	-	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	$\checkmark_2$	$\checkmark_1$
	expt	$\checkmark_1$	$\checkmark_1$	-	$\checkmark_1$	$\checkmark_1$	$\checkmark_2$	$\checkmark_1$
	sum	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	-	$\checkmark_1$	$\checkmark_2$	$\checkmark_1$
	sum_sq	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	-	$\checkmark_2$	$\checkmark_1$
	mult	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	$\checkmark_1$	-	$\checkmark_1$
	fib	$(\checkmark_2)$	$(\checkmark_2)$	$\times$	$\times$	$\times$	$\times$	-

QuickSpec		
Lemma	Valid	Invalid
$\times$	10	5
$\times$	17	4
$\times$	OoM	OoM
$\times$	7	2
$\times$	7	1
$\times$	200	20
$\times$	OoM	OoM