Truth-as-Simulation: towards a coalgebraic perspective on logic and games

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It has been said that *modal logic is the logic of dynamics*, as the natural language describing bisimulation-invariant properties of transition systems. On the other hand, a current trend in the semantics of programming languages is based on taking *coalgebras* as standard models for dynamical systems, generalizing (various notions of) transition systems. A natural problem that arises from this work is to *generalize modal logics to coalgebraic logics for various functors.* There are at least two different directions to explore: first, extending modal logic concepts, methods and ideas to the coalgebraic context; second, applying coalgebraic methods and concepts back to the study of modal logics and of logic in general. Some work along the first line has been done by L. Moss, M. Rossiger, A. Kurz and B. Jacobs, but there still remains a whole area of open problems concerning finding the coalgebraic analogues of classical constructions in model theory of modal logic. The second line of research has not even been touched until now, as far as I know.

This is a summary of my on-going work in this area, some results, some projects and some open questions. My work builds on, improves and simplifies the one done by L. Moss. He introduced a language that generalizes a fragment of standard modal logic, strong enough to characterize coalgebras up to bisimilarity (generalizing the Hennessey-Milner property). He uses the standard Aczel-Milner definition of coalgebraic bisimulation. I am using an equivalent definition in terms of *relators*, which I find to be find simpler, more natural, easier to use and to generalize. In fact, I also use a generalization of this notion, namely the notion of *coalgebraic simulation*, proposed by A. Thuijs as a generalization of the standard notion of simulation for non-deterministic transition systems and for (deterministic) automata. I propose a logic that can capture this relation of similarity, and I prove analogues of classical characterization, definability, preservation and interpolation theorems for this logic. I also study the associated semantical (pebble) games, under the name of *simulation games.* An interesting observation is that, if we endow the language with a natural coalgebra structure (with respect to a more complex functor), then truth (the satisfaction relation) for this logic can be seen as the largest simulation relation between the model (the given coalgebra) and the language. My project is to pursue in general this *coinductive definition of truth*, which can be seen as a *dual of Tarski's inductive definition.* Such a notion can be applied to self-referential or non-inductive languages. Also, one can see some non-compositional logical semantics, namely some of the so-called *game semantics*, as instances of this coinductive notion. This is natural: games in general are coinductive objects, but it is interesting that the truth-defining games can be understood as simulation games.

Given a functor F on Set, an F*-coalgebra* is a pair (A, α) of some set A and some function $\alpha : A \longrightarrow FA$. A *coalgebraic morphism* between F-coalgebras (A, α), (B, β) is a function $f : A \longrightarrow B$ such that $(Ff) \circ \alpha = \beta \circ f$. An F*-relator*is an operator Γ, associating to each relation $R \subseteq A \times B$ between sets A, B some relation $\Gamma R \subseteq FA \times FB$, and satisfying certain conditions. A special example of an F-relator is the *strong relator* Γ_F, which is the unique relator that preserves identities: $\Gamma_F(1_A) = 1_{FA}$ (and which exists iff F preserves weak pullbacks). Every relator Γ has a *dual* or *transposed* relator $\Gamma^{\sim}$, given by:

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$\Gamma^{\sim}R =: (\Gamma(R^{\sim}))^{\sim}$, where in general $R^{\sim}$ is the converse of the relation R. Given an F-relator Γ, Thuijs defines a Γ-*simulation* between two F-coalgebras (A, α),(B, β) as a relation $R \subseteq A \times B$ such that: if aRb then $\alpha(a)(\Gamma R)\beta(b)$. Two coalgebras are Γ-*similar* if there exists a simulation relation between them. One can easily see that a $\Gamma^{\sim}$-simulation is simply the converse of a Γ-simulation. Two coalgebras are said to be Γ-equivalent if they are both Γ-similar and $\Gamma^{\sim}$-similar. In certain conditions, Γ-equivalence is equivalent to F-bisimilarity (in the Aczel-Mendler sense). Also, if we take Γ to be the (unique) strong F-relator Γ_F, then the notion of Γ_F-simulation is equivalent to F-bisimulation. On the other hand, the standard notion of simulation for transition systems can be also obtained as a particular, if we take a natural (non-strong) relator for the powerset functor.

The language of our coalgebraic logic, for a given functor F, is obtained as the least set L of formulas such that: if Φ is a set of formulas in L and $X \in F\Phi$, then $\bigwedge \Phi, \bigvee \Phi, \Box X, \Diamond X$ are formulas in L. (No negation is needed!). Formulas which do not contain any occurence of $\Box$ are called *existential*, while formulas not containing $\Diamond$ are called *universal*. The semantics of conjunction and disjunction is as usual, while for the modalities we put: $a \models \Box X$ iff $\alpha(a)(\Gamma \models)X$, and $a \models \Diamond X$ iff $\alpha(a)(\Gamma^{\sim} \models)X$. Truth is invariant under Γ-equivalence. Moreover, one can prove analogues of classical preservation, characterization (and interpolation) theorems of modal logic: (1) Similarity is equivalent to the relation of preservation of existential formulas. Dual (converse) similarity is equivalent to preservation of universal formulas. Γ-equivalence is equivalent to the preservation of all coalgebraic formulas. (2) For every state $a \in A$ of some F-coalgebra (A, α), there exists an (existential) formula θ, such that a state b satisfies θ iff there exists a simulation R with aRb. Similar statements hold for converse simulation and Γ-equivalence. (3) A formula is preserved by every Γ-simulation iff it is equivalent to an existential formula. (4) A class of F-coalgebras is closed under Γ-similarity iff it can be axiomatized by an existential theory. Similarly, a class is closed under Γ-equivalence iff it is axiomatizable in our language.

An interesting observation is that the truth-condition for $\Box$ resembles the definition of simulation (and the condition for $\Diamond$ - the one of converse simulation). This resemblance is the basis of my current work. The language can be naturally endowed with a coalgebraic structure, but with respect to a more complex functor $\overline{F}$. The models (i.e. F-coalgebras (A, α)) can also be naturally seen as $\overline{F}$-coalgebras. And one can check that *truth is the largest simulation relation between models and language.* This observation suggests an alternative, coinductive definition of truth. Such a definition can then be easily seen to be meaningful for richer languages: fix-point and self-referential languages, and in general any language that can be seen as a coalgebra (or, in particular, as a transition system). Some of the proofs of classical results become more transparent in this general setting: for instance, preservation of existential sentences under embeddings (simulations) becomes a simple consequence of the transitivity of similarity. Moroever, this definition gives a new perspective on *games*, used by some authors to define truth in a non-compositional way. One can see that game definitions of truth are actually coinductive definitions, and some of them are simulation games. On the other hand, games of the Ehrenfeust-Fraisse type have been used for various logics (sometimes under the name of "pebble games") to decide whether two structures are "similar" (elementary equivalent, modally equivalent, bisimilar, potentially isomorphic etc.). The coalgebraic perspective might give a unified view on these various uses of games in logic: the analogues of pebble games for coalgebras are simulation (and bisimulation) games, but the truth game turned out to also be a simulation game. So the pebble games can be seen (at least in the case of coalgebras) to be games *of the same type* as the ones used in game semantics.

References:

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