A type theory for classical arithmetic

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Abstract

Since the work of T. Griffin and C. Murthy, there has been great interest shown by the LICS community into investigations of the 'classical formulae–as–types correspondence'. The literature contains successful extensions of Griffin's idea of modelling cut elimination in NK by appeal to ideas used in providing access to continuations in functional programming to providing well-behaved proof semantics for NK and extending these representations to the predicate calculus.

Less success has met attempts to provide such a correspondence for theories of induction, and approaches to classical witness extraction have relied on conservation results between classical and intuitionistic arithmetic for P^0_2 sentences rather than trying to provide a direct acount of the formulae–as–types correspondences for classical arithmetic.

In this talk we shall examine a new approach to modelling classical induction in constructive type theory based upon a rereading of Martin-Löf's slogan 'constructions–as–demonstrations' in the context of Parigot's lambda-mu calculus.

Rescuing constructive proof theory from the constructivists An immediate difficulty faces any claim to provide a constructive characterisation of classical mathematics: how are we to characterise constructive content independently of certain theses about what mathematics is according to the constructivists? We examine an alternative formulation of constructive content based upon reading off the meaning of logical constants from their rules, an idea that goes back to Gentzen and which has been illuminatingly developed by Dag Prawitz and Michael Dummett.

Semantics without set theory We show how dependent types, proof reductions and Martin-Löf's intensional equality may be used to give a *proof semantics* of intuitionistic arithmetic. We discuss how the success of this account depends upon the blurring of the distinction between proof and construction which is a consequence of the formulae-as-types correspondence.

Formulae-as-types correspondence for classical logic We examine the formulae–as–types correspondence for classical logic through Michel Parigot's $\lambda\mu$ calculus. We extend our account of Prawitz's inversion principle to the new construtors of the $\lambda\mu$ calculus, which in the context of our earlier discussion warrants the claim that this calculus is a constructive account of classical logic.

Difficulties with classical induction We show how the presence of the new constructors of Parigot's calculus endanger our account of induction. We discuss a proposal of Parigot to avoid this difficulty, and argue that it fails to provide a computational account of induction.

Classical type theory We develop an account of arithmetic type theory which extends our earlier account of intuitionistic arithmetic, and show its soundness. We note the limitations of our method with respect to modelling classical arithmetic, agreeing with William Tait that it is the conjunction of the axiom of choice and the principle of the excluded middle which is incompatible with the presence of constructive content, and not either principle alone.