A WEAK CONSTRUCTIVE SET THEORY WITH INACCESSIBLE SETS

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In this talk we shall present a constructive set theory with inaccessible sets but without $\in$ - Induction (foundation). We show that this set theory has a limited proof theoretic strength, even though it has an extensive expressive power. This result underlines a characteristic of inaccessible set axioms which is opaque in classical Zermelo Fraenkel set theory, mainly that the consistency strength of inaccessible set axioms heavily depends on the context in which they are embedded.

Constructive set theory originated from the work of Myhill [8] and was introduced as a natural formalism which relates to Bishop's constructive mathematics [4] as **ZFC** relates to classical Cantorian mathematics. It is characterised by the use of the same standard first order language as classical set theory and it makes no explicit use of the notions of construction or constructive object. Therefore the conventions, practice and ideas of the set theoretical presentation of ordinary mathematics may be used in the development of constructive mathematics as well. Moreover, its theorems are theorems of classical set theory too.

Various systems of intuitionistic and constructive set theory have been introduced in the literature so far. We shall be concerned with Constructive Zermelo Fraenkel set theory, **CZF** [1, 2, 3], as it has a natural interpretation in Martin Löf Type Theory by means of which its notion of set receives a constructive meaning.

We shall denote with $\mathbf{CZF}^-$ the system **CZF** minus $\in$ - Induction and we shall extend $\mathbf{CZF}^-$ by an axiom for inaccessible sets. In **ZFC** inaccessible sets are of the form V_κ where κ is a strongly inaccessible cardinal and V_κ denotes the κ - th level of the von Neumann hierarchy. Inaccessible sets figure prominently in category theory as Grothendieck universes and are related to universes in type theory. Let **INAC** be the statement that for every set there is an inaccessible set containing it. $\mathbf{CZF}^- + \mathbf{INAC}$ is a mathematically rich theory in which one can easily formalize Bishop style constructive mathematics and a great deal of category theory. $\mathbf{CZF}^- + \mathbf{INAC}$ also has a realizability interpretation in type theory which gives its theorems a direct computational meaning. The main result presented here is that the proof theoretic ordinal of $\mathbf{CZF}^- + \mathbf{INAC}$ is a surprisingly small ordinal known as the Feferman - Schütte ordinal Γ_0.

To put this result into perspective we note that $\mathbf{CZF}^- + \mathbf{INAC}$ plus the principle of excluded middle, $\mathbf{EM}$, has the strength of $\mathbf{ZFC} + \forall\alpha\,\exists k\,(\alpha < k \;\wedge\; k\ \textit{is a strong}$ *inaccessible cardinal*). It is also worth noting that the strength of the whole theory $\mathbf{CZF}$ plus $\mathbf{INAC}$ exceeds that of all systems considered in Reverse Mathematics. In fact, the techniques of [9] can be used to show that the strength of $\mathbf{CZF} + \mathbf{INAC}$ is the same as that of the fragment of second order arithmetic known as $\Delta^1_2 - \mathbf{CA} + \mathbf{BI}$.

In our investigations we have been led by an analogy with important results obtained in the classical case of Kripke Platek set theory, $\mathbf{KP}$. Research by Jäger and Pohlers [5] has supplied a proof theoretic analysis for an extension of $\mathbf{KP}$, denoted $\mathbf{KPi}$, which is obtained from Kripke Platek set theory by adding an axiom stating the existence of many admissible sets. The upshot is that $\mathbf{KPi}$ has the same proof theoretic strength as $\Delta^1_2 - \mathbf{CA} + \mathbf{BI}$. Jäger [6] has furthermore shown that if restrictions are imposed on the principle of induction (i.e. foundation), then the proof theoretic strength of the theories considered strongly decreases. In particular, if foundation is completely omitted from $\mathbf{KPi}$ then the proof theoretic strength of the system obtained is measured by the ordinal Γ_0.

We have therefore started to investigate the system $\mathbf{CZF}$ looking for similar outcomes in this constructive context. The results for $\mathbf{KPi}$ referred to above have been attained by use of partial cut elimination. Unfortunately, this technique is not suitable for $\mathbf{CZF}$ and restrictions. In fact, even when completely omitting the principle of $\in$ - Induction from $\mathbf{CZF}$, we are still left with the axioms of Collection which have an unrestricted quantifier complexity. Therefore a different approach is needed in this case.

The strategy adopted here is to introduce a realizability interpretation for systems of constructive set theory into suitable classical theories for which the proof theoretic strength may be determined by use of cut elimination techniques. The proof theoretic strength of the classical theory will then provide an upper bound for the constructive set theory. Regarding the realizability interpretation, we have to depart substantially from Aczel's interpretation of constructive set theory in $\mathbf{MLTT}$ ([3]). That interpretation, in fact, makes use of the W type in order to interpret the Regular Extension Axiom ($\mathbf{REA}$), of which $\mathbf{INAC}$ is a strengthening, and would therefore require a far too strong theory. We shall instead introduce a theory of iterated fixed point definitions with ordinals, called $\widehat{\mathbf{ID}}^*$, whose proof theoretic strength we have shown to be Γ_0. In $\widehat{\mathbf{ID}}^*$ we shall build up two hierarchies: $(\mathbb{U}_\alpha)_\alpha$ of universes and $(\mathbb{V}_\alpha)_\alpha$ of types of iterative sets. In addition, we shall define a bisimulation relation on $\mathbb{V}_\alpha$ in the style of [7], and interpret equality between sets as maximum bisimulations among types in $\mathbb{V}_\alpha$, giving rise to a non-well-founded membership relation. This will allow us to define a realizability interpretation for $\mathbf{CZF}^- + \mathbf{INAC}$ in $\widehat{\mathbf{ID}}^*$, providing an upper bound for $\mathbf{CZF}^- + \mathbf{INAC}$.

The lower bound is obtained by interpreting $\mathbf{ATR_0^i}$, an intuitionistic version of the subsystem $\mathbf{ATR_0}$ of second order arithmetic, in $\mathbf{CZF^-} + \mathbf{INAC}$ and by showing that Γ_0 is a lower bound for $\mathbf{ATR_0^i}$. We rely on the literature for a well ordering proof in such a theory, in a similar way as in [10].

This will enable us to conclude that Γ_0 is a measure for the strength of $\mathbf{CZF^-} + \mathbf{INAC}$.

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