A Tableaux System for a Fragment of the Hyperset Theory

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Abstract

Non-well-founded phenomena arise in a variety of ways in Informatics. The need to formalise with circular phenomena has motivated the development of the theory of Non-Well-Founded Sets or Hypersets. The talk introduces a decidable tableau calculus for a fragment of the Hyperset Theory.

1 Introduction

Non-well-founded phenomena have motivated works on Non-Well-Founded Sets or Hypersets[1] by Aczel [1] and Barwise [3, 4] respectively. Their works provide an elegant mathematical tool to formalise phenomena that are characterised by circular behaviours, e.g., the works on CSS by Robin Milner [14, 15, 16] and the philosophical problem of the liar paradox by Jon Barwise and John Etchemendy [2].

Hyperset Theory by the AntiFoundation Axiom and the Solution Lemma provides a powerful and elegant mathematical tool to formalise those circular phenomena that cannot be expressed by the standard axiom system ZFC of axiomatic Set Theory [20]. The application of Set Theory has been extensively investigated in the past decades. One research field, namely Computable Set Theory [6, 7, 8], has been related to the decidability problem of Set Theory.

Despite the existence of many phenomena in Informatics characterised by a circular beings, there has been little attention to the decidability problem, hence the automation, within Hyperset Theory.

2 A Tableaux System for a Fragment of the Hyperset Theory

In Computable Set Theory [6, 7, 8], previous works have proved the decidability problem for many fragments of Set Theory. An initial fragment named Multi-level Syllogistic (MLS) has been proved to be decidable in [13]. Several extensions of MLS were proved decidable, among them Multi-Level Syllogistic

[1] I will use the name Hypersets for the rest of the paper.

with Singleton (MLSS), which extends MLS with the singleton operator. An efficient decision procedure for MLSS has been stated as tableau calculus in [5].

In [12] we have proved the decision problem for a fragment of the Hyperset Theory. The work was originally motivated by the decision problem in presence of the AntiFoundation Axiom, hence the decision problem of Hyperset Theory. As Hyperset Theory extends the classical Set Theory, the work extends some decidability results within Set Theory to the Hyperset Theory. We have defined the language *HMLSS*[2] and proved that the satisfiability problem for *HMLSS* in the hypersets universe is decidable. This result can be seen as an extension of the Solution Lemma. It is enough to think to a formula as a system. In other words, finding a model to satisfy a given *HMLSS*-formula is like to solve an equations system. This problem has already been solved in [13] for the case of the well-founded sets universe. This problem is taken into account in [18] within the hypersets context but using different techniques. The decidability result is the basis to define a decidable tableaux system for *HMLSS*. We have proved the correctness and completeness of the tableaux system. In conclusion, we have provided a sound and complete tableau calculus for the fragment *HMLSS* of Hyperset Theory. The tableau calculus is based on a decision procedure for *HMLSS*. Moreover, the tableau calculus terminates by a saturation strategy. The tableau system for *HMLSS* has been implemented in SETL2 [21, 22].

3 Further work

I will discuss further work to extend the *HMLSS*-tableaux system. The following points summarises the working program.

- Extension of the *HMLSS* language to obtain more expressive formulae (e.g., explicit quantification, functions, powerset,). All further extensions of the *HMLSS* language should not violate the decidability result. In particular we intend to extend the *HMLSS* language by the following points
  - iterative membership operator and other extensions similarly defined for the case of Well-Founded Sets (e.g., [5, 9, 10, 11]).
  - the predicate is_wf to decide whether or not a set is well-founded
  - the function Wfd with domain in the hypersets universe and image in the well-founded sets universe, the function returns the well-founded part of a given set.

  All above extensions will be provided by an equivalent tableau calculus and an implementation. The tableau calculus will be the kernel for a system to provide proofs and counterexamples within the Hyperset Theory.

- Comparison of the tableau calculus with other tableau systems (e.g., [19]). The comparison aims to point out similarities and to define possible scenarios of integration with other systems.

- The tableau calculus for *HMLSS* has already been implemented, see [12], in SETL2 [21, 22]. We plan to develop a complete toolset to support the theoretical results. The toolset will improve our ability to investigate possible scenarios of applications. For example, the toolset can be used to
  - investigate cases of non-well-founded reasoning in programming languages (e.g., [17])
  - investigate possible scenarios of applications in Software Engineering.

[2] HMLSS: Hyperset Multi-Level Syllogistic with Singleton.

We believe that the decidability of fragments of Hyperset Theory can give rise to many applications in Informatics in which there are various examples of non-well-founded phenomena.

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