Compositional Characterizations of λ-terms using Intersection Types (Extended abstract)

Yoko Motohama*

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The *intersection-types discipline* was introduced in [5] as a tool of overcoming the limitations of Curry's type assignment system. Subsequently it was used in [3] as a tool for proving Scott's conjecture concerning the *completeness* of the set-theoretic semantics for *simple types.*

Very early on, however, it was realized that intersection type theories are a very expressive tool for giving *compositional* characterizations (i.e. characterizations based on properties of proper subterms) of *evaluation* properties of λ-terms. There are two seminal results in this respect.

The first result is that the Ω-free fragment of intersection-types allows one to type *all* and *only* the *strongly normalizing terms.* This is largely a folklore result; the first published proof appears in [15].

The second result is the *filter model* construction based on the intersection type theory $\Sigma^{\mathcal{BCD}}$, carried out in [3]. This result shows that there is a very tight connection between intersection types and *compact elements* in ω-algebraic denotational models of λ-calculus. This connection later received a categorically principled explanation by Abramsky in the broader perspective of "domain theory in logical form"[1].

Since then, the number of intersection type theories, used for the fine study of the denotational semantics of untyped λ-calculus, has increased considerably (e.g. [7, 6, 11, 9, 2, 14, 10]).

An intersection type theory Σ^{∇} includes always the basic set for $\leq$ and possibly other special purpose axioms and rules. In Figure 1 we give the basic set of axioms and rules. In Figure 2 appears a list of special purpose axioms and rules which have been considered in the literature. We introduce in Figure 3 a list of significant intersection type theories.

(refl)	$A \leq A$	(idem)	$A \leq A \cap A$
(incl$_L$)	$A \cap B \leq A$	(incl$_R$)	$A \cap B \leq B$
(mon)	$\dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$	(trans)	$\dfrac{A \leq B \quad B \leq C}{A \leq C}$

Figure 1: The basic set of axioms and rules.

Before giving the crucial notion of *intersection type assignment system*, we need to introduce some preliminary definitions and notations.

Definition 1 *i) A* ∇-basis *is a set of statements of the shape $x{:}B$, where $B \in \mathbf{T}^{\nabla}$, all whose variables are distinct.*

ii) An intersection type assignment system $\lambda\cap^{\nabla}$ *relative to Σ^{∇} is a formal system for deriving judgments of the form $\Gamma \vdash^{\nabla} M : A$, where the* subject M *is an untyped λ-term, the* predicate A *is in $\mathbf{T}^{\nabla}$, and Γ is a ∇-basis.*

*Dipartimento di Informatica, corso Svizzera 185, 10149 Torino, Italy, yoko@di.unito.it

(Ω)	$A \leq \Omega$
$(\Omega\text{-}\eta)$	$\Omega \leq \Omega \to \Omega$
$(\Omega\text{-}lazy)$	$A \to B \leq \Omega \to \Omega$
$(\to\text{-}\cap)$	$(A \to B) \cap (A \to C) \leq A \to B \cap C$
(η)	$\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}$
$(\omega\text{-}Scott)$	$\Omega \to \omega \sim \omega$
$(\omega\text{-}Park)$	$\omega \to \omega \sim \omega$
$(\omega\varphi)$	$\omega \leq \varphi$
$(\varphi \to \omega)$	$\varphi \to \omega \sim \omega$
$(\omega \to \varphi)$	$\omega \to \varphi \sim \varphi$

Figure 2: Some special purpose Axioms and Rules concerning $\leq$.

$\mathbb{C}^{\mathcal{B}a}$	$=$	$\mathbb{C}_\infty$	$\mathcal{B}a$	$=$	$\{(\to\text{-}\cap), (\eta)\}$
$\mathbb{C}^{\mathcal{AO}}$	$=$	$\{\Omega\}$	$\mathcal{AO}$	$=$	$\mathcal{B}a \cup \{(\Omega), (\Omega\text{-}lazy)\}$
$\mathbb{C}^{\mathcal{BCD}}$	$=$	$\{\Omega\} \cup \mathbb{C}_\infty$	$\mathcal{BCD}$	$=$	$\mathcal{B}a \cup \{(\Omega), (\Omega\text{-}\eta)\}$
$\mathbb{C}^{\mathcal{S}c}$	$=$	$\{\Omega, \omega\}$	$\mathcal{S}c$	$=$	$\Sigma^{\mathcal{BCD}} \cup \{(\omega\text{-}Scott)\}$
$\mathbb{C}^{\mathcal{P}a}$	$=$	$\{\Omega, \omega\}$	$\mathcal{P}a$	$=$	$\Sigma^{\mathcal{BCD}} \cup \{(\omega\text{-}Park)\}$
$\mathbb{C}^{\mathcal{CDZ}}$	$=$	$\{\Omega, \varphi, \omega\}$	$\mathcal{CDZ}$	$=$	$\Sigma^{\mathcal{BCD}} \cup \{(\omega\varphi), (\varphi\to\omega), (\omega\to\varphi)\}$

Figure 3: Type Theories: atoms, axioms and rules.

Definition 2 (Basic Type Assignment System) *Let* Σ^∇ *be a type theory. The* basic type assignment system $\lambda\cap^\nabla_\mathcal{B}$ *is a formal system for deriving judgments of the shape* $\Gamma \vdash^\nabla_\mathcal{B} M : A$. *Its rules are the following*

$$(Ax)\ \frac{x{:}A \in \Gamma}{\Gamma \vdash^\nabla_\mathcal{B} x{:}A} \qquad (\to\mathrm{I})\ \frac{\Gamma, x{:}A \vdash^\nabla_\mathcal{B} M : B}{\Gamma \vdash^\nabla_\mathcal{B} \lambda x.M : A \to B}$$

$$(\to\mathrm{E})\ \frac{\Gamma \vdash^\nabla_\mathcal{B} M : A \to B \quad \Gamma \vdash^\nabla_\mathcal{B} N : A}{\Gamma \vdash^\nabla_\mathcal{B} MN : B} \qquad (\cap\mathrm{I})\ \frac{\Gamma \vdash^\nabla_\mathcal{B} M : A \quad \Gamma \vdash^\nabla_\mathcal{B} M : B}{\Gamma \vdash^\nabla_\mathcal{B} M : A \cap B}$$

$$(\leq_\nabla)\ \frac{\Gamma \vdash^\nabla_\mathcal{B} M : A \quad A \leq_\nabla B}{\Gamma \vdash^\nabla_\mathcal{B} M : B}$$

If $\Omega \in \mathbb{C}^\nabla$, in line with the intended set-theoretic interpretation of Ω as the universe, we extend the Basic Type Assignment System with a suitable axiom for Ω:

Definition 3 (Ω-type Assignment System) *Let* Σ^∇ *be a type theory with* $\Omega \in \mathbb{C}^\nabla$. *The axioms and rules of the* Ω-type assignment system $\lambda\cap^\nabla_\Omega$ *are those of the Basic type Assignment System, together with the further axiom:*

$$(Ax\text{-}\Omega) \quad \Gamma \vdash^\nabla_\Omega M : \Omega.$$

In this talk we address the problem of investigating *uniformly* the use of intersection type theories, and corresponding type assignment systems, for giving a *compositional* characterization of *evaluation* properties of λ-terms.

In particular we discuss *termination* properties such as *strong normalization, normalization, head normalization, weak head normalization.*

Definition 4 (Normalization property)

i) M is a normal form*,* $M \in \mathsf{NF}$*, if* M *cannot be further reduced;*

ii) M is strongly normalizing*,* $M \in \mathsf{SN}$*, if all reductions starting at* M *are finite;*

iii) M has a normal form*,* $M \in \mathsf{N}$*, if* M *reduces to a normal form;*

iv) M has a head normal form*,* $M \in \mathsf{HN}$*, if* M *reduces to a term of the form* $\lambda\vec{x}.y\vec{M}$ *(where possibly* y *appears in* $\vec{x}$*);*

v) M has a weak head normal form*,* $M \in \mathsf{WN}$*, if* M *reduces to an abstraction or to a term starting with a free variable.*

We consider also the *persistent* versions of such notions. *Persistently normalizing* terms have been introduced in [4].

Definition 5 (Persistent normalization property)

i) A term M *is* persistently normalizing*,* $M \in \mathsf{PN}$*, if* $M\vec{N} \in \mathsf{N}$ *for all terms* $\vec{N}$ *in* N.

ii) A term M *is a* persistently normalizing normal form*,* $M \in \mathsf{PNF}$*, if* $M \in \mathsf{PN} \cap \mathsf{NF}$.

iii) A term M *is* persistently head normalizing*,* $M \in \mathsf{PHN}$*, if* $M\vec{N} \in \mathsf{HN}$ *for all terms* $\vec{N}$.

iv) A term M *is* persistently weak normalizing*,* $M \in \mathsf{PWN}$*, if* $M\vec{N} \in \mathsf{WN}$ *for all terms* $\vec{N}$.

By way of example we consider also another evaluation property, unrelated to termination, namely reducibility to a *closed term.*

Definition 6 (Closable term) M *is* closable*,* $M \in \mathsf{C}$*, if* M *reduces to a closed term.*

Before giving the main theorem a last definition is necessary.

Definition 7 *A type theory* $\Sigma^{\bigtriangledown}$ *is an* arrow*-type theory if* $\Omega \in \mathbb{C}^{\bigtriangledown}$ *and for all* $\psi \in \mathbb{C}^{\bigtriangledown}$ *either* $\psi{\sim_{\bigtriangledown}}\Omega$ *or* $\exists I, \{A_i, B_i\}_{i\in I}.\psi{\sim_{\bigtriangledown}} \bigcap_{i\in I}(A_i{\rightarrow}B_i)$. ■

We put to use *intersection type disciplines* to give a *compositional* characterization of evaluation properties of λ-terms.

Theorem 8 (Characterization)

1 Normalization properties

i) (strongly normalizing terms) A λ*-term* $M \in \mathsf{SN}$ *if and only if for all type theories* $\Sigma^{\bigtriangledown}$ *there exist* $A \in \mathbf{T}^{\bigtriangledown}$ *and a* $\bigtriangledown$*-basis* Γ *such that* $\Gamma \vdash^{\bigtriangledown} M : A$*. Moreover in the system* $\lambda\cap^{\mathcal{B}a}_{\mathcal{B}}$ *the terms satisfying the latter property are precisely the strongly normalizing ones.*

ii) (normalizing terms) A λ*-term* $M \in \mathsf{N}$ *if and only if for all type theories* $\Sigma^{\bigtriangledown}$ *such that* $\{\Omega\} \subset \mathbb{C}^{\bigtriangledown}$*, there exist* $A \in \mathbf{T}^{\bigtriangledown}$ *and a* $\bigtriangledown$*-basis* Γ *such that* $\Gamma \vdash^{\bigtriangledown}_{\Omega} M : A$ *and* $\Omega \notin A, \Gamma$*. Moreover in the system* $\lambda\cap^{\mathcal{BCD}}_{\Omega}$ *the terms satisfying the latter property are precisely the ones which have a normal form. Furthermore, in the system* $\lambda\cap^{\mathcal{CDZ}}_{\Omega}$ *the terms typable with type* φ*, in the* $\mathcal{CDZ}$*-basis all of whose predicates are* ω*, are precisely the ones which have a normal form.*

iii) (head normalizing terms) A λ*-term* $M \in \mathsf{HN}$ *if and only if for all type theories* $\Sigma^{\bigtriangledown}$ *such that* $\Omega \in \mathbb{C}^{\bigtriangledown}$*, and for all* $A \in \mathbf{T}^{\bigtriangledown}$ *there exist a* $\bigtriangledown$*-basis* Γ *and two integers* m, n *such that* $\Gamma \vdash^{\bigtriangledown}_{\Omega} M : (\Omega^m \rightarrow A)^n \rightarrow A$*. Moreover in the system* $\lambda\cap^{\mathcal{BCD}}_{\Omega}$ *the terms satisfying the latter property are precisely the ones which have a head normal form.*

iv) (weak head normalizing terms) A λ-term $M \in \mathsf{WN}$ if and only if for all type theories $\Sigma^\triangledown$ such that $\Omega \in \mathbb{C}^\triangledown$, there exists a $\triangledown$-basis Γ such that $\Gamma \vdash_\Omega^\triangledown M : \Omega \to \Omega$. Moreover in the system $\lambda\cap_\Omega^{\mathcal{AO}}$ the terms satisfying the latter property are precisely the ones which have a weak head normal form.

2 Persistent normalization properties

i) (persistently normalizing terms) A λ-term $M \in \mathsf{PN}$ if and only if for all arrow-type theories $\Sigma^\triangledown$ and all $A \in \mathbf{T}^\triangledown$ there exists a $\triangledown$-basis Γ such that $\Gamma \vdash_\Omega^\triangledown M : A$. Moreover in the system $\lambda\cap_\Omega^{\mathcal{CDZ}}$ the terms typable with type ω, in the $\mathcal{CDZ}$-basis all of whose predicates are ω, are precisely the persistently normalizing ones.

ii) (persistently head normalizing terms) A λ-term $M \in \mathsf{PHN}$ if and only if for all type theories $\Sigma^\triangledown$ such that $\Omega \in \mathbb{C}^\triangledown$ and all $A \in \mathbf{T}^\triangledown$ there exist a $\triangledown$-basis Γ and an integer n such that $\Gamma \vdash_\Omega^\triangledown M : \Omega^n \to A$. Moreover in the system $\lambda\cap_\Omega^{\mathcal{S}c}$ the terms typable with type ω, in the $\mathcal{S}c$-basis all of whose predicates are ω, are precisely the persistently head normalizing ones.

iii) (persistently weak normalizing terms) A λ-term $M \in \mathsf{PWN}$ if and only if for all type theories $\Sigma^\triangledown$ such that $\Omega \in \mathbb{C}^\triangledown$ and all integers n there exists a $\triangledown$-basis Γ such that $\Gamma \vdash_\Omega^\triangledown M : \Omega^n \to \Omega$. Moreover in the system $\lambda\cap_\Omega^{\mathcal{AO}}$ the terms satisfying the latter property are precisely the persistently weak normalizing ones.

3 Closability *(closed terms)A λ-term $M \in \mathsf{C}$ if and only if for all type theories $\Sigma^\triangledown$ such that $\Omega \in \mathbb{C}^\triangledown$ and $\omega \sim_\triangledown \omega \to \omega$ for some $\omega \in \mathbb{C}^\triangledown$, M is typable with type ω in the empty $\triangledown$-basis. Moreover in the system $\lambda\cap_\Omega^{\mathcal{P}a}$ the terms satisfying the latter property are precisely the terms which reduce to closed terms.*

Many of the characterization results that we give are indeed inspired by earlier semantical work on filter models of the untyped λ-calculus, but they are rather novel in spirit. We focus, in fact, on *proof-theoretic* properties of intersection type assignment systems *per se*. Most of our characterizations are therefore new, to our knowledge, or else they streamline, strengthen, or generalize earlier results in the literature.

The *completeness* part of the characterizations is proved uniformly for all the properties. We use a very elementary presentation of the technique of *logical relations* phrased in terms of a *set-theoretical* semantics of intersection types over suitable kinds of *stable* sets. This technique generalizes Krivine's [12] and Mitchell's [13] proof methods for *strong normalization*, to other evaluation properties.

A full treatment can be founded in [8].

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