Quine's New Foundations NF

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Overview

- Foundations of Mathematics (ZF, TT)
- Quine's New Foundations
- Properties of New Foundations
- Hailperin's finite axiomatization of NF
- NF as higher-order arithmetic

Foundations of Mathematics (I)

Mathematics is about *counting*, *measuring*, and about *collections*. All these concepts are captured in axiomatic systems of set-theory.

'Naïve' set-theory:

(Extensionality)

$$\forall x \forall x' [\forall y (y \in x \leftrightarrow y \in x') \to x = x'];$$

(Comprehension)

$$\forall x \exists y \forall z [z \in y \leftrightarrow \varphi(z, x)],$$

for *every* formula $\varphi(z, x)$ in the language of set-theory;

(Infinity)

Peano Arithmetic/Infinity.

Foundations of Mathematics (II)

Russell argued that Frege's *Grundlagen der Arithmetik* based on the system above is inconsistent by looking at

$$\exists y \forall z (z \in y \leftrightarrow z \notin z)\,.$$

Many solutions were proposed on how to avoid this and related paradoxes. Among them are

- Zermelo-Fränkel set-theory (ZF);
- Russell's theory of ramified types (RTT);
- type theories, combinatory logic, intuitionism, finitism . . . .

Zermelo-Fränkel set-theory (I)

(Extensionality)

$$\forall x \forall x' (\forall y (y \in x \leftrightarrow y \in x') \rightarrow x = x');$$

(Emptyset)

$$\exists x \forall y \neg y \in x;$$

(Pairing)

$$\forall x \forall x' \exists y \forall z (z \in y \leftrightarrow z = x \vee z = x');$$

(Union)

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists x'. z \in x' \in x);$$

(Powerset)

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x);$$

Zermelo-Fränkel set-theory (II)

(Separation (Aussonderungsaxiom))

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z));$$

(Infinity)

$$\exists x (\emptyset \in x \wedge \forall y \in x . y \cup \{y\} \in x),$$

where $\emptyset$ abbreviates the empty-set, and $\{y\}$ the singleton set containing just y;

(Collection)

$$\forall x (\forall y \in x \exists w \varphi(w, y, x) \rightarrow \exists z \forall y \in x \exists w \in z . \varphi(w, y, x));$$

(Foundation)

$$\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x).$$

Russell's theory of types (I)

Russell's ramified type theory used in *Principia Mathematica* by Russell and Whitehead* is quite complicated. The following type theory is a simplification due to Ramsey:

- Basic types X_n are indexed by the natural numbers;
- There are membership relations $\in_n \hookrightarrow X_n \times X_{n+1}$ for each n; and
- equality symbols $=_n \hookrightarrow X_n \times X_n$ for each n.

*B.A.W. Russell and A.N. Whitehead. *Principia Mathematica.* Cambridge University Press, Cambridge 1910, 1912, 1913.

Russell's theory of types (II)

The axioms are (Extensionality), (Comprehension), plus axioms like (Infinity), (Choice), etc.

(Extensionality)

$$\forall x^{n+1} \forall x'^{n+1} (\forall y^n (y^n \in_n x^{n+1} \leftrightarrow y^n \in_n x'^{n+1}) \\ \rightarrow x^{n+1} = x'^{n+1});$$

(Comprehension)

$$\exists y^{n+1} \forall z^n (z^n \in_n y^{n+1} \leftrightarrow \varphi(z^n))$$

for every (well-formed) formula $\varphi(z^n)$ of the underlying type theory.

Russell's theory of types (III)

- Two canonical models:

$$\emptyset, P(\emptyset), PP(\emptyset), PPP(\emptyset), \ldots$$

$$\mathbb{N}, P(\mathbb{N}), PP(\mathbb{N}), PPP(\mathbb{N}), \ldots .$$

  The first model shows that in type theory (Infinity) is not derivable from (Extensionality) and (Comprehension).

- Russell's paradox is avoided by *syntactically* forbidding terms like $x \in x$.

- *In practice* the type indices are *omitted* in *Principia Mathematica*. According to the authors they can easily be restored.

Quine's New Foundations

Idea: Retain the axioms of TT, but abandon types formally.

The axioms are:

(NF1)

$$\forall x \forall x' (\forall y (y \in x \leftrightarrow y \in x') \rightarrow x = x');$$

(NF2)

$$\exists y \forall z (z \in y \leftrightarrow \varphi(z));$$

for $\varphi(z)$ *stratified*, a formula obtained from a formula in type theory by deleting all type indices.

Properties (I)

(i) Since $x = x$ is stratified there exists a universal set

$$V = \{x \mid x = x\}$$

in (NF). This by itself does not explain the difficulty of proving consistency of (NF).

(ii) The universe V is a Boolean algebra (closed under binary intersections and complements). NF does prove (Union).

(iii) Cardinality captures the *true* meaning of having the same size: card$(x) = \{y \mid \exists f. f\colon x \cong y\}$, and this set (!) does exist in (NF).

(iv) (NF) postulates explicitly the existence of sets definable by stratified formulae, but other sets also exist, like $x \cup \{x\}$ for a given set x.

Properties (II)

(v) The sets (!) of all groups, posets, lattices, Boolean algebras, rings, categories, ..., exist.

(vi) Neither V, the whole universe, nor the (small) category of all categories are Cartesian closed. This means for example that for two sets x and y the set of all functions from x to y exists, but it does *not* have the expected properties.

(vii) $V = P(V)$, so Cantor's theorem fails (provided that (NF) is consistent).

(viii) Call a set *Cantorian* if $x \cong \{\{y\} \mid y \in x\}$. Cantorian sets behave as one expects from (ZF).

Properties (III)

(ix) (NF) is finitely axiomatizable.

(x) (NF_n) is the system (NF1)+($NF2_n$), where the latter means (NF2) restricted to formulae using at most n successive type indices in a stratification.

By the previous remark, (NF) $\equiv$ (NF_n) for some large n. In fact, (NF) $\equiv$ (NF_4) and (NF_3) is consistent.*

(xi) (NF_2) is essentially equi-consistent with (NFU), New Foundations with urelements (see below).

*V.N. Grishin. Consistency of a fragment of Quine's NF system. *Soviet Mathematics Doklady*, 10:1387–1390, 1969. V.N. Grishin. The equivalence of Quine's NF to one of its fragments (in Russian). *Nauchno-tekhnicheskaya Informatsiya*, 1:22-24, 1972. The latter proves that (NF) is equivalent to (NF_5), but this bound was later improved.

NF as foundations for mathematics

- *Yes*, since both Rosser and recently Holmes have written books in which they develop basic mathematics in systems based on (NF), or, to be more precise, on (NF) with atoms.

- *Who cares.* If one believes in (ZFC) then (NF) is a strange system that simply does not model mathematics correctly. If one believes in (NF), then the only importance is that (NF) is not inconsistent. Relative consistency to (ZF) is only of academic interest.

- *No*, simply because there are too many facts about (NF) that do not agree with our intuition

Specker's results (I)

(TNT) is the theory of negative types (better: positive and negative types). Clearly,

$$\mathrm{con}(\mathsf{TNT}) \leftrightarrow \mathrm{con}(\mathsf{TT})\,.$$

To $\varphi \in \mathcal{L}_{\mathsf{TNT}}$ associate $\varphi^* \in \mathcal{L}_{\mathsf{TNT}}$ by raising each type index by 1.

$$\mathsf{TNT} \vdash \varphi \qquad \text{iff} \qquad \mathsf{TNT} \vdash \varphi^*\,.$$

Stronger, we might ask for $\mathsf{TNT} \vdash \varphi \leftrightarrow \varphi^*$.

Specker's results (II)

Theorem. (Specker.*) (NF) *is consistent if and only if* (TNT) $\cup$ $\{\varphi \leftrightarrow \varphi^* \mid \varphi$ *closed*$\}$ *is.*

Theorem. (Specker.†) NF $\vdash \neg$AC *and* (NF) *proves thus the axiom of infinity.*

Specker also showed that (NF) cannot have a standard model, that is, a model (N, E) in which E is the ordinary membership relation.

*E. Specker. Dualität. *Dialectica*, 12:451-465, 1958.

†E. Specker. The axiom of choice in Quine's new foundations for mathematical logic. *Proceedings of the National Academy of Sciences of the USA*, 39:972–975, 1953.

New Foundations with urelements

We relax (Extensionality) to allow atoms ('Urelemente'):

(NF1A)

$$\forall x \forall x' ([\exists z. z \in x \wedge \forall y (y \in x \leftrightarrow y \in x')] \to x = x').$$

(NFU) is the theory (NF1A)+(NF2).

Theorem. (Jensen.*) (NFU) *is consistent. It is consistent to add* (Infinity) *and* (AC) *to* (NFU)*.*

*R.B. Jensen. On the consistency of a slight (?) modification of Quine's NF. *Synthese*, 19:250-263, 1969.

Mac Lane set-theory with bounded separation

$\mathbb{S}$ is the theory with axioms

- (Extensionality);
- (Pairing), (Union), (Powerset);
- (Weak Aussonderungsaxiom), which is separation for *bounded* formulae.

Theorem. (Jensen.) *If* $\mathbb{S}$ *(*$\mathbb{S}$+(Infinity), $\mathbb{S}$+(Infinity)+(AC)*) is consistent, then so are the following systems* (NFU) *(*(NFU)+(Infinity), (NFU)+(Infinity)+(AC)*).*

In a different direction Kaye showed recently that (NF) is $\mathbb{S}$ + the existence of a universal set.

Current research and references

- Consistent sub-systems of New Foundations;
- Intuitionistic NF, Predicative NF, . . . ;
- Relative consistency results.

`http://math.idbsu.edu/~holmes/holmes/nf.html`.

Hailperin's finite axiomatization of NF (I)

(P1) $\forall u \forall v \exists \beta \forall x [x \in \beta \leftrightarrow x \in u \mid x \in v]$

(P2) $\forall \alpha \exists \beta \forall x \forall y [\langle \iota x, \iota y \rangle \in \beta \leftrightarrow \langle x, y \rangle \in \alpha]$

(P3) $\forall \alpha \exists \beta \forall x \forall y \forall z [\langle x, y, z \rangle \in \beta \leftrightarrow \langle x, y \rangle \in \alpha]$

(P4) $\forall \alpha \exists \beta \forall x \forall y \forall z [\langle x, z, y \rangle \in \beta \leftrightarrow \langle x, y \rangle \in \alpha]$

(P5) $\forall \alpha \exists \beta \forall x \forall y [\langle y, x \rangle \in \beta \leftrightarrow x \in \alpha]$

(P6) $\forall \alpha \exists \beta \forall x [x \in \beta \leftrightarrow \forall u \langle u, \iota x \rangle \in \alpha]$

Hailperin's finite axiomatization of NF (II)

(P7) $\forall\alpha\exists\beta\forall x\forall y[\langle y,x\rangle \in \beta \leftrightarrow \langle x,y\rangle \in \alpha]$

(P8) $\exists\beta\forall x[x \in \beta \leftrightarrow \exists y(x = \iota y)]$

(P9) $\exists\beta\forall x\forall y[\langle \iota x,y\rangle \in \beta \leftrightarrow x \in y]$

Here $|$ denotes the Sheffer stroke (NAND), ιy is the singleton set $\{y\}$, and $\langle x,y\rangle$ denotes $\{\{x\},\{x,y\}\}$.

Standard reading of these axioms

Hailperin's reading of these axioms is as follows:

> *Axiom P1 has already been treated. For any relation α, P2 provides us with the corresponding relation with is of one higher type.* Axioms P3, P4, and P5 enable us, among other things, to "raise" the degree of a relation. Axiom P6 eventually gives us the domain of a relation, P7 gives us the converse of a relation, and P8 gives us the class of all unit classes, that is the cardinal 1. Finally, P9 states that $x \in y$ determines a relation—but one which is inhomogeneous.**

*The term is used here merely to render some aid to the intuitive understanding of the axioms. [Hailperin's original footnote.]

A new finite axiomatization

- We assume one-sorted first-order logic with $\in$ the only non-logical symbol.

Our starting point is a form of logicism:

- Objects that are *constructible* out of existing objects *with the help of logic* do exist.

  Since the underlying logic is first-order logic this means that objects constructible using the language of first-order logic (with parameters) do exists.

The propositional axioms

(B1) $\forall u \forall \exists \beta \forall x[x \in \beta \leftrightarrow x \in u \wedge x \in v]$

(B2) $\forall \alpha \exists \beta \forall x[x \in \beta \leftrightarrow \neg x \in \alpha]$

(B3) $\exists \beta \forall x . x \in \beta$

(B4) $\exists \beta \forall x . \neg x \in \beta$

The last two axioms provide us with the universal set

$$V = \{x \mid x = x\} = \{x \mid \top\}$$

and with an atom $\{x \mid \bot\}$. In the presence of the extensionality axiom this atom is unique and denoted $\emptyset$.

The first-order axioms

To handle first-order quantification we need the following axiom:

$$\forall\alpha\exists\beta\forall x[x \in \beta \leftrightarrow \exists y.\langle x, y\rangle \in \alpha]$$

However, this axiom is not enough since we also have to talk about n-tuples. We need at least the following additional axioms:

(B5) $\forall\alpha\exists\beta\forall x[x \in \beta \leftrightarrow \exists y_i, \ldots, \exists y_n\langle y_1, \ldots, y_i, x, y_{i+1}, \ldots, y_n\rangle \in \alpha]$ for every n and every $1 \leq i \leq n+1$;

(B6) $\forall\alpha\exists\beta\forall x_1, \ldots \forall x_n[\langle x_1, \ldots, x_n\rangle \in \beta \leftrightarrow \langle x_{\pi 1}, \ldots, x_{\pi n}\rangle \in \alpha]$ for every permutation $\pi \in S_n$;

(B7) $\forall\alpha\exists\beta\forall x\forall x_1, \ldots, \forall x_n[\langle x, x_1, \ldots, x_n\rangle \in \beta \leftrightarrow \langle x_1, \ldots, x_n\rangle \in \alpha]$

Remark

We haven't said what $\langle -, \ldots, - \rangle$ is, and how these predicates relate to each other!

> For the moment we will assume that $\langle -, - \rangle$ is primitive, and that the $\langle -, \ldots, - \rangle$ are *defined* using $\langle -, - \rangle$.

First results (I)

Proposition. *The system* (B1)–(B7) *is consistent and finitely axiomatizable.*

Proposition. *Let M be a model of* (B1)–(B7). *Let φ be a formula in a relational language $\mathcal{L}_{\omega\omega}(R, R_1, \ldots, R_n)$. Then for every $a_1, \ldots, a_n$ in M the model satisfies*

$$\exists\beta\forall x[x \in \beta \leftrightarrow \varphi(x, a_1, \ldots, a_n)]\,.$$

Conversely, if some interpretation M of the language $\mathcal{L}_{\omega\omega}(\in, \langle -, - \rangle)$ satisfies this principle then M is a model of (B1)–(B7).

First results (II)

The previous proposition is not true if the equality symbol is allowed in formulae. For this we need another axiom:

(B8) $\exists\beta\forall x[x \in \beta \leftrightarrow \exists y.x = \langle y, y\rangle]$

Proposition. *The previous proposition remains true for formulae of the relational language* $\mathcal{L}_{\omega\omega}(=, R, R_1, \ldots, R_n)$ *and the axiom system* (B1)–(B8)*.*

Remaining axioms

(B9) $\exists\beta\forall x[x \in \beta \leftrightarrow \exists y.x = \{y\}]$

(B10) Infinity, i.e., the existence of the natural numbers.

If we add axiom extensionality as (B0) then we can prove the following theorem:

Theorem. *The axioms* (B0)–(B10) *are consistent. These axioms with* $\langle -, - \rangle$ *as primitive relation symbol are equi-consistent with* (B'0)–(B'10) *where* $\langle -, - \rangle$ *is* defined *as* $\{\{-_1\}, \{-_1, -_2\}\}$.

(B11) Powerset

Proposition. *The axioms* (B0)–(B11) *are a (finite) axiomatization of Quine's NF.*

The finite axiomatization (I)

(B0) Extensionality

(B1) $\forall u \forall \exists \beta \forall x [x \in \beta \leftrightarrow x \in u \wedge x \in v]$

(B2) $\forall \alpha \exists \beta \forall x [x \in \beta \leftrightarrow \neg x \in \alpha]$

(B3) $\exists \beta \forall x . x \in \beta$

(B4) $\exists \beta \forall x . \neg x \in \beta$

(B5) $\forall \alpha \exists \beta \forall x [x \in \beta \leftrightarrow \exists y_i, \ldots, \exists y_n \langle y_1, \ldots, y_i, x, y_{i+1}, \ldots, y_n \rangle \in \alpha]$

for every n and every $1 \leq i \leq n+1$;

The finite axiomatization (II)

(B6) $\forall\alpha\exists\beta\forall x_1, \ldots \forall x_n[\langle x_1, \ldots, x_n\rangle \in \beta \leftrightarrow \langle x_{\pi 1}, \ldots, x_{\pi n}\rangle \in \alpha]$

for every permutation $\pi \in S_n$;

(B7) $\forall\alpha\exists\beta\forall x\forall x_1, \ldots, \forall x_n[\langle x, x_1, \ldots, x_n\rangle \in \beta \leftrightarrow \langle x_1, \ldots, x_n\rangle \in \alpha]$

(B8) $\exists\beta\forall x[x \in \beta \leftrightarrow \exists y.x = \langle y, y\rangle]$

(B9) $\exists\beta\forall x[x \in \beta \leftrightarrow \exists y.x = \{y\}]$

(B10) Infinity, i.e., the existence of the natural numbers.

(B11) Powerset

Remarks

- To attack our axioms one has to attack either extensionality, first-order logic, infinity, or the powerset axiom.

- As it turns out, (NF) is a form of higher-order arithmetic.

- Our axiom system does *not* give a consistency proof of (NF).

- The system allows to define various natural subsystems of (NF), and our reading of the axioms shows that some subsystems of (NF) (like (B0)–(B4), (B5)–(B11)) studied in the literature are not of interest.

- In Hailperin's axiomatization the axiom of infinity is hiding in (P2).