Explicit Extensions in (Typed) λ-calculi

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Item Notation/Lambda Calculus à la de Bruijn

- $\mathcal{I}(\lambda x.B) = [x]\mathcal{I}(B)$ and $\mathcal{I}(AB) = (\mathcal{I}(B))\mathcal{I}(A)$
- $\mathcal{I}((\lambda x.(\lambda y.xy))z) \equiv (z)[x][y](y)x$. The items are (z), $[x]$, $[y]$ and (y).
- *applicator wagon* (z) and *abstractor wagon* $[x]$ occur NEXT to each other.
- A term is a wagon followed by a term.
- (β) $(\lambda x.A)B \to_\beta A[x := B]$ becomes
- (β) $(B)[x]A \to_\beta A[x := B]$ or $(B)[x]A \to_\beta [x := B]A$
- Sometimes, de Bruijn wrote: (β) $(B)[x]A \to_\beta (B)[x][x := B]A$

Redexes in Item Notation

Classical Notation		*Item Notation*	
$(\underline{(\lambda_x.(\lambda_y.\lambda_z.zd)c)b})a$	$\to_\beta$	$(a)\underline{(b)[x]}(c)[y][z](d)z$	$\to_\beta$
$(\underline{(\lambda_y.\lambda_z.zd)c})a$	$\to_\beta$	$(a)\underline{(c)[y]}[z](d)z$	$\to_\beta$
$\underline{(\lambda_z.zd)a}$	$\to_\beta ad$	$\underline{(a)[z]}(d)z$	$\to_\beta (d)a$

$$(a)\ (b)\ [x]\ (c)\ [y]\ [z]\ (d)\ \ z$$

Figure 1: Redexes in item notation

Well-balanced segments

- The "bracketing structure" of $t = ((\lambda_x.(\lambda_y.\lambda_z. - -)c)b)a)$, is compatible with '$\{_1\ \{_2\ \{_3\ \}_2\ \}_1\ \}_3$', where '$\{_i$' and '$\}_i$' match.

- $(a)(b)[x](c)[y][z](d)$ has the bracketing structure $\{\{\ \}\{\ \}\}$.

- Define a well-balanced segment $\overline{s}$ to be a segment of partnered () and [] pairs that match like '$\{$' and '$\}$'.

- Let $\overline{s} \equiv (a)(b)[x](c)[y][z](d)$. Then: (a), (b), $[x]$, (c), $[y]$, and $[z]$, are the *partnered* main items of $\overline{s}$. (d) is a *bachelor* item. $(a)(b)[x](c)[y][z]$ is *well-balanced*.

Generalised reduction

- (general β) $(b)\overline{s}[v]a \leadsto_\beta \overline{s}\{a[v := b]\}$ if $\overline{s}$ is well-balanced

- Many step general β-reduction $\twoheadrightarrow\!\!\!\!\sim_\beta$ is the reflexive transitive closure of $\leadsto_\beta$.

- $t \equiv (a)(b)[x](c)[y][z](d)z \leadsto_\beta$
  $(b)[x](c)[y]\{((d)z)[z := a]\} \equiv$
  $(b)[x](c)[y](d)a$

Lemma 1. *If $a \to_\beta b$ then $a \leadsto_\beta b$. And, If $a \leadsto_\beta b$ then $a =_\beta b$.*

Corollary 1. *If $a \twoheadrightarrow\!\!\!\!\sim_\beta b$ then $a =_\beta b$.* $\square$

Theorem 1. *The general β-reduction is Church-Rosser. I.e. If $a \twoheadrightarrow\!\!\!\!\sim_\beta b$ and $a \twoheadrightarrow\!\!\!\!\sim_\beta c$, then there exists d such that $b \twoheadrightarrow\!\!\!\!\sim_\beta d$ and $c \twoheadrightarrow\!\!\!\!\sim_\beta d$.*

Term reshuffling

- $(a)(b)[x](c)[y][z](d)z$ can be easily rewritten as $(b)[x](c)[y](a)[z](d)z$ by moving the item (a) to the right.

- I.e., we can keep the old β-axiom and we can contract redexes in any order.

- difficult to describe how $((\lambda_x.(\lambda_y.\lambda_z.zd)c)b)a$, is rewritten as $(\lambda_x.(\lambda_y.(\lambda_z.zd)a)c)b$.

$$(b)\ [x]\ (c)\ [y]\ (a)\ [z]\ (d)\ \ z$$

Figure 2: Term reshuffling in item notation

Uses of Generalised reduction and term reshuffling?

- Regnier's *premier redex* in [Reg 92] is a *generalised redex*. [Reg 94] shows that the perpetual reduction strategy finds the longest reduction path when the term is SN. Vidal in [Vid 89] and Sabry in [SF 92] used extended redexes.

- [KTU 94] uses some generalised reduction to show that typability in ML is equivalent to acyclic semi-unification.

- [Nederpelt 73] and [dG 93] and [KW 95a] use generalised reduction and/or term reshuffling to reduce strong normalisation for β-reduction to weak normalisation for related reductions.

- [KW 94] uses amongst other things, generalised reduction and term reshuffling to reduce typability in the rank-2 restriction of system F to the problem of acyclic semi-unification.

- [AFM 95] uses a form of term-reshuffling (which they call "let-C") as a part of an analysis of how to implement sharing in a real language interpreter in a way that directly corresponds to a formal calculus.

- The above description can be found in [KN 95]. Also, [KN 95] showed that generalised reduction makes more redexes visible and hence allows for more flexibility in reducing a term.

- [BKN 96] showed that with generalised reduction one may indeed avoid size explosion without the cost of a longer reduction path and that λ-calculus can be elegantly extended with definitions which result in shorter type derivation.

- [Kam 00] shows that generalised reduction is the first relation for which both conservation and postponement of k-redexes hold. [Kam 00] shows that generalised reduction has PSN.

Partnered and Bachelor Items

"partnered" and "bachelors" items help categorize the main items of a term:

Lemma 2. *Let $\overline{s}$ be the body of a term a. Then the following holds:*

1. *Each bachelor main abstraction item in $\overline{s}$ precedes each bachelor main application item in $\overline{s}$.*

2. *$\overline{s}$ minus all bachelor main items equals a* well-balanced *segment.*

3. *The removal from $\overline{s}$ of all main reducible couples, leaves behind $[v_1]\ldots[v_n](a_1)\ldots(a_m)$, the segment consisting of all bachelor main abstraction and application items.*

4. *If $\overline{s} \equiv \overline{s_1}(b)\overline{s_2}[v]\overline{s_3}$ where $[v]$ and (b) match, then $\overline{s_2}$ is well-balanced.*

Corollary 2. *For each non-empty segment $\overline{s}$, there is a unique* partitioning *in segments $\overline{s_0}, \overline{s_1}, \cdots, \overline{s_n}$, such that $\overline{s} \equiv \overline{s_0}\,\overline{s_1} \cdots \overline{s_n}$ and:*

1. *$\forall 0 \leq i \leq n$, $\overline{s_i}$ is well-balanced in $\overline{s}$ for even i and $\overline{s_i}$ is bachelor in $\overline{s}$ for odd i.*

2. *If $\overline{s_i}$ and $\overline{s_j}$ for $0 \leq i, j \leq n$ are bachelor abstraction resp. application segments, then $\overline{s_i}$ precedes $\overline{s_j}$ in $\overline{s}$.*

3. *If $i \geq 1$ then $\overline{s_{2i}} \not\equiv \emptyset$.* $\square$

This is actually a very nice corollary. It tells us a lot about the structure of our terms.

Example

$\overline{s} \equiv [x][y](a)[z][x'](b)(c)(d)[y'][z'](e)$, has the following partitioning:

- well-balanced segment $\overline{s_0} \equiv \emptyset$
- bachelor segment $\overline{s_1} \equiv [x][y]$,
- well-balanced segment $\overline{s_2} \equiv (a)[z]$,
- bachelor segment $\overline{s_3} \equiv [x'](b)$,
- well-balanced segment $\overline{s_4} \equiv (c)(d)[y'][z']$,
- bachelor segment $\overline{s_5} \equiv (e)$.

Using () everywhere

- We will replace (a) by $(a\delta)$.
- We will replace $[x]$ by (λ_x) or $(\varepsilon\lambda_x)$; and $[x : A]$ by $(A\lambda_x)$.
- New items: substitution items $(A\sigma_x)$ and typing items $(\Gamma\tau)$.
- For example:

(β) $\qquad (B\delta)(\lambda_x)A \to_\beta (B\delta)(\lambda_x)(B\sigma_x)A$

Type Theory in Item Notation

- $\mathcal{T} = * \mid \Box \mid V \mid \mathcal{T}\mathcal{T} \mid \pi_{V:\mathcal{T}}.\mathcal{T}$

- $(\beta) \qquad (\lambda_{x:B}.A)C \to_\beta A[x := C]$

- $\mathcal{I}$ which translates terms from classical notation to item notation such that:

$$
\begin{array}{lcll}
\mathcal{I}(A) & = & A & \text{if } A \in \{*, \Box\} \cup V \\
\mathcal{I}(\pi_{x:A}.B) & = & (\mathcal{I}(A)\pi_x)\mathcal{I}(B) & \\
\mathcal{I}(AB) & = & (\mathcal{I}(B)\delta)\mathcal{I}(A) &
\end{array}
$$

- $(\beta) \qquad (\lambda_{x:B}.A)C \to_\beta A[x := C]$

- $(\beta) \qquad (C\delta)(B\lambda_x)A \to_\beta (C\sigma_x)A$

Trees

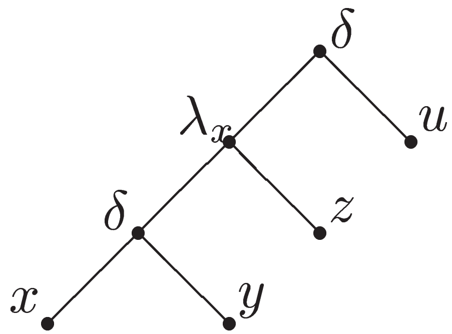


Figure 3: binary tree of $(\lambda_{x:z}.xy)u$

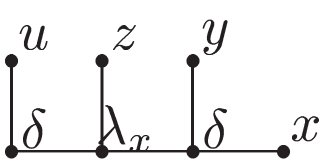


Figure 4: layered tree of $(\lambda_{x:z}.xy)u$

$$\mathcal{I}((\lambda_{x:z}.xy)u) \equiv (u\delta)(z\lambda_x)(y\delta)x$$

Compatibility

- In Classical notation:
  - $$\frac{A_1 \to A_2}{A_1 B \to A_2 B} \qquad \frac{B_1 \to B_2}{A B_1 \to A B_2}$$
  -
  - $$\frac{A_1 \to A_2}{\pi_{x:A_1}.B \to \pi_{x:A_2}.B} \qquad \frac{B_1 \to B_2}{\pi_{x:A}.B_1 \to \pi_{x:A}.B_2}$$
- In Item notation:
  - $$\frac{A_1 \to A_2}{(A_1 \omega) B \to (A_2 \omega) B} \qquad \frac{B_1 \to B_2}{(A \omega) B_1 \to (A \omega) B_2}$$

Restrictions of terms

The restriction $t \restriction x^\circ$ of a term t to a variable occurrence x° in t is a term consisting of precisely those "parts" of t that may be relevant for this x°, especially as regards binding, typing and substitution.

- the type of x° in t is the type of x° in $t \restriction x^\circ$,
- the λ's relevant to x° in t appear also in $t \restriction x^\circ$ and have the same binding relation to x°,
- If in t, any substitution for x° is possible, then it is also possible in $t \restriction x^\circ$.

Example of term restriction

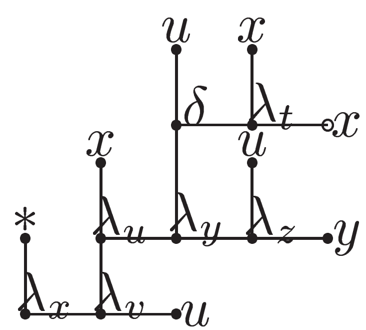
- $t \equiv (*\lambda_x)(x\lambda_v)(x\delta)(*\lambda_y)((x\lambda_z)y^\circ\delta)(y\lambda_u)u$.

- Only $(*\lambda_x)$, $(x\lambda_v)$, $(x\delta)$, $(*\lambda_y)$ and $(x\lambda_z)$ are of importance for y°.
  - y° is in the scope of $(*\lambda_x), (x\lambda_v), (*\lambda_y)$ and $(x\lambda_z)$.
  - The x is a candidate for substitution for y°, due to the presence of the $\delta\lambda$-segment $(x\delta)(*\lambda_y)$ meaning that the x will substitute y in $((x\lambda_z)y^\circ\delta)(y\lambda_u)u$.
  - Nothing else in t is relevant to y°.

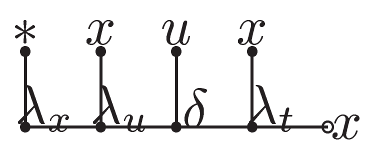
- $t \upharpoonleft y^\circ$ is $(*\lambda_x)(x\lambda_v)(x\delta)(*\lambda_y)(x\lambda_z)$. Remove everything to the right of y°: $(*\lambda_x)(x\lambda_v)(x\delta)(*\lambda_y)((x\lambda_z)$. Remove all extra parentheses.

- If t is written $\lambda_{x:*}.\lambda_{v:x}.(\lambda_{y:*}.(\lambda_{u:y}.u)\lambda_{z:x}.y^\circ)x$ then $t \upharpoonleft y^\circ$ is less obvious.

restriction trees



$$t \equiv (*\lambda_x)((x\lambda_u)((u\delta)(x\lambda_t)x^\circ\lambda_y)(u\lambda_z)y\lambda_v)u$$



$$t \restriction x^\circ \equiv (*\lambda_x)(x\lambda_u)(u\delta)(x\lambda_t)x^\circ$$

Figure 5: A term and its restriction to a variable

Definition of term restriction

Definition 1. $x^\circ \upharpoonleft x^\circ \equiv x$ *and* $(t_1\omega)t_2 \upharpoonleft x^\circ \equiv \begin{cases} t_1 \upharpoonleft x^\circ & \text{if } x^\circ \textit{ occurs in } t_1 \\ (t_1\omega)(t_2 \upharpoonleft x^\circ) & \text{if } x^\circ \textit{ occurs in } t_2 \end{cases}$

Let t be $(*\lambda_x)((x\lambda_u)((u\delta)(x\lambda_t)x^\circ\lambda_y)(u\lambda_z)y\lambda_v)u$.

$$\begin{aligned}
\text{Then } t \upharpoonleft x^\circ &\equiv ((*\lambda_x)((x\lambda_u)((u\delta)(x\lambda_t)x^\circ\lambda_y)(u\lambda_z)y\lambda_v)u) \upharpoonleft x^\circ \\
&\equiv (*\lambda_x)(((x\lambda_u)((u\delta)(x\lambda_t)x^\circ\lambda_y)(u\lambda_z)y\lambda_v)u \upharpoonleft x^\circ) \\
&\equiv (*\lambda_x)((x\lambda_u)((u\delta)(x\lambda_t)x^\circ\lambda_y)(u\lambda_z)y \upharpoonleft x^\circ) \\
&\equiv (*\lambda_x)(x\lambda_u)(((u\delta)(x\lambda_t)x^\circ\lambda_y)(u\lambda_z)y \upharpoonleft x^\circ) \\
&\equiv (*\lambda_x)(x\lambda_u)((u\delta)(x\lambda_t)x^\circ \upharpoonleft x^\circ) \\
&\equiv (*\lambda_x)(x\lambda_u)(u\delta)((x\lambda_t)x^\circ \upharpoonleft x^\circ) \\
&\equiv (*\lambda_x)(x\lambda_u)(u\delta)(x\lambda_t)(x^\circ \upharpoonleft x^\circ) \\
&\equiv (*\lambda_x)(x\lambda_u)(u\delta)(x\lambda_t)x
\end{aligned}$$

Describing normal forms in a substitution calculus

[KR 95] provided λs, a calculus of substitution à la de Bruijn, which remains as close as possible to the classical λ-calculus. The set of terms, noted Λs, of the λs-calculus is given as follows:

$$\Lambda s ::= \mathbb{N} \mid \Lambda s \Lambda s \mid \lambda \Lambda s \mid \Lambda s \, \sigma^i \Lambda s \mid \varphi^i_k \Lambda s \qquad where \quad i \geq 1 \, , \quad k \geq 0 \, .$$

The set of open terms, noted Λs_{op} is given as follows:

$$\Lambda s_{op} ::= \mathbf{V} \,|\, \mathbb{N} \,|\, \Lambda s_{op} \Lambda s_{op} \,|\, \lambda \Lambda s_{op} \,|\, \Lambda s_{op} \sigma \Lambda s_{op} \,|\, \varphi^i_k \Lambda s_{op} \qquad where \quad i \geq 1 \, , \quad k \geq 0$$

The λs-calculus

$$
\begin{array}{llcl}
\sigma\text{-generation} & (\lambda a)\, b & \longrightarrow & a\, \sigma^1\, b \\
\sigma\text{-}\lambda\text{-transition} & (\lambda a)\, \sigma^i b & \longrightarrow & \lambda(a\, \sigma^{i+1}\, b) \\
\sigma\text{-app-transition} & (a_1\, a_2)\, \sigma^i b & \longrightarrow & (a_1\, \sigma^i b)\, (a_2\, \sigma^i b) \\
\sigma\text{-destruction} & \mathtt{n}\, \sigma^i b & \longrightarrow & \begin{cases} \mathtt{n}-\mathtt{1} & \text{if } \; n > i \\ \varphi^i_0\, b & \text{if } \; n = i \\ \mathtt{n} & \text{if } \; n < i \end{cases} \\
\varphi\text{-}\lambda\text{-transition} & \varphi^i_k(\lambda a) & \longrightarrow & \lambda(\varphi^i_{k+1}\, a) \\
\varphi\text{-app-transition} & \varphi^i_k(a_1\, a_2) & \longrightarrow & (\varphi^i_k\, a_1)\, (\varphi^i_k\, a_2) \\
\varphi\text{-destruction} & \varphi^i_k\, \mathtt{n} & \longrightarrow & \begin{cases} \mathtt{n}+\mathtt{i}-\mathtt{1} & \text{if } \; n > k \\ \mathtt{n} & \text{if } \; n \leq k \end{cases}
\end{array}
$$

We use λs to denote this set of rules.

The λs_e-*calculus*

The λs_e-*calculus* is obtained by adding the following rules to those of the λs-calculus.

σ-σ-transition	$(a\sigma b)\,\sigma^j\,c$	$\longrightarrow$	$(a\,\sigma^{j+1}\,c)\,\sigma\,(b\,\sigma^{j-i+1}\,c)$	if	$i \leq j$
σ-φ-transition 1	$(\varphi^i_k\,a)\,\sigma^j\,b$	$\longrightarrow$	$\varphi^{i-1}_k\,a$	if	$k < j < k+i$
σ-φ-transition 2	$(\varphi^i_k\,a)\,\sigma^j\,b$	$\longrightarrow$	$\varphi^i_k(a\,\sigma^{j-i+1}\,b)$	if	$k+i \leq j$
φ-σ-transition	$\varphi^i_k(a\,\sigma^j\,b)$	$\longrightarrow$	$(\varphi^i_{k+1}\,a)\,\sigma^j\,(\varphi^i_{k+1-j}\,b)$	if	$j \leq k+1$
φ-φ-transition 1	$\varphi^i_k\,(\varphi^j_l\,a)$	$\longrightarrow$	$\varphi^j_l\,(\varphi^i_{k+1-j}\,a)$	if	$l+j \leq k$
φ-φ-transition 2	$\varphi^i_k\,(\varphi^j_l\,a)$	$\longrightarrow$	$\varphi^{j+i-1}_l\,a$	if	$l \leq k < l+j$

We use λs_e to denote this set of rules.

s_e-normal forms in classical notation

It is cumbersome to describe s_e-normal forms of open terms. But this description is needed to show the weak normalisation of the s_e-calculus. In classical notation, an open term is an s_e-normal form iff one of the following holds:

- $a \in \mathbf{V} \cup \mathbb{N}$, i.e. a is a variable or a de Bruijn number.
- $a = b\,c$, where b and c are s_e-normal forms.
- $a = \lambda b$, where b is an s_e-normal form.
- $a = b\,\sigma^j c$, where c is an s_e-nf and b is an s_e-nf of the form X, or $d\,\sigma^i e$ with $j < i$, or $\varphi^i_k d$ with $j \leq k$.
- $a = \varphi^i_k b$, where b is an s_e-nf of the form X, or $c\,\sigma^j d$ with $j > k+1$, or $\varphi^j_l c$ with $k < l$.

s_e-normal forms in item notation

The s_e-nf's can be described in item notation by the following syntax:

$$NF ::= \mathbf{V} \mid \mathrm{I\!N} \mid (NF\,\delta)NF \mid (\lambda)NF \mid \overline{s}\,\mathbf{V}$$

where $\overline{s}$ is a normal $\sigma\varphi$-segment whose bodies belong to NF. $a\,\sigma^i b = (b\,\sigma^i)a$ and $\varphi^i_k a = (\varphi^i_k)a$. $(b\,\sigma^i)$ and (φ^i_k) are called σ- and φ-items respectively. b and a are the *bodies* of these respective items.

A *normal $\sigma\varphi$-segment* $\overline{s}$ is a sequence of σ- and φ-items such that every pair of adjacent items in $\overline{s}$ are of the form:

$(\varphi^i_k)(\varphi^j_l)$ and $k < l$
$(b\,\sigma^i)(c\,\sigma^j)$ and $i < j$
$(\varphi^i_k)(b\,\sigma^j)$ and $k < j - 1$
$(b\,\sigma^j)(\varphi^i_k)$ and $j \leq k$.

Types

- At the end of the nineteenth century, types did not play a role in mathematics or logic, unless at the meta-level, in order to distinguish between different 'classes' of objects.

- Frege's formalization of logical reasoning, as explained in the *Begriffsschrift* ([Frege 1879]), was untyped.

- Only after the discovery of Russell's paradox, undermining Frege's work, one may observe various formulations of typed theories.

- The first theory which aimed at avoiding the paradoxes using types was that of Russell and Whitehead, as exposed in their famous *Principia Mathematica* ([Whitehead and Russell 1910]).

- Church was the first to define a type theory 'as such', almost a decade after he developed a theory of functionals which is nowadays called λ-*calculus* ([Church 1932]).

- This calculus was used for defining a notion of computability that turned out to be of the same power as Turing-computability or general recursiveness.

- However, the original, untyped version did not work as a foundation for mathematics.

- In order to come round the inconsistencies in his proposal for logic, Church developed the 'simple theory of types' $\lambda_{\rightarrow}$ ([Church 1940]).

- From then till the present day, research on the area has grown and one can find various reformulations of type theories.

- A taxonomy of type systems has recently been given by Barendregt ([Bar 92]).

- Church's $\lambda_{\rightarrow}$ is the lowest system on the Cube.

- $\lambda_{\rightarrow}$ has, apart from *type variables*, so-called *arrow-types* of the form $A \rightarrow B$.

- In higher type theories, arrow-types are replaced by dependent products $\Pi_{x:A}.B$, where B may contain x as a free variable, and thus may *depend on* x. Example: $\Pi_{A:*}.\lambda_{x:A}.x$

- This means that abstraction can be over types, similarly to the abstraction over terms: $\lambda_{x:A}.b$.

Barendregt Cube

(axiom) $<> \vdash_\beta * : \square$

(start rule) $$\frac{\Gamma \vdash_\beta A : S}{\Gamma.\lambda_{x:A} \vdash_\beta x : A}\ x \not\in \Gamma$$

(weakening rule) $$\frac{\Gamma \vdash_\beta A : S \qquad \Gamma \vdash_\beta D : E}{\Gamma.\lambda_{x:A} \vdash_\beta D : E}\ x \not\in \Gamma$$

(application rule) $$\frac{\Gamma \vdash_\beta F : \Pi_{x:A}.B \qquad \Gamma \vdash_\beta a : A}{\Gamma \vdash_\beta Fa : B[x := a]}$$

(abstraction rule) $$\frac{\Gamma.\lambda_{x:A} \vdash_\beta b : B \qquad \Gamma \vdash_\beta \Pi_{x:A}.B : S}{\Gamma \vdash_\beta \lambda_{x:A}.b : \Pi_{x:A}.B}$$

(conversion rule) $$\frac{\Gamma \vdash_\beta A : B \qquad \Gamma \vdash_\beta B' : S \qquad B =_\beta B'}{\Gamma \vdash_\beta A : B'}$$

(formation rule) $$\frac{\Gamma \vdash_\beta A : S_1 \qquad \Gamma.\lambda_{x:A} \vdash_\beta B : S_2}{\Gamma \vdash_\beta \Pi_{x:A}.B : S_2}\ \text{if } (S_1, S_2) \text{ is a rule}$$

System	*Allowed* (S_1, S_2) *rules*			
$\lambda_\to$	$(*, *)$			
$\lambda 2$	$(*, *)$	$(\Box, *)$		
λP	$(*, *)$		$(*, \Box)$	
$\lambda P2$	$(*, *)$	$(\Box, *)$	$(*, \Box)$	
$\lambda \underline{\omega}$	$(*, *)$			$(\Box, \Box)$
$\lambda \omega$	$(*, *)$	$(\Box, *)$		$(\Box, \Box)$
$\lambda P \underline{\omega}$	$(*, *)$		$(*, \Box)$	$(\Box, \Box)$
$\lambda P \omega = \lambda C$	$(*, *)$	$(\Box, *)$	$(*, \Box)$	$(\Box, \Box)$

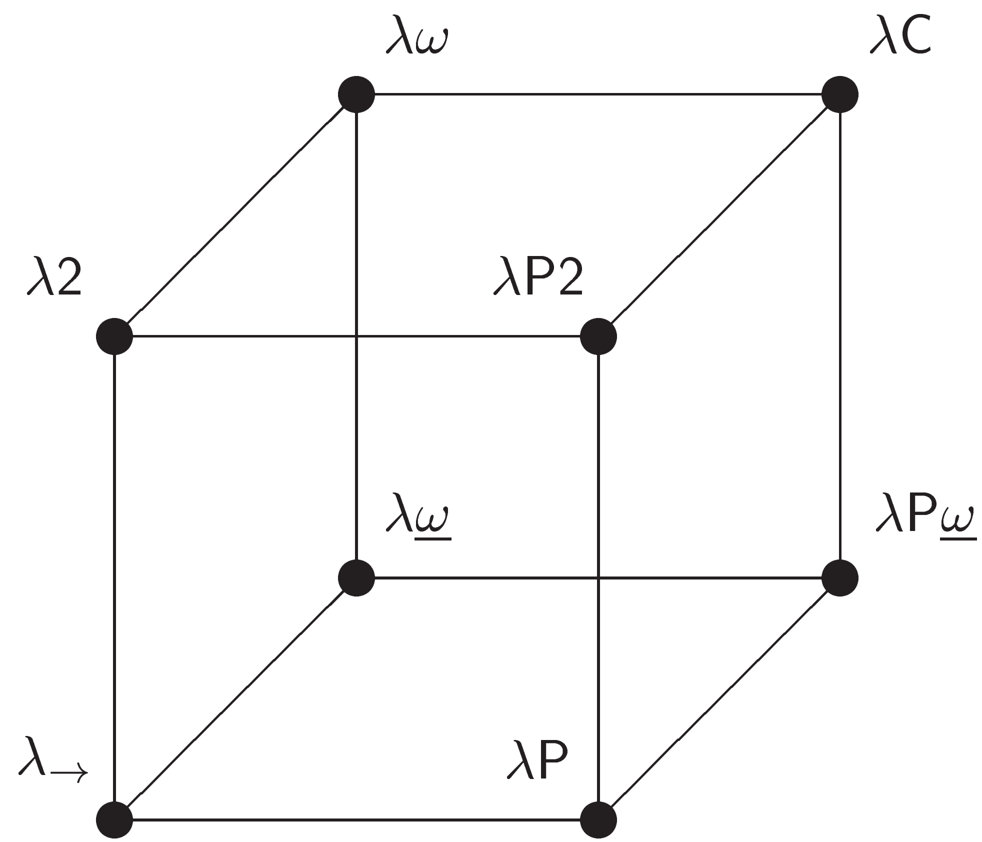


Figure 6: The Cube

Example derivation

Take $\Gamma \equiv \lambda_{\beta:*}.\lambda_{y:\beta}$. In $\lambda 2$, using the rules $(*,*)$ and $(\Box,*)$ we have:

$\Gamma \vdash_{\lambda 2} y : \beta : * : \Box$	
$\Gamma.\lambda_{\alpha:*} \vdash_{\lambda 2} \alpha : *$	(start)
$\Gamma.\lambda_{\alpha:*}.\lambda_{x:\alpha} \vdash_{\lambda 2} x : \alpha : *$	(start resp weakening)
$\Gamma.\lambda_{\alpha:*} \vdash_{\lambda 2} \Pi_{x:\alpha}.\alpha : *$	(formation rule $(*,*)$)
$\Gamma.\lambda_{\alpha:*} \vdash_{\lambda 2} \lambda_{x:\alpha}.x : \Pi_{x:\alpha}.\alpha$	(abstraction)
$\Gamma \vdash_{\lambda 2} \Pi_{\alpha:*}.\Pi_{x:\alpha}.\alpha : *$	(formation rule $(\Box,*)$)
$\Gamma \vdash_{\lambda 2} \lambda_{\alpha:*}.\lambda_{x:\alpha}.x : \Pi_{\alpha:*}\Pi_{x:\alpha}.\alpha$	(abstraction)
$\Gamma \vdash_{\lambda 2} (\lambda_{\alpha:*}.\lambda_{x:\alpha})\beta : \Pi_{x:\beta}.\beta$	(application, we already knew $\Gamma \vdash_{\lambda 2} \beta : *$)
$\Gamma \vdash_{\lambda 2} (\lambda_{\alpha:*}.\lambda_{x:\alpha}.x)\beta y : \beta$	(application, we already knew $\Gamma \vdash_{\lambda 2} y : \beta$)

It is not possible to derive this judgement in $\lambda_{\rightarrow}$ as $(\Box,*)$ is needed.

The system $\lambda_{\rightarrow}$

(axiom) $<> \vdash_\beta * : \Box$

(start rule) $$\frac{\Gamma \vdash_\beta A : S}{\Gamma.\lambda_{x:A} \vdash_\beta x : A}\ x \not\in \Gamma$$

(weakening rule) $$\frac{\Gamma \vdash_\beta A : S \qquad \Gamma \vdash_\beta D : E}{\Gamma.\lambda_{x:A} \vdash_\beta D : E}\ x \not\in \Gamma$$

(application rule) $$\frac{\Gamma \vdash_\beta F : A \to B \qquad \Gamma \vdash_\beta a : A}{\Gamma \vdash_\beta Fa : B}$$

(abstraction rule) $$\frac{\Gamma.\lambda_{x:A} \vdash_\beta b : B \qquad \Gamma \vdash_\beta A \to B : S}{\Gamma \vdash_\beta \lambda_{x:A}.b : A \to B}$$

(conversion rule) $$\frac{\Gamma \vdash_\beta A : B \qquad \Gamma \vdash_\beta B' : S \qquad B =_\beta B'}{\Gamma \vdash_\beta A : B'}$$

(formation rule) $$\frac{\Gamma \vdash_\beta A : * \qquad \Gamma.\lambda_{x:A} \vdash_\beta B : *}{\Gamma \vdash_\beta \Pi_{x:A}.B : *}$$

The system $\lambda_{\to}$ revised

(start rule) $$\frac{\Gamma \vdash_\beta A : S}{\Gamma.\lambda_{x:A} \vdash_\beta x : A}\; x \not\in \Gamma$$

(weakening rule) $$\frac{\Gamma \vdash_\beta A : S \qquad \Gamma \vdash_\beta D : E}{\Gamma.\lambda_{x:A} \vdash_\beta D : E}\; x \not\in \Gamma$$

(application rule) $$\frac{\Gamma \vdash_\beta F : A \to B \qquad \Gamma \vdash_\beta a : A}{\Gamma \vdash_\beta Fa : B}$$

(abstraction rule) $$\frac{\Gamma.\lambda_{x:A} \vdash_\beta b : B}{\Gamma \vdash_\beta \lambda_{x:A}.b : A \to B}$$

Desirable Properties: See [Bar 92]

If $\Gamma \vdash A : B$ then A and B are legal *expressions* and Γ is a legal *context*.

Theorem 2. *(The Church Rosser Theorem CR, for $\twoheadrightarrow_\beta$) If $A \twoheadrightarrow_\beta B$ and $A \twoheadrightarrow_\beta C$ then there exists D such that $B \twoheadrightarrow_\beta D$ and $C \twoheadrightarrow_\beta D$*

Lemma 3. *Correctness of types for $\vdash_\beta$)*
If $\Gamma \vdash_\beta A : B$ then $(B \equiv \Box$ or $\Gamma \vdash_\beta B : S$ for some sort $S)$.

Theorem 3. *(Subject Reduction SR, for $\vdash_\beta$ and $\twoheadrightarrow_\beta$)*
If $\Gamma \vdash_\beta A : B$ and $A \twoheadrightarrow_\beta A'$ then $\Gamma \vdash_\beta A' : B$

Theorem 4. *(Strong Normalisation with respect to $\vdash_\beta$ and $\twoheadrightarrow_\beta$)*
For all $\vdash_\beta$-legal terms M, we have SN$_{\twoheadrightarrow_\beta}(M)$. I.e. M is strongly normalising with respect to $\twoheadrightarrow_\beta$.

Π-reduction: See [KN 96a]

- Once we allow abstraction over types, it would be nice to discuss the reduction rules which govern these types.

- We want: $(\lambda_{x:A}.b)C \to_\beta b[x := C]$, as well as $(\Pi_{x:A}.B)C \to_\beta B[x := C]$.

- This strategy of permitting Π-*application* $(\Pi_{x:A}.B)C$ in term construction is not commonly used, yet is desirable for the following reasons:

- (2) below is more elegant and uniform than (1).

$$\text{If } f : \Pi_{x:A}.B \text{ and } a : A, \text{ then } fa : B[x := a] \tag{1}$$

$$\text{If } f : \Pi_{x:A}.B \text{ and } a : A, \text{ then } fa : (\Pi_{x:A}.B)a. \tag{2}$$

- With Π-reduction, one introduces a *compatibility property* for the typing of applications:

$$M : N \Rightarrow MP : NP.$$

This is in line with the compatibility property for the typing of abstractions, which *does* hold in general:

$$M : N \Rightarrow \lambda_{y:P} M : \Pi_{y:P} N.$$

- $$\begin{array}{lcll} A : *, b : A, a : A & \vdash & a : A & \text{(start)} \\ A : *, b : A & \vdash & (\lambda_{a:A}.a) : (\Pi_{a:A}.A) & \text{(abstraction)} \\ A : *, b : A & \vdash & (\lambda_{a:A}.a)b : (\Pi_{a:A}.A)b & \text{(application)} \\ A : *, b : A & \vdash & (\lambda_{a:A}.a)b : A & \text{(conversion)} \end{array}$$

- The ability to divide two important questions of typing. $\Gamma \vdash A : B$ becomes:

1. Is A typable in Γ? $\Gamma \vdash A$.
2. Is B the type of A in Γ? How does $\tau(\Gamma, A)$ and B compare.

- In a compiler, Π-reduction allows to separate the type finder from the evaluator since $\vdash$ no longer mentions substitution. One first extracts the type and only then evaluates it.

- This is related to some work of Peyton-Jones in his language Henk.

Extending the Cube with Π-reduction: See [KN 96a]

$\beta\Pi$-reduction $\rightarrow_{\beta\Pi}$, is the least compatible relation generated out of the following axiom:

$$(\beta\Pi) \qquad (\pi_{x:B}.A)C \rightarrow_{\beta\Pi} A[x := C]$$

$\twoheadrightarrow_{\beta\Pi}$ is the reflexive transitive closure of $\rightarrow_{\beta\Pi}$. $=_{\beta\Pi}$ is the least equivalence relation generated by $\twoheadrightarrow_{\beta\Pi}$.

$$\text{(new application rule)} \quad \frac{\Gamma \vdash_{\beta\Pi} F : \Pi_{x:A}.B \qquad \Gamma \vdash_{\beta\Pi} a : A}{\Gamma \vdash_{\beta\Pi} Fa : (\Pi_{x:A}.B)a}$$

$$\text{(new conversion rule)} \quad \frac{\Gamma \vdash_{\beta\Pi} A : B \qquad \Gamma \vdash_{\beta\Pi} B' : S \qquad B =_{\beta\Pi} B'}{\Gamma \vdash_{\beta\Pi} A : B'}$$

Barendregt Cube with Π-reduction

(axiom) $<> \vdash_{\beta\Pi} * : \Box$

(start rule) $$\frac{\Gamma \vdash_{\beta\Pi} A : S}{\Gamma.\lambda_{x:A} \vdash_{\beta\Pi} x : A}\ x \notin \Gamma$$

(weakening rule) $$\frac{\Gamma \vdash_{\beta\Pi} A : S \qquad \Gamma \vdash_{\beta\Pi} D : E}{\Gamma.\lambda_{x:A} \vdash_{\beta\Pi} D : E}\ x \notin \Gamma$$

(new application rule) $$\frac{\Gamma \vdash_{\beta\Pi} F : \Pi_{x:A}.B \qquad \Gamma \vdash_{\beta\Pi} a : A}{\Gamma \vdash_{\beta\Pi} Fa : (\Pi_{x:A}.B)a}$$

(abstraction rule) $$\frac{\Gamma.\lambda_{x:A} \vdash_{\beta\Pi} b : B \qquad \Gamma \vdash_{\beta\Pi} \Pi_{x:A}.B : S}{\Gamma \vdash_{\beta\Pi} \lambda_{x:A}.b : \Pi_{x:A}.B}$$

(new conversion rule) $$\frac{\Gamma \vdash_{\beta\Pi} A : B \qquad \Gamma \vdash_{\beta\Pi} B' : S \qquad B =_{\beta\Pi} B'}{\Gamma \vdash_{\beta\Pi} A : B'}$$

(formation rule) $$\frac{\Gamma \vdash_{\beta\Pi} A : S_1 \qquad \Gamma.\lambda_{x:A} \vdash_{\beta\Pi} B : S_2}{\Gamma \vdash_{\beta\Pi} \Pi_{x:A}.B : S_2}$$ if (S_1, S_2) is a rule

Generation Lemma

Lemma 4. *(Generation Lemma for $\vdash_\beta$)*

- *If* $\Gamma \vdash_\beta \Pi_{x:A}.B : C$ *then* $\Gamma \vdash_\beta A : S_1$ *and* $\Gamma.\lambda_{x:A} \vdash_\beta B : S_2$ *and* (S_1, S_2) *is a rule,* $C =_\beta S_2$ *and.....*

- *If* $\Gamma \vdash_\beta Fa : C$ *then* $\Gamma \vdash_\beta F : \Pi_{x:A}.B$ *and* $\Gamma \vdash_\beta a : A$ *and* $C =_\beta B[x := a]$ *and .....*

- ....

In Generation lemma for $\vdash_{\beta\Pi}$ for application case, we replace $B[x := a]$ by $(\Pi_{x:A}.B)a$ and change β to to $\beta\Pi$ everywhere.

Correctness of types fails for Π-reduction even in $\lambda_{\to}$

Lemma 5. *For any* A, B, C, S *we have* $\Gamma \not\vdash_{\beta\Pi} (\Pi_{x:A}.B)C : S$.

Proof: *If* $\Gamma \vdash_{\beta\Pi} (\Pi_{x:A}.B)C : S$ *then by generation*, $\Gamma \vdash_{\beta\Pi} \Pi_{x:A}.B : \Pi_{x:A'}.B'$ *and again by generation*, $\Gamma.\lambda_{x:A} \vdash_{\beta\Pi} B : S' \wedge S' =_{\beta\Pi} \Pi_{x:A'}.B'$. *Absurd.* $\Box$

The new legal terms of the form $(\Pi_{x:B}.C)A$ imply the failure of Correctness of types Lemma 3 for $\vdash_{\beta\Pi}$ even in $\lambda_{\to}$.

- $\Gamma \vdash_{\beta\Pi} A : B$ may not imply $B \equiv \Box$ or $\Gamma \vdash_{\beta\Pi} B : S$ for some sort S.
- E.g., if $\Gamma \equiv \lambda_{z:*}.\lambda_{x:z}$ then $\Gamma \vdash_{\beta\Pi} (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$, but $\Gamma \not\vdash_{\beta\Pi} (\Pi_{y:z}.z)x : S$ from Lemma 5.

We have a weak correctness of types:

Lemma 6. *If* $\Gamma \vdash_{\beta\Pi} A : B$ *and* B *is not a* Π*-redex then* $(B \equiv \Box$ *or* $\Gamma \vdash_{\beta\Pi} B : S$ *for some sort* $S)$.

Proof: *By a trivial induction on the derivation of* $\Gamma \vdash_{\beta\Pi} A : B$ *noting that the application rule does not apply as* $(\Pi_{x:A}.B)a$ *is not a* Π*-redex.* $\Box$

Failure of correctness of types implies failure of Subject Reduction even in $\lambda_{\rightarrow}$:

- In $\lambda_{\rightarrow}$, we have: $\lambda_{z:*}.\lambda_{x:z} \not\vdash_{\beta\Pi} x : (\Pi_{y:z}.z)x$.
- Otherwise,by generation: $\lambda_{z:*}.\lambda_{x:z} \vdash_{\beta\Pi} (\Pi_{y:z}.z)x : S$, which is absurd by Lemma 5.
- Yet in $\lambda_{\rightarrow}$, we have: $\lambda_{z:*}.\lambda_{x:z} \vdash_{\beta\Pi} (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$.

Relating $\vdash_{\beta\Pi}$ and $\vdash_\beta$ and Weak SR

For A $\vdash_{\beta\Pi}$-legal, let $\hat{A}$ be $C[x := D]$ if $A \equiv (\Pi_{x:B}.C)D$ and A otherwise.

Lemma 7.

1. *If* $\Gamma \vdash_{\beta\Pi} A : B$ *then* $\Gamma \vdash_\beta A : \hat{B}$.

2. *If* $\Gamma \vdash_\beta A : B$ *then* $\Gamma \vdash_{\beta\Pi} A : B$.

Lemma 8. *(Weak Subject Reduction for* $\vdash_{\beta\Pi}$ *and* $\twoheadrightarrow_{\beta\Pi}$*)*

1. *If* $\Gamma \vdash_{\beta\Pi} A : B$ *and* $A \twoheadrightarrow_{\beta\Pi} A'$ *then* $\Gamma \vdash_{\beta\Pi} A' : \hat{B}$

2. *If* $\Gamma \vdash_{\beta\Pi} A : B$ *and* $A \twoheadrightarrow_{\beta\Pi} A'$ *and* B *is* $\vdash_\beta$*-legal then* $\Gamma \vdash_{\beta\Pi} A' : B$

Canonical typing

There are reasons why separating the questions "what is the type of a term" (via τ) and "is the term typable" (via $\vdash$), is advantageous. Here are some:

- The canonical type of A is easy to calculate.

- $\tau(A)$ plays the role of a preference type for A. The preference type of $A \equiv \lambda_{x:*}.(\lambda_{y:*}.y)x$ is $\tau(<>, A) \equiv \Pi_{x:*}.(\Pi_{y:*}.*)x$ which $\twoheadrightarrow_{\beta\Pi}$ to $\Pi_{y:*}.*$, the type traditionally given to A.

- The conversion rule is no longer needed as a separate rule in the definition of

$\vdash$. It is accommodated in our application rule:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash AB} \text{ if } \tau(\Gamma, A) =_{\beta\Pi} \Pi_{x:C}.D \text{ and } \tau(\Gamma, B) =_{\beta\Pi} C$$

It will be the case that $\tau(\Gamma, AB) \equiv \tau(\Gamma, A)B =_{\beta\Pi} (\Pi_{x:C}.D)B \to_{\beta\Pi} D[x := B]$ and so indeed $\tau(\Gamma, AB) =_{\beta\Pi} D[x := C]$.

- Higher degrees:If we use λ^1 for Π and λ^2 for λ then we can aim for a possible generalization. In fact, we can extend our system by incorporating more different λ's. For example, with an infinity of λ's, viz. λ^0, λ^1, λ^2, λ^3 ..., we replace $\tau(\Gamma, \lambda_{x:A}.B) \equiv \Pi_{x:A}.\tau(\Gamma.\lambda_{x:A}, B)$ and $\tau(\Gamma, \Pi_{x:A}.B) \equiv \tau(\Gamma.\lambda_{x:A}, B)$ by the following:

$$\tau(\Gamma, \lambda^{i+1}_{x:A}.B) \equiv \lambda^i_{x:A}.\tau(\Gamma.\lambda_{x:A}, B), \text{for } i = 0, 1, 2, \ldots \text{ where } \lambda^0_{x:A}.B \equiv B$$

There may be circumstances in which one desires to have more "layers" of λ's. As an example we refer to [de Bruijn 74].

- This notion enables one to separate the judgement $\Gamma \vdash A : B$ in two: $\Gamma \vdash A$ and $\tau(\Gamma, A) = B$.

$$
\begin{array}{lcll}
\tau(\Gamma, *) & \equiv & \square & \\
\tau(\Gamma, x) & \equiv & A \text{ if } (A\lambda_x) \in \Gamma & \\
\tau(\Gamma, (a\delta)F) & \equiv & (a\delta)\tau(\Gamma, F) & \\
\tau(\Gamma, (A\lambda_x)B) & \equiv & (A\Pi_x)\tau(\Gamma(A\lambda_x), B) & \text{if } x \notin dom(\Gamma) \\
\tau(\Gamma, (A\Pi_x)B) & \equiv & \tau(\Gamma(A\lambda_x), B) & \text{if } x \notin dom(\Gamma)
\end{array}
$$

- In usual type theory:
  - the type of $(*\lambda_x)(x\lambda_y)y$ is $(*\Pi_x)(x\Pi_y)x$ and
  - the type of $(*\Pi_x)(x\Pi_y)x$ is $*$.

- With our τ, we get the same result:
  - $\tau(<>, (*\lambda_x)(x\lambda_y)y) \equiv (*\Pi_x)\tau((*\lambda_x), (x\lambda_y)y) \equiv (*\Pi_x)(x\Pi_y)\tau((*\lambda_x)(x\lambda_y), y) \equiv (*\Pi_x)(x\Pi_y)x$ and
  - $\tau(<>, (*\Pi_x)(x\Pi_y)x) \equiv \tau((*\lambda_x), (x\Pi_y)x) \equiv \tau((*\lambda_x)(x\lambda_y), x) \equiv *$

Let $\Gamma_0 \equiv <>$, $\Gamma_1 \equiv (*\lambda_z)$, $\Gamma_2 \equiv (*\lambda_z)(*\lambda_y)$, $\Gamma_3 \equiv \Gamma_2(*\lambda_x)$. We want to find the canonical type of $(*\Pi_z)(B\delta)(*\lambda_y)(y\delta)(*\lambda_x)x$ in Γ_0.

$$
\begin{array}{ccccccccccccc}
(\Gamma_0\tau) & (*\Pi_z) & & (B\delta) & & (*\lambda_y) & & (y\delta) & & (*\lambda_x) & & x & \to_\tau \\
& & (\Gamma_1\tau) & (B\delta) & & (*\lambda_y) & & (y\delta) & & (*\lambda_x) & & x & \to_\tau \\
& & & (B\delta) & (\Gamma_1\tau) & (*\lambda_y) & & (y\delta) & & (*\lambda_x) & & x & \to_\tau \\
& & & (B\delta) & & (*\Pi_y) & (\Gamma_2\tau) & (y\delta) & & (*\lambda_x) & & x & \to_\tau \\
& & & (B\delta) & & (*\Pi_y) & & (y\delta) & (\Gamma_2\tau) & (*\lambda_x) & & x & \to_\tau \\
& & & (B\delta) & & (*\Pi_y) & & (y\delta) & & (*\Pi_x) & (\Gamma_3\tau) & x & \to_\tau \\
& & & (B\delta) & & (*\Pi_y) & & (y\delta) & & (*\Pi_x) & & * &
\end{array}
$$

New Typability

(⊢-axiom) $<>\vdash *$

(⊢-start rule) $\dfrac{\Gamma \vdash A}{\Gamma(A\lambda_x) \vdash x}$ if vc

(⊢-weakening rule) $\dfrac{\Gamma \vdash A \qquad \Gamma \vdash D}{\Gamma(A\lambda_x) \vdash D}$ if vc

(⊢-application rule) $\dfrac{\Gamma \vdash F \qquad \Gamma \vdash a}{\Gamma \vdash (a\delta)F}$ if ap

(⊢-abstraction rule) $\dfrac{\Gamma(A\lambda_x) \vdash b \qquad \Gamma \vdash (A\Pi_x)B}{\Gamma \vdash (A\lambda_x)b}$ if ab

(⊢-formation) $\dfrac{\Gamma \vdash A \qquad \Gamma(A\lambda_x) \vdash B}{\Gamma \vdash (A\Pi_x)B}$ if fc

- vc (variable condition): $x \notin \Gamma$ and $\tau(\Gamma, A) \twoheadrightarrow_{\beta\Pi} S$ for some S

- ap (application condition): $\tau(\Gamma, F) =_{\beta\Pi} (A\Pi_x)B$ and $\tau(\Gamma, a) =_{\beta\Pi} A$ for some A, B.

- ab (abstraction condition): $\tau(\Gamma(A\lambda_x), b) =_{\beta\Pi} B$ and $\tau(\Gamma, (A\Pi_x)B) \twoheadrightarrow_{\beta\Pi} S$ for some S.

- fc (formation condition): $\tau(\Gamma, A) \twoheadrightarrow_{\beta\Pi} S_1$ and $\tau(\Gamma(A\lambda_x), B) \twoheadrightarrow_{\beta\Pi} S_2$ for some rule (S_1, S_2).

Properties of $\vdash$

Define $\overline{A}$ to be the $\beta\Pi$-normal form of A.

Lemma 9. *If $\Gamma \vdash A$ then $\downarrow \overline{\tau(\Gamma, A)}$ and $\Gamma \vdash_\beta A : \overline{\tau(\Gamma, A)}$*

Lemma 10. *(Subject Reduction for $\vdash$ and τ)*
$\Gamma \vdash A \wedge A \twoheadrightarrow_{\beta\Pi} A' \Rightarrow [\Gamma \vdash A' \wedge \tau(\Gamma, A) =_{\beta\Pi} \tau(\Gamma, A')]$

Theorem 5. *(Strong Normalisation for $\vdash$)*
If A is $\Gamma^{\vdash}$-legal, then $SN_{\twoheadrightarrow_\beta}(A)$.

Lemma 11. $\Gamma \vdash_\beta A : B \iff \Gamma \vdash A$ *and* $\tau(\Gamma, A) =_{\beta\Pi} B$ *and* B *is $\vdash_\beta$-legal type.*

Properties of the Cube with generalised reduction

$$\begin{tikzcd}
& C(\text{CR, SN, SR}) \arrow[dl] \arrow[dr] & \\
C_{\leadsto_\beta}(\text{CR, SN}) \arrow[dr] & & C_{def}(\text{CR, SN, SR}) \arrow[dl] \\
& C_{\leadsto_\beta def}\ (\text{CR, SN, SR}) &
\end{tikzcd}$$

Figure 7: Properties of the Cube with generalised reduction

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