Themes in the λ-calculus

Fairouz Kamareddine
Department of Mathematics and Computing Science
Eindhoven University of Technology
Eindhoven, the Netherlands

August 1992

The type free λ-calculus (i.e. where terms are only $E ::= x \mid (E_1 E_2) \mid (\lambda x.E_1)$) is powerful enough to contain all the polymorphic and self referential nature of properties. Logic however, due to the paradoxes, is absent from the type free λ-calculus. Attempts at extending the calculus with logic are many yet neither of them tackles the proof theory of the resulting system. In fact even basic questions such as *completeness, cut elimination theorems* and *extensionality* axioms are not addressed. In fact, once we extend the type free λ-calculus with logic, many of these issues become very hard to tackle.

In this paper, we present the type free λ-calculus with logic λ_L which is to some extent influenced by the work of Aczel, Feferman, Flagg and Myhill. We then explore the proof theory of λ_L, and present versions of *the deduction theorem* and *the cut elimination* theorem which hold in our calculus. The structure of the lambda terms and the study of their normal form, play a crucial role in the proof theory and those people familiar with Dana Scott's work on combinators and classes can get a feeling of what might be involved.

From the proof theory of the new calculus we move to the metatheory and show the *completeness* of λ_L. Furthermore, we interpret the work of Bunder and Barendregt on Illative Combinatory Algebras, ICA, in our framework, showing that ICA is weaker than λ_L. Finally, we touch on the issue of adding *extensionality axioms* to λ_L, showing that this will lead to a paradox. We are hence consistent with the Gordeev's theorem which states that any theory as strong as Feferman's T_0 cannot be extended consistently with extensionality axioms.