Texts in Mathematics 8

A Primer of Mathematical Analysis and the Foundations of Computation

Solutions to Exercises

Fairouz Kamareddine, Jonathan Seldin

Appendix C

Solutions to Exercises

C.1 Solutions for Chapter 1

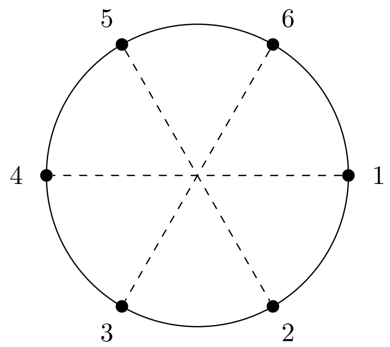
Solution C.1. [Of Exercise 1.1.] Say x = age of Diophantus, then his son was born when Diophantus had $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5$ years. The son lived half his father's life which is $\frac{x}{2}$. Diophantus lived for four more years after his son died. So, the entire life of Diophantus is the period up to his son's birth + the life of his son + 4 which is: $x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4$, hence $x = 84$. So Diophantus lived till 84, his son lived till 42. When his son died, Diophantus was 80 years old.

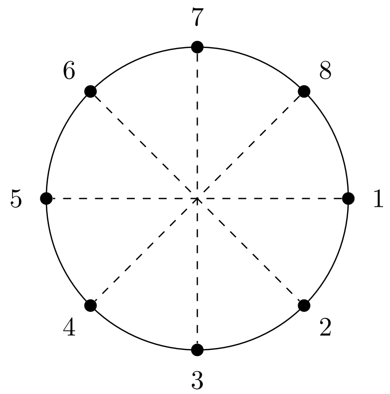
Solution C.2. [Of Exercise 1.2.] $\dfrac{n \times 5 + 20}{5} - n = 4.$

Solution C.3. [Of Exercise 1.3.] $\dfrac{n \times 6 + 48}{6} - n = 8.$

Solution C.4. [Of Exercise 1.4.] $\dfrac{n \times m + m \times k}{m} - n = k.$

Solution C.5. [Of Exercise 1.5.] As can be seen from the pictures below, if we divide the cake into 6 pieces (as in the ballroom), instead of 8 pieces (as in the seminar room), we get a larger piece of cake if we are in the ballroom. This is a general phenomena. If $m < n$ then $\frac{1}{m} > \frac{1}{n}$. We will see this in more details in later chapters.





Solution C.6. [Of Exercise 1.6.] In step 4, we divided by $(a-b)$ which is 0 and hence the division is not allowed. Furthermore, if $a=b=0$ then many other steps are false (give these steps).

Solution C.7. [Of Exercise 1.7.] The false step here is that we passed from the squares to the numbers themselves in step 6. As we saw in Section 1.2, each positive number has two squares roots, one positive and one negative. Hence, the fact that the squares of two numbers are equal does not mean that the numbers are equal.

Solution C.8. [Of Exercise 1.8.] The second interest rate of 10% of the total value added to the sum at the end of every six months is better. This can be seen as follows:

	1st Year		2nd Year	
	Int 1	Int 2	Int 1	Int 2
Sum after 1st Half of Year	40,000	44,000	48,000	53,240
Sum after 2nd Half of Year	48,000	48,400	57,600	58,564
Total interest for Year	8,000	8,400	9,600	10,164

Solution C.9. [Of Exercise 1.9.] The given so-called proof is not a proof of anything. It is true that since all sides involve positive numbers, we can square all sides and still get the same inequalities. But this has no implication on the truth or falsity of the inequalities we started from. That is, although $81 < 84 < 100$ is true, this does not imply that either of $\sqrt{3}+\sqrt{7} < \sqrt{20}$ and $\sqrt{3}+\sqrt{7} > \sqrt{19}$ is true. In order to prove that both $\sqrt{3}+\sqrt{7} < \sqrt{20}$ and $\sqrt{3}+\sqrt{7} > \sqrt{19}$ hold (which they do), we follow the proof by contradiction method as follows:

- Assume $\sqrt{19} \geq \sqrt{3} + \sqrt{7}$.

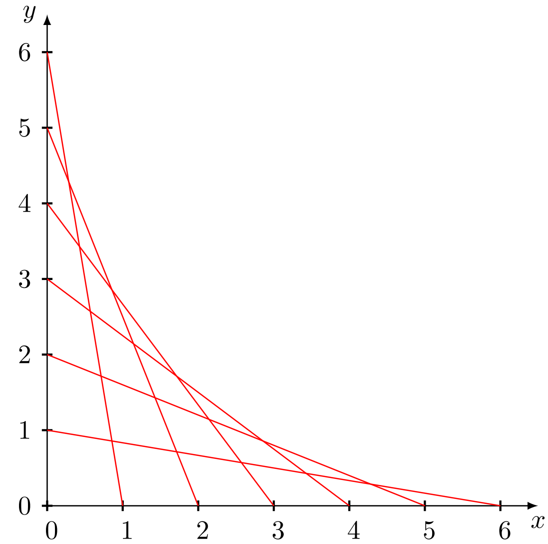
Then	19	$\geq$	$10 + 2\sqrt{21}$.
Hence	9	$\geq$	$2\sqrt{21}$.
So	81	$\geq$	84.
Contradiction.			
Hence	$\sqrt{19}$	$<$	$\sqrt{3} + \sqrt{7}$.

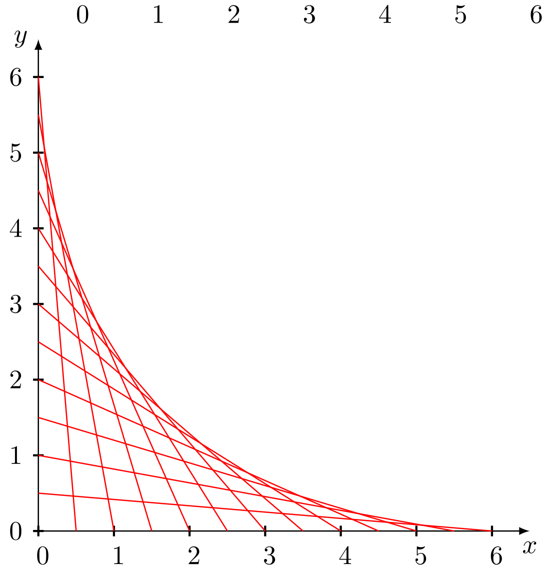
- Assume $\sqrt{3} + \sqrt{7} \geq \sqrt{20}$.

Then	$10 + 2\sqrt{21}$	$\geq$	20.
Hence	$2\sqrt{21}$	$\geq$	10.
So	84	$\geq$	100.
Contradiction.			
Hence	$\sqrt{3} + \sqrt{7}$	$<$	$\sqrt{20}$.

Solution C.10. [Of Exercise 1.10.] The problem here is that in the third step we considered $P_1, P_2, \cdots, P_k$ and $P_2, \cdots, P_k, P_{k+1}$ to have at least one common element. If the inductive case we are trying to prove in step 3 above involves $k + 1 < 3$ (say $k = 1$) then when looking at P_1, P_2 and splitting it into the two subcollections P_1 and P_2 which have no common elements, then we will not be able to deduce that the age of P_1 is the same as the age of P_2. For example, we know that the property holds for $k = 1$ and so proving it for $k + 1 = 2$ by taking the sets $\{P1\}$ and $\{P_2\}$ where the age of P_1 is 50 years and the age P_2 is 20 years does not allow us to say that all elements of $\{P_1, P_2\}$ have the same age.

Solution C.11. [Of Exercise 1.11.] The first requested figure is on the left below. The second figure is on the right. In both figures, the boundary is a collection of straight lines but where the collection is leaning towards a curve. In the second figure, the curve is more pronounced. The more we reduce the length of the connecting lines, the more curve-like the boundary becomes.





Solution C.12. [Of Exercise 1.12.] Let a and b be the sides of the rectangle. Then, the area of the rectangle is $A = ab$ and the perimeter is $P = 2(a+b)$. Since $(a+b)^2 - (a-b)^2 = 4ab = 4A$ then $4A = (\frac{P}{2})^2 - (a-b)^2$. That is, $A = (\frac{P}{4})^2 - (\frac{a-b}{2})^2$. Now, since $(\frac{P}{4})^2$ and $(\frac{a-b}{2})^2$ are positive, to maximise A we need to minimise $(\frac{a-b}{2})^2$. Hence to maximise A we need to have $a = b$. Therefore, the rectangle with maximum area whose perimeter is P is the square whose side is $\frac{P}{4}$.

Solution C.13. [Of Exercise 1.13.]

1⇒2) Assume 1 holds. If by contradiction 2 does not hold, then let C be a circle whose area is A and perimeter is P and let B be a planar shape whose area is A and whose perimeter is $P' < P$. Now, let C' be the circle C whose perimeter is P' (and hence the area A' of C' is strictly smaller than A). By 1, since the perimeters of C' and B are equal, the area of B is smaller than the area of C'. That is, A is smaller than A'. Contradiction.

2⇒1) Assume 2 holds. If by contradiction 1 does not hold, then let C be a circle whose area is A and perimeter is P and let B be a planar shape whose area is $A' > A$ and whose perimeter is P. Now, let C' be the circle C whose area is A' and hence its parameter is $P' > P$. By 2, since the areas of C' and B are equal, the perimeter of C' is smaller than that of A'. That is, P' is smaller than P. Contradiction.

Solution C.14. [Of Exercise 1.14.]

1. $(1+7+7^2+7^3+7^4)(7-1) = 7+7^2+7^3+7^4+7^5-(1+7+7^2+7^3+7^4) = 7^5 - 1$. Hence $1 + 7 + 7^2 + 7^3 + 7^4 = \frac{7^5 - 1}{7 - 1}$.

2. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = (1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1})(2 - 1) = 2 + 2^2 + 2^3 + \cdots + 2^n - (1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}) = 2^n - 1$.
   Hence $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^n - 1$.

Solution C.15. [Of Exercise 1.15.]

1. In the Moscow Papyrus, the numbers used for h, a, and b were resp. 6, 4, and 2, and the answer given for V was 56. If we fill these numbers in $V = \frac{1}{3}h(a^2 + ab + b^2)$ we get indeed that $56 = \frac{1}{3}6(4^2 + 4 \times 2 + 2^2)$.

2. (a): $(a^2 + ab + b^2)(a - b) = a^3 + a^2b + ab^2 - (a^2b + ab^2 + b^3) = a^3 - b^3$.
   Hence, $a^2 + ab + b^2 = \frac{a^3 - b^3}{a - b}$.
   (b): Without loss of generality, assume that $a = b + 1$. Then,
   $a^2 + ab + b^2 = (b + 1)^2 + b(b + 1) + b^2 = 3b^2 + 3b + 1 = a^3 - b^3$.
   Hence, $a^2+ab+b^2 = a^3 - b^3$ when a and b are two consecutive integers.

3. Left to the reader.

C.2 Solutions for Chapter 2

Solution C.16. [Of Exercise 2.1.] According to Dichotomy, the runner needs to reach half of the 100 km (that is 50 km), then half of the remaining 50 km (that is 25 km) and then half of the remaining 25 km and so on. According to Dichotomy, the runner can never reach his destination because to do so, he would need to cross an infinite sequence of points $100/2, 100/4, 100/8, 100/16, 100/32, \cdots$. The runner needs to complete an infinite serie of tasks which is impossible.

Solution C.17. [Of Exercise 2.2.] According to Dichotomy, the frog needs to reach half the way to the pond, then half of the remaining way and then half of the remaining way and so on. According to Dichotomy, the frog can never reach the pond because to do so, it would need to cross an infinite sequence of points. The frog needs to complete an infinite series of tasks which is impossible.

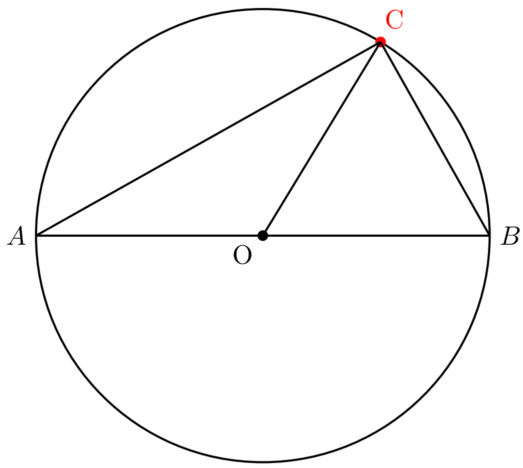
Solution C.18. [Of Exercise 2.3.] At the start, Achilles (A) is at point 0 and the tortoise (T) is at point 100. By the time A reaches point 100, T will have reached point 110. By the time A does another 10 meters and reached point 110, T will have 1 meter and reached point 111. By the time A does another 1 meters and reached point 111, T will have done another 0.1 meter and reached point 111.1. This process continues ad infinitum and we always see that T is ahead of A.

Here are the positions of T resp. A:

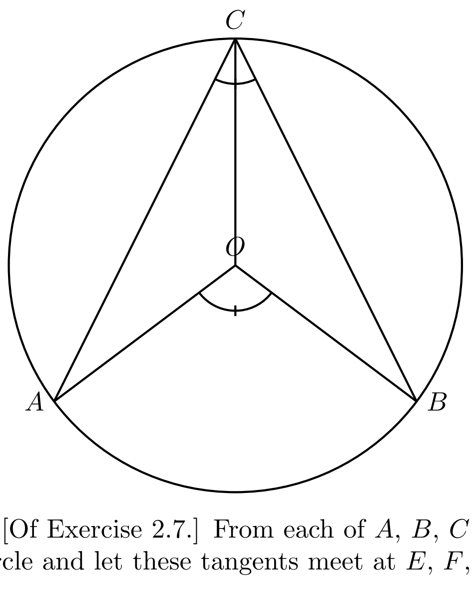
T	100	110	111	$111+\frac{1}{10}$	$111+\frac{1}{10}+\frac{1}{100}$	$111+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}$	$\cdots$
A	0	100	110	111	$111+\frac{1}{10}$	$111+\frac{1}{10}+\frac{1}{100}$	$\cdots$

Solution C.19. [Of Exercise 2.4.] $\angle BAC = 30°$ and $\angle ACB = 90°$.

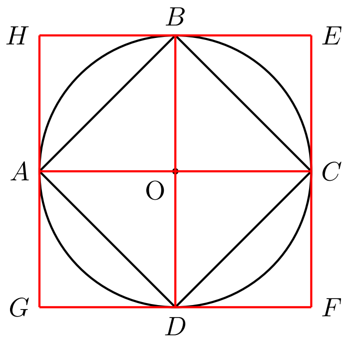
Solution C.20. [Of Exercise 2.5.] We will show that $\angle ACB = 90°$. The proof of $\angle ADB = 90°$ is similar. It is clear that the two triangles AOC and BOC are isosceles since $OA = OB = OC$.
Hence, $\angle OAC = \angle OCA$ and $\angle OCB = \angle OBC$.
But, $\angle OAC + \angle OCA + \angle AOC = \angle OBC + \angle OCB + \angle BOC = 180°$.
So, $2\angle OCA + \angle AOC + 2\angle OCB + \angle BOC = 360°$.
Since $2\angle OCA + 2\angle OCB = 2\angle ACB$ and $\angle AOC + \angle BOC = \angle AOB = 180°$, we get $2\angle ACB + 180° = 360°$. Hence, $\angle ACB = 90°$.



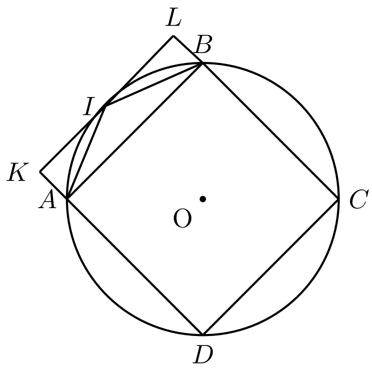
Solution C.21. [Of Exercise 2.6.] It is clear that the two triangles AOC and BOC are isosceles since $OA = OB = OC$.
Hence, $\angle OAC = \angle OCA$ and $\angle OCB = \angle OBC$.
But, $\angle OAC + \angle OCA + \angle AOC = \angle OBC + \angle OCB + \angle BOC = 180°$ and $\angle AOB + \angle AOC + \angle COB = 360°$.
Solving these equations we get: $\angle AOB + (180° - 2\angle ACO) + (180° - 2\angle OCB) = 360°$.
Hence, $\angle AOB = 2(\angle ACO + \angle OCB) = 2\angle ACB$.



Solution C.22. [Of Exercise 2.7.] From each of A, B, C and D, draw the tangent to the circle and let these tangents meet at E, F, G and H. Then, each of $OBEC$, $OAHB$, $OCFD$, $ODGA$ and $EFGH$ is a square. Moreover, the area of the square $EFGH$ is twice the area of the square $ABCD$. So, area of square $ABCD = \frac{1}{2}$ area of square $EFGH > \frac{1}{2}$ area of circle $ABCD$ and we are done.

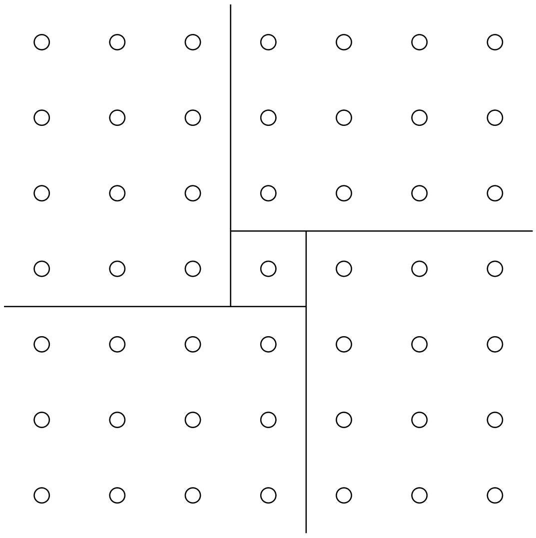


Solution C.23. [Of Exercise 2.8.] Let us draw the tangent to the circle at point I and the tangent join BC at L and AD at K. We see that the area of the parallelogram $ABLK$ is twice that of the triangle ABI. Hence, the area of the triangle $ABI = \frac{1}{2}$ the area of parallelogram $ABLK > \frac{1}{2}$ the part of the circle enclosing triangle ABI.

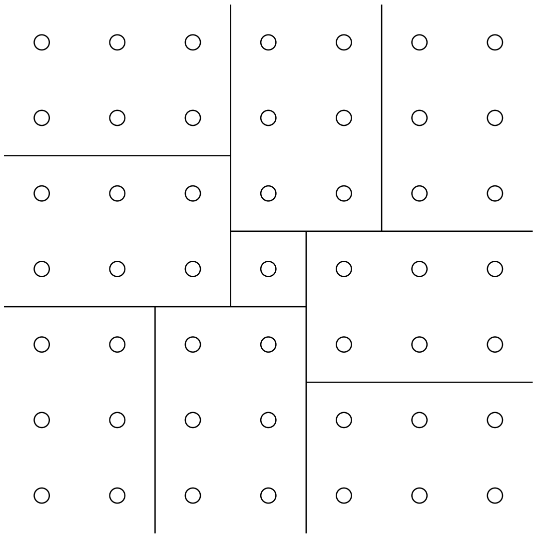


Solution C.24. [Of Exercise 2.9.] For each of the 16 triangles, if the sides are of length a, b and c where the side of length c is opposite the right angle, then $c = \sqrt{a^2 + b^2}$. For the smallest triangle, both sides adjacent to the right angle are of length 1 and hence, the remaining side is of length $\sqrt{2}$. Similarly, for the second smallest triangle, since it has two sides of lengths 1 and $\sqrt{2}$ resp., the length of the third side is $\sqrt{3}$. And so on we establish the length of all the sides opposite the right angle.

Solution C.25. [Of Exercise 2.10.]

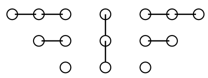


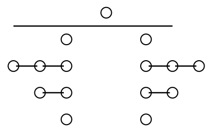
Solution C.26. [Of Exercise 2.11.]



Solution C.27. [Of Exercise 2.12.]

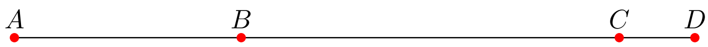
- **Proposition 23:** On the left hand side diagram, we have odd additions of odd numbers. On the right hand side, we have 1 + addition of even numbers which is odd since by Euclid IX Proposition 21, any addition of even numbers returns even.



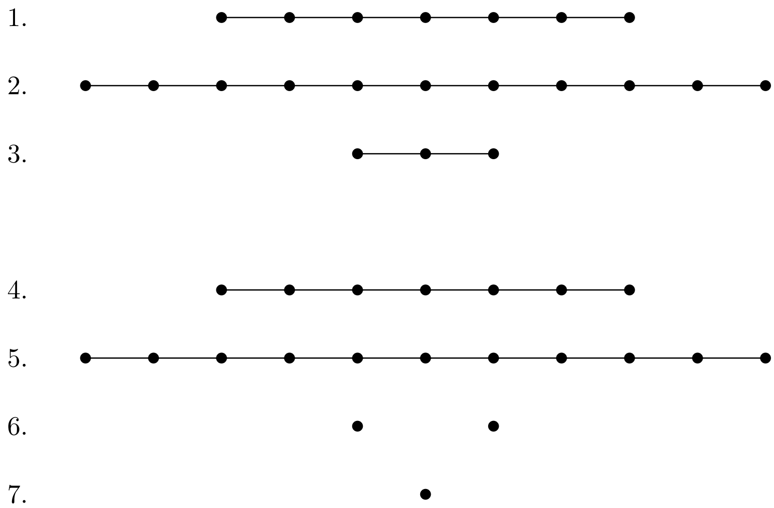


You could also show Proposition 23 using other numbers and giving more details as follows:

Assume the numbers are AB, BC, CD where the numbers are 7, 11 and 3.



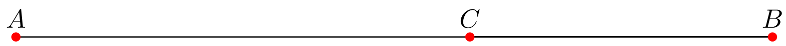
Following Knorr, the first 3 lines of the diagram below represent these 3 numbers AB (line 1), BC (line 2) and CD (line 3). The 4 following lines represent resp. the numbers AB (line 4), BC (line 5), CD minus unit (line 6), and then the unit (line 7). By Proposition 22, the addition of lines 4 and 5 is even and the number at line 6 is even and hence by Proposition 21, the total of lines 4, 5 and 6 is even. But the number at line 7 is unit, and hence the total at lines 4, 5, 6 and 7 is odd (since it is an even number + 1).



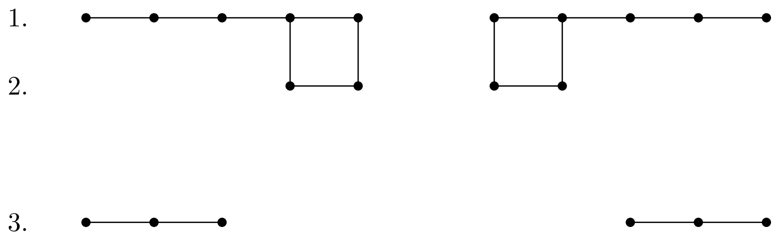
- **Proposition 24:** We start on the left hand side diagram and end on the right hand side.

Again here, we will use other numbers to demonstrate the proposition again and giving more details:

Assume the numbers are AB and BC where the numbers are 10 and 4.



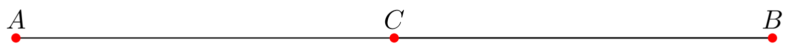
Following Knorr, the first 2 lines of the diagram below represent these 2 numbers AB (line 1) and BC (line 2). The vertical lines show subtraction, after which we are left with the number at line 3 which has a half part and hence is even by definition.



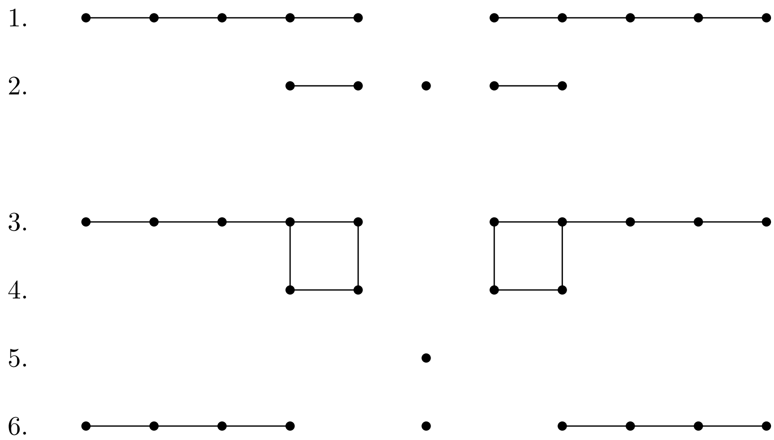
- **Proposition 25:** We start on the left hand side diagram and end on the right hand side.

Another example here is as follows:

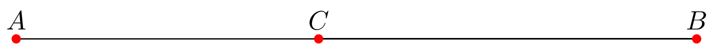
Assume the numbers are AB and BC where the numbers are 10 and 5.



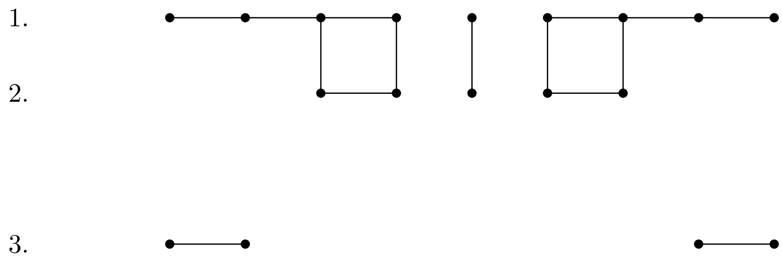
Following Knorr, the first 2 lines of the diagram below represent these 2 numbers AB (line 1) and BC (line 2). Line 4 represents the number BC from which a unit is subtracted. The unit is now in line 5. The vertical lines show subtraction, after which we are left with the number at line 6 which does not have a half part and hence is odd by definition.



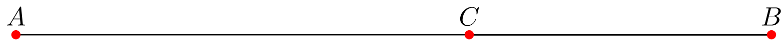
- **Proposition 26**: Assume the numbers are AB and BC where the numbers are 9 and 5.



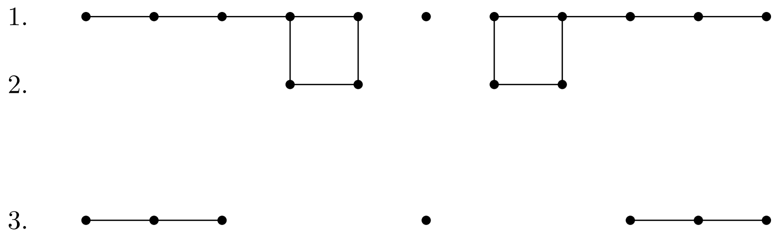
Following Knorr, the first 2 lines of the diagram below represent these 2 numbers AB (line 1) and BC (line 2). The vertical lines show subtraction, after which we are left with the number at line 3 which has a half part and hence is even by definition.



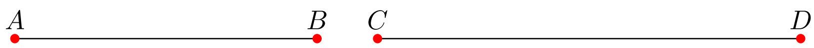
- **Proposition 27**: Assume the numbers are AB and BC where the numbers are 11 and 4.



Following Knorr, the first 2 lines of the diagram below represent these 2 numbers AB (line 1) and BC (line 2). The vertical lines show subtraction, after which we are left with the number at line 3 which has a 1 unit more than an even number and hence is odd by definition. If an odd number is multiplied by an odd number, then the product is odd.



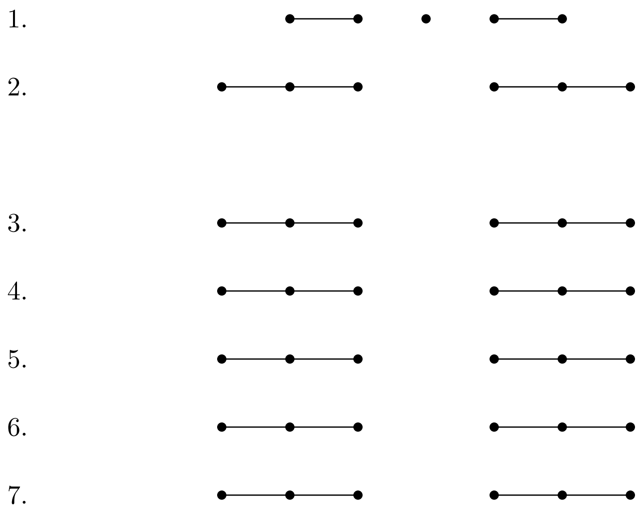
- **Proposition 28**: Assume the numbers are AB and BC where the numbers are 5 and 6.



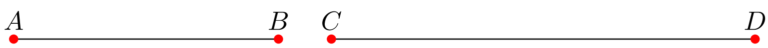
Let us recall here the definition of multiplication from Euclid (Definition 15, book VII).

> A number is said to **multiply** a number when that number which it multiplies is added to itself as many times as there are units in the other, and thus some number is produced.

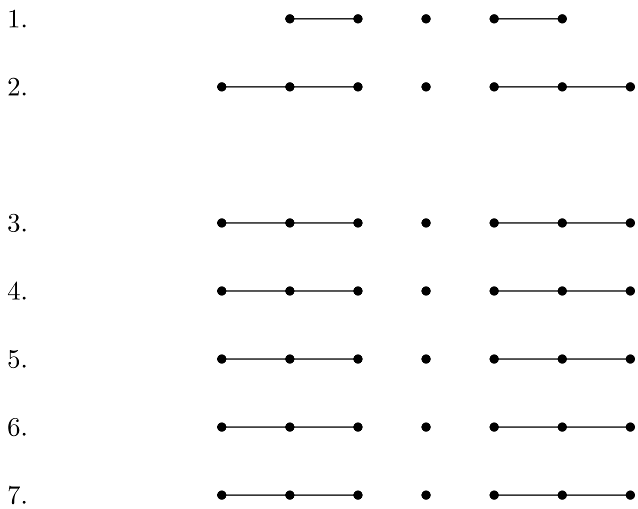
Following Knorr, the first 2 lines of the diagram below represent these 2 numbers AB (line 1) and CD (line 2). The remaining 5 lines represent the addition of the second number to itself as many times as there are units in the first number. Since we are adding even numbers, by Proposition IX, 21, the result is even.



- **Proposition 29**: Assume the numbers are AB and BC where the numbers are 5 and 7.



Following Knorr, the first 2 lines of the diagram below represent these 2 numbers AB (line 1) and CD (line 2). The remaining 5 lines represent the addition of the second number to itself as many times as there are units in the first number. Since we are adding odd numbers, an odd number of times, by Proposition IX, 23, the result is odd.



Solution C.28. [Of Exercise 2.14.]

1. $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, $(7, 24, 25)$, $(20, 21, 29)$.

2. In a Pythagorean triple (a, b, c), we know that $c > a$ and $c > b$. Hence, $c \neq 1$. Also, $c \neq 2$ since otherwise by Theorem 2.5.5, both a and b need to be even which is impossible since they can only be 1. Furthermore, $c \neq 3$ since by Theorem 2.5.8, one of a, b needs to be even (2) and the other needs to be odd (1) and we can check that $1^2 + 2^2 \neq 3^2$. Moreover, $c \neq 4$ because by Corollary 2.5.7, both a and b need to be multiples of 4 and hence both need to be 4 which is absurd. Now we check if $c = 5$. By Theorem 2.5.8, one of a, b needs to be odd and the other even. So the only choices are: (1 and 4) or (2 and 3) or (3 and 4). A quick check would demonstrate that $(1, 4, 5)$ and $(2, 3, 5)$ are not a Pythagorean triples but $(3, 4, 5)$ is. Furthermore, this is a primitive Pythagorean triple and 5 is the smallest integer c where (a, b, c) a Pythagorean triple.

3. Since $(3, 4, 5)$ is a Pythagorean triple, then for any number $k > 1$, $(3k, 4k, 5k)$ is a Pythagorean triple. Pick 100 triples from these.

4. If (a, b, c) is a primitive Pythagorean triple and both a and b are even then by Theorem 2.5.9, c is even. This means that all of a, b, and c are divisible by 2 which contradicts the primitivity of the Pythagorean triple. Hence, a and b cannot both be even.

5. If (a, b, c) is a primitive Pythagorean triple and both a and b are odd then by Theorem 2.5.10, c is even. This contradicts Theorem 2.5.5, which states that when c is even then both a and b must be even. Hence a and b cannot both be odd.

6. Assume (a, b, c) is a primitive Pythagorean triple and a is odd. By Theorems 2.5.10 and 2.5.5, c is odd and b is even. If $c + b$ and $c - b$ have a common factor $d > 1$ then for some k, k', $c + b = kd$ and $c - b = k'd$. Hence $2c = d(k + k')$ and $2b = d(k - k')$. So, d is a common factor for $2c$ and $2d$. Since b and c do not share any common factors, $d = 2$. Hence $(c+b)(c-b) = c^2 - b^2 = a^2 = d^2kk' = 4kk'$. So, a^2 is even which contradicts Theorem 2.4.2 which states that since a is odd, a^2 must be one more than a multiple of 4 which is odd. Hence, $c + b$ and $c - b$ have no common factor $d > 1$.

7. Assume (a, b, c) is a primitive Pythagorean triple. If c is even then by Theorem 2.5.5, all of a, b and c are even and have 2 as a common factor contradicting the primitivity of (a, b, c). Hence c is odd.

Solution C.29. [Of Exercise 2.15.] $(m^2-n^2)^2+4m^2n^2 = m^4+n^4+2m^2n^2 = (m^2 + n^2)^2$. Hence, $(m^2 - n^2, 2mn, m^2 + n^2)$ is a Pythagorean triple.

C.3 Solutions for Chapter 3

Solution C.30. [Of Exercise 3.1.]

1. (a) R_1 is reflexive on $\mathbb{N}^+$ because for any m in $\mathbb{N}^+$, by reflexivity of $=$, $m \cdot m = m \cdot m$ and so, mR_1m.

   (b) R_1 is symmetric on $\mathbb{N}^+$ because for any m, n in $\mathbb{N}^+$, if mR_1n then $m \cdot m = n \cdot n$ and by symmetricity of $=$, $n \cdot n = m \cdot m$ and hence nR_1m.

   (c) R_1 is transitive on $\mathbb{N}^+$ because for any m, n, p in $\mathbb{N}^+$, if mR_1n and nR_1p then $m \cdot m = n \cdot n$ and $n \cdot n = p \cdot p$ and by transitivity of $=$, $m \cdot m = p \cdot p$ and hence mR_1p.

   (d) R_1 is an equivalence relation because it is reflexive, symmetric, and transitive.

2. (a) R_2 is reflexive on $\mathbb{N}^+$ because for any m in $\mathbb{N}^+$, by reflexivity of $=$, $m + m = m + m$ and so, mR_2m.

   (b) R_2 is symmetric on $\mathbb{N}^+$ because for any m, n in $\mathbb{N}^+$, if mR_2n then $m + m = n + n$ and by symmetricity of $=$, $n + n = m + m$ and hence nR_2m.

   (c) R_2 is transitive on $\mathbb{N}^+$ because for any m, n, p in $\mathbb{N}^+$, if mR_2n and nR_2p then $m + m = n + n$ and $n + n = p + p$ and by transitivity of $=$, $m + m = p + p$ and hence mR_2p.

   (d) R_2 is an equivalence relation because it is reflexive, symmetric, and transitive.

3. (a) R_3 is not reflexive on $\mathbb{N}^+$. For example, choose $m = 1$ then there is no p in $\mathbb{N}^+$ such that $m \cdot p = m + p$. In fact, for any p in $\mathbb{N}^+$, $m \cdot p = 1 \cdot p = p$ and $m + p = 1 + p$. There is no p in $\mathbb{N}^+$ such that $p = 1 + p$. Hence $1 \not\!R_3 1$.

   (b) R_3 is not symmetric on $\mathbb{N}^+$. For example, choose $m = 5$ and $n = 4$. Then, there is $p = 1$ such that $m \cdot p = n + p$. That is: $5 \cdot 1 = 4 + 1$.

   But, there is no p such that $n \cdot p = 4 \cdot p = 5 + p$. There is no p in $\mathbb{N}^+$ such that $3p = 5$. Hence, $5R_34$ but $4 \not\!R_3 5$.

   (c) R_3 is not transitive on $\mathbb{N}^+$. For example, choose $m = 5$, $n = 4$ and $k = 3$. Then, $5R_34$ (take $p = 1$) and $4R_33$ (again take $p = 1$). But there is no p in $\mathbb{N}^+$ such that $4p = 3$ and so there is no p in $\mathbb{N}^+$ such that $5 \cdot p = 3 + p$. Hence $5 \not\!R_3 3$.

(d) R_3 is not an equivalence relation on $\mathbb{N}^+$ because it is not all of reflexive, symmetric, and transitive. In fact, it is none of them.

4. (a) R_4 is not reflexive on $\mathbb{N}^+$. For example, choose $m = 1$ then there is no p in $\mathbb{N}^+$ such that $m \cdot m = m + p$. In fact, for any p in $\mathbb{N}^+$, $m \cdot m = 1 \cdot 1 = 1$ and $1 + p \neq p$. Hence $1 \not R_4 1$.

   (b) R_4 is not symmetric on $\mathbb{N}^+$. For example, choose $m = 5$ and $n = 1$. Then, there is $p = 4$ such that $m \cdot n = n + p$. That is: $5 \cdot 1 = 1 + 4$.
   But, there is no p such that $n \cdot m = 1 \cdot 5 = 5 + p$. There is no p in $\mathbb{N}^+$ such that $5 = 5 + p$. Hence, $5 R_4 1$ but $1 \not R_4 5$.

   (c) Transitivity of R_4 is left as an exercise.

   (d) equivalence of R_4 is left as an exercise.

Solution C.31. [Of Exercise 3.2.] Let $\mathbf{a} = \left[\frac{m}{n}\right]$, $\mathbf{b} = \left[\frac{p}{q}\right]$ and $\mathbf{c} = \left[\frac{r}{s}\right]$ be rational numbers.

$$\begin{aligned}
\mathbf{a} \cdot_r (\mathbf{b} +_r \mathbf{c}) &= \left[\frac{m}{n}\right] \cdot_r \left(\left[\frac{p}{q}\right] +_r \left[\frac{r}{s}\right]\right) \\
&= \left[\frac{m}{n}\right] \cdot_r \left[\frac{ps + qr}{qs}\right] \\
&= \left[\frac{m(ps + qr)}{n(qs)}\right] \\
&= \left[\frac{n(mps + mqr)}{n(nqs)}\right] \\
&= \left[\frac{(mp)(ns) + (nq)(mr)}{(nq)(ns)}\right] \\
&= \left[\frac{mp}{nq}\right] +_r \left[\frac{mr}{ns}\right] \\
&= \left[\frac{m}{n}\right] \cdot_r \left[\frac{p}{q}\right] +_r \left[\frac{m}{n}\right] \cdot_r \left[\frac{r}{s}\right] \\
&= \mathbf{a} \cdot_r \mathbf{b} +_r \mathbf{a} \cdot_r \mathbf{c}.
\end{aligned}$$

Solution C.32. [Of Exercise 3.3.]

1.
$$m_r +_r n_r = \left[\frac{m}{1}\right] +_r \left[\frac{n}{1}\right] = \left[\frac{m + n}{1}\right] = (m + n)_r.$$

2.
$$m_r \cdot_r n_r = \left[\frac{m}{1}\right] \cdot_r \left[\frac{n}{1}\right] = \left[\frac{mn}{1}\right] = (mn)_r.$$

Solution C.33. [Of Exercise 3.4.] Let $\mathbf{a} = \left[\frac{m}{n}\right]$ and $\mathbf{b} = \left[\frac{p}{q}\right]$. By Theorem 3.2.21, we know that $\mathbf{a}^{-1} = \left[\frac{n}{m}\right]$ and $\mathbf{b}^{-1} = \left[\frac{q}{p}\right]$.

1. By Theorem 3.2.21, we know that $(\mathbf{a}^{-1})^{-1} = \left[\frac{m}{n}\right]$. Hence, $(\mathbf{a}^{-1})^{-1} = \mathbf{a}$.
2. $\mathbf{ab} = \left[\frac{m}{n}\right] \cdot_r \left[\frac{p}{q}\right] = \left[\frac{mp}{nq}\right]$ and By Theorem 3.2.21, $(\mathbf{ab})^{-1} = \left[\frac{nq}{mp}\right]$. But $\mathbf{a}^{-1}\mathbf{b}^{-1} = \left[\frac{n}{m}\right]\left[\frac{q}{p}\right] = \left[\frac{nq}{mp}\right]$. Hence, $(\mathbf{ab})^{-1} = \mathbf{a}^{-1}\mathbf{b}^{-1}$.
3. By definition, $\frac{\mathbf{a}}{\mathbf{b}} = \mathbf{a} \cdot_r \mathbf{b}^{-1}$. Hence, by 2., resp. 1., $\left(\frac{\mathbf{a}}{\mathbf{b}}\right)^{-1} = \mathbf{a}^{-1} \cdot_r (\mathbf{b}^{-1})^{-1} = \mathbf{a}^{-1} \cdot_r \mathbf{b} = \mathbf{b} \cdot_r \mathbf{a}^{-1} = \frac{\mathbf{b}}{\mathbf{a}}$
4. By 3 above, $\left(\frac{\mathbf{a}}{\mathbf{b}}\right)^{-1} \cdot_r \mathbf{a} = \frac{\mathbf{b}}{\mathbf{a}} \cdot_r \mathbf{a} =^{\text{Associativity of } \cdot_r} (\mathbf{b} \cdot_r \mathbf{a}^{-1}) \cdot_r \mathbf{a} = \mathbf{b} \cdot_r (\mathbf{a}^{-1} \cdot_r \mathbf{a}) =^{\text{Inverse for } \cdot_r} \mathbf{b} \cdot_r 1_r =^{\text{Identity for } \cdot_r} \mathbf{b}$.
5. Left to the reader to prove that $(\mathbf{ab})^{-1} \cdot_r \mathbf{a} = \mathbf{b}^{-1}$.

Solution C.34. [Of Exercise 3.5.] Let three rationals $\mathbf{a} = \left[\frac{m}{n}\right]$, $\mathbf{b} = \left[\frac{p}{q}\right]$, and $\mathbf{c} = \left[\frac{r}{s}\right]$.

1. If $\mathbf{a} +_r \mathbf{b} = \mathbf{a} +_r \mathbf{c}$ then
   $\left[\frac{m}{n}\right] +_r \left[\frac{p}{q}\right] = \left[\frac{m}{n}\right] +_r \left[\frac{r}{s}\right]$, hence
   $\left[\frac{mq+np}{nq}\right] = \left[\frac{ms+nr}{ns}\right]$ and so
   $\frac{mq+np}{nq} \asymp \frac{ms+nr}{ns}$ and
   $(mq+np)ns = mqns + npns = nq(ms+nr) = nqms + nqnr$. So, $mqns + npns = nqms + nqnr$ and by commutativity, associativity and cancellation we get $ps = qr$ and finally, $\mathbf{b} = \left[\frac{p}{q}\right] = \left[\frac{r}{s}\right] = \mathbf{c}$.
2. If $\mathbf{a} \cdot_r \mathbf{b} = \mathbf{a} \cdot_r \mathbf{c}$ then
   $\left[\frac{m}{n}\right] \cdot_r \left[\frac{p}{q}\right] = \left[\frac{m}{n}\right] \cdot_r \left[\frac{r}{s}\right]$, hence $\left[\frac{mp}{nq}\right] = \left[\frac{mr}{ns}\right]$ and so
   $\frac{mp}{nq} \asymp \frac{mr}{ns}$ and $mpns \asymp nqmr$, so $ps \asymp qr$ and $\frac{p}{q} \asymp \frac{r}{s}$. Finally, $\mathbf{b} = \left[\frac{p}{q}\right] = \left[\frac{r}{s}\right] = \mathbf{c}$.

Solution C.35. [Of Exercise 3.6.] If we start with nonzero even naturals, then the identity for multiplication property (property 10 of Figure 3.1) would fail since there is no identity e on nonzero even naturals such that $e \cdot m = m \cdot e = e$. However, all other properties 1..9 hold, simply replace *nonzero natural* by *nonzero even natural*. Then, all definitions except Definition 3.2.18 remain exactly the same except that *nonzero natural* is replaced by *nonzero even natural*. All lemmas and theorems (except Theorem 3.2.20) hold exactly as they are, just make sure that whenever you see *nonzero natural*, simply replace it by *nonzero even natural*.

As for Definition 3.2.18, it should be replaced by

Definition C.3.1. *The* nonzero natural rational n_r*, which corresponds to the nonzero even natural* n *is defined by*

$$n_r = \left[\frac{2n}{2}\right].$$

And, Theorem 3.2.20 remains the same except for the definition of 1_r which should be $1_r = \left[\frac{2}{2}\right]$.

This exercise illustrates the fact that if we start with a natural number system without a multiplicative identity, the rational number system we get adds one.

Solution C.36. [Of Exercise 3.8.] Assume $m \ominus n \in \alpha$, $p \ominus q \in \beta$ and $r \ominus s \in \gamma$. Then

$$\begin{aligned}
\alpha \cdot_i (\beta +_i \gamma) &= [m \ominus n] \cdot_i [(p \ominus q) +_c (r \ominus s)] \\
&= [m \ominus n] \cdot_i [(p + r) \ominus (q + s)] \\
&= [(m(p + r) + n(q + s)) \ominus (m(q + s) + n(p + r))] \\
&= [(mp + nq + mr + ns) \ominus (mq + np + nr + ms)] \\
&= [((mp + nq) + (mr + ns)) \ominus ((mq + np) + (nr + ms))] \\
&= [(mp + nq) \ominus (mq + np)] +_i [(mr + ns) \ominus (nr + ms)] \\
&= [(m \ominus n) \cdot_i (p \ominus q)] +_i [(m \ominus n) \cdot_i (r \ominus s)] \\
&= \alpha \cdot_i \beta +_i \alpha \cdot_i \gamma.
\end{aligned}$$

Solution C.37. [Of Exercise 3.9.] Assume $m \ominus n \in \alpha$, $p \ominus q \in \beta$ and $r \ominus s \in \gamma$.

- Cancellation for addition:
  $\alpha +_i \beta = \alpha +_i \gamma \Longrightarrow$
  $[m \ominus n] +_i [p \ominus q] = [m \ominus n] +_i [r \ominus s] \Longrightarrow$
  $[(m \ominus n) +_c (p \ominus q)] = [(m \ominus n) +_c (r \ominus s)] \Longrightarrow$
  $[(m + p) \ominus (n + q)] = [(m + r) \ominus (n + s)] \Longrightarrow$
  $((m + p) \ominus (n + q)) \cong ((m + r) \ominus (n + s)) \Longrightarrow$
  $m + p + n + s = n + q + m + r \Longrightarrow$

$p + s = q + r \Longrightarrow$
$p \ominus q \cong r \ominus s \Longrightarrow$
$[p \ominus q] = [r \ominus s] \Longrightarrow \beta = \gamma.$

- Cancellation for multiplication: Assume $\alpha \neq 0$.
  $\alpha \cdot_i \beta = \alpha \cdot_i \gamma \Longrightarrow$
  $[m \ominus n] \cdot_i [p \ominus q] = [m \ominus n] \cdot_i [r \ominus s] \Longrightarrow$
  $[(m \ominus n) \cdot_c (p \ominus q)] = [(m \ominus n) \cdot_c (r \ominus s)] \Longrightarrow$
  $[(mp + nq) \ominus (mq + np)] = [(mr + ns) \ominus (ms + nr)] \Longrightarrow$
  $((mp + nq) \ominus (mq + np)) \cong ((mr + ns) \ominus (ms + nr)) \Longrightarrow$
  $mp + nq + ms + nr = mq + np + mr + ns \Longrightarrow$
  $m(p + s) + n(q + r) = m(q + r) + n(p + s) \Longrightarrow$
  - If $m > n$ then $m = n + t$ and $n(p + s) + t(p + s) + n(q + r) = n(q + r) + t(q + r) + n(p + s) \Longrightarrow$
    $t(p + s) = t(q + r) \Longrightarrow$
    $p + s = q + r \Longrightarrow$
    $p \ominus q \cong r \ominus s \Longrightarrow$
    $[p \ominus q] = [r \ominus s] \Longrightarrow \beta = \gamma.$
  - Case $m < n$ is similar to above case.

Solution C.38. [Of Exercise 3.10.]]

1. $m_i +_i n_i = [(p+m) \ominus p] +_i [(p+n) \ominus p] = [((p+m) \ominus p) +_c ((p+n) \ominus p)] = [(p + m + p + n) \ominus (p + p)] = [(p + p + m + n) \ominus (p + p)] = (m + n)_i.$

2. $m_i \cdot_i n_i = [(p+m) \ominus p] \cdot_i [(p+n) \ominus p] = [((p+m) \ominus p) \cdot_c ((p+n) \ominus p)] = [((p + m)(p + n) + pp) \ominus ((p + m)p + p(p + n))] = [p(m + n) + 2pp + mn) \ominus (p(m + n) + 2pp)] = (mn)_i.$

Solution C.39. [Of Exercise 3.11.] Assume $\alpha = [m \ominus n]$.
$1_i \cdot_i \alpha = [(p + 1) \ominus p] \cdot_i [m \ominus n] = [((p + 1) \ominus p) \cdot_c (m \ominus n)] = [(m(p + 1) + pn) \ominus (pm + n(p + 1))] = [(pm + pn + m) \ominus [(pm + pn + n)] = [m \ominus n].$
Similarly we can show that $\alpha \cdot_i 1_i = \alpha$.

Solution C.40. [Of Exercise 3.12.] We do the case for $(\mathbb{N}^+, +)$ and leave the other case to the reader.
By definition of $\mathbb{N}^+$ as given at the start of Chapter 3, the laws of closure, commutative, associative and cancellations all hold for $+$ on $\mathbb{N}^+$. Hence, $(\mathbb{N}^+, +)$ is a commutative cancellation semigroup.

Solution C.41. [Of Exercise 3.13.] Since $y \circ z = w \circ y$ then by commutativity $z \circ y = w \circ y$ and by cancellation $z = w$. Hence, since $x \circ z = y \circ w$ and $z = w$ then by cancellation $x = y$.

x and z (hence y and w) do not need to be equal. For example, in $(\mathbb{N}^+, \cdot)$ we can take $x = y = 10$ and $z = w = 5$.

Solution C.42. [Of Exercise 3.14.] By Definition 3.4.1 and Theorems 3.4.18, 3.4.19 and 3.4.20, we only need to show the cancellation law for $\circ_d$ on S_d. Let $\mathfrak{a} = [(x, y)]$, $\mathfrak{b} = [(x', y')]$, and $\mathfrak{c} = [(u, v)]$ be elements in S_d such that $\mathfrak{a} \circ_d \mathfrak{c} = \mathfrak{b} \circ_d \mathfrak{c}$. Then
$[(x, y) * (u, v)] = [(x', y') * (u, v)]$ and hence $[(x \circ u, y \circ v)] = [(x' \circ u, y' \circ v)]$. So, $x \circ u \circ y' \circ v = y \circ v \circ x' \circ u$ and hence $x \circ y' = y \circ x'$ and so, $[(x, y)] = [(x', y')]$. Therefore, $\mathfrak{a} = \mathfrak{b}$.

This means, $(S_d, \circ_d)$ is a commutative cancellation semigroup.

Solution C.43. [Of Exercise 3.15.] $(\mathbb{Q}^+, \cdot_r)$ is a commutative cancellation semigroup by Theorems 3.2.14, 3.2.1.5. and 3.2.16 and Exercise 3.5.
$(\mathbb{Z}, +_i)$ is a commutative cancellation semigroup by Theorems 3.3.13, 3.3.14 and 3.3.15 and Exercise 3.9.
$(\mathbb{Q}^+, +_r)$ and $(\mathbb{Z}, \cdot_i)$ are also commutative cancellation semigroups for the same reasons as that $(\mathbb{Q}^+, \cdot_r)$ and $(\mathbb{Z}, +_i)$.

Solution C.44. [Of Exercise 3.16.] We build $(\mathbb{Q}^+, \cdot_r)$ from $(\mathbb{N}^+, \cdot)$. We know by Lemma 3.4.2 that $(\mathbb{N}^+, \cdot)$ is a commutative cancellation semigroup.

1. We write (x, y) as $\frac{x}{y}$.

2. We build a congruence $\asymp$ on $\mathbb{N}^+ \times \mathbb{N}^+$ based on $(\mathbb{N}^+, \cdot)$ as follows:

   $(x, y) \asymp (u, v)$ iff $x \cdot v = y \cdot u$. In our notation, $\frac{x}{y} \asymp \frac{u}{v}$ iff $x \cdot v = y \cdot u$.

   By Theorem 3.4.9 $\asymp$ is an equivalence relation.

3. The operation $\cdot_f$ on $\mathbb{N}^+ \times \mathbb{N}^+$ inherited from $\cdot$ is defined by
   $(x, y) \cdot_f (u, v) = (x \cdot u, y \cdot v)$. In our notation, $\frac{x}{y} \cdot_f \frac{u}{v} = \frac{x \cdot u}{y \cdot v}$.

4. The **value** of $\frac{x}{y}$ is $[\frac{x}{y}]_\asymp = \{\frac{u}{v} : \frac{u}{v} \asymp \frac{x}{y}\}$. We define $\mathbb{Q}^+ = \{[\frac{x}{y}]_\asymp : x, y \in \mathbb{N}^+\}$.

5. The operation $\cdot_r$ corresponding to $\cdot$ is defined as follows: If $\mathfrak{a} = [\frac{x}{y}]_\asymp$ and $\mathfrak{b} = [\frac{u}{v}]_\asymp$, define $\mathfrak{a} \cdot_r \mathfrak{b} = [\frac{x}{y} \cdot_f \frac{u}{v}]_\asymp = [\frac{x \cdot u}{y \cdot v}]_\asymp$.

6. **Closure Law**. For all $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Q}^+$, $\mathfrak{a} \cdot_r \mathfrak{b}$ is an element of $\mathbb{Q}^+$ uniquely determined by $\mathfrak{a}$ and $\mathfrak{b}$. See Theorem 3.4.18.

7. **Commutative Law**. For all $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Q}^+$, we have $\mathfrak{a} \cdot_r \mathfrak{b} = \mathfrak{b} \cdot_r \mathfrak{a}$. See Theorem 3.4.19.

8. **Associative Law**. For all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathbb{Q}^+$, we have $(\mathfrak{a} \cdot_r \mathfrak{b}) \cdot_r \mathfrak{c} = \mathfrak{a} \cdot_r (\mathfrak{b} \cdot_r \mathfrak{c})$. See Theorem 3.4.20.

9. **Cancellation law for $\cdot_r$ on** $\mathbb{Q}^+$. For all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathbb{Q}^+$, if $\mathfrak{a}\cdot_r\mathfrak{c} = \mathfrak{b}\cdot_r\mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$. See Exercise 3.14.

10. $(\mathbb{Q}^+, \cdot_r)$ **is a commutative cancellation semigroup**. See Exercise 3.14.

11. $\mathbb{N}^+$ **is a subset of** $\mathbb{Q}^+$. For each x in $\mathbb{N}^+$, $x_r = [\frac{y \cdot x}{y}]_\asymp$ is in $\mathbb{Q}^+$. See Definition 3.4.22 and Lemma 3.4.21.

12. **Identity for** $\mathbb{Q}^+$. Define 1_r to be $[\frac{x}{x}]_\asymp$ for some x in $\mathbb{N}^+$. For all $\mathfrak{a}$ in $\mathbb{Q}^+$, we have $1_r \cdot_r \mathfrak{a} = \mathfrak{a} \cdot_r 1_r = \mathfrak{a}$. See Theorem 3.4.26.

13. **Inverses for Dyads**. If $\mathfrak{a} = [\frac{x}{y}]_\asymp$, define $\mathfrak{a}^{-1}$ to be $[\frac{y}{x}]_\asymp$. We have $\mathfrak{a} \cdot_r \mathfrak{a}^{-1} = 1_r = \mathfrak{a}^{-1} \cdot_r \mathfrak{a}$. See Theorem 3.4.29.

We build $(\mathbb{Z}, +_i)$ from $(\mathbb{N}^+, +)$. We know by Lemma 3.4.2 that $(\mathbb{N}^+, +)$ is a commutative cancellation semigroup.

1. We write (x, y) as $x \ominus y$.

2. We build a congruence $\cong$ on $\mathbb{N}^+ \times \mathbb{N}^+$ based on $(\mathbb{N}^+, +)$ as follows:

   $(x, y) \cong (u, v)$ iff $x + v = y + u$. In our notation, $x \ominus y \cong u \ominus v$ iff $x + v = y + u$.

   By Theorem 3.4.9 $\cong$ is an equivalence relation.

3. The operation $+_c$ on $\mathbb{N}^+ \times \mathbb{N}^+$ inherited from $+$ is defined by $(x, y) +_c (u, v) = (x + u, y + v)$. In our notation, $x \ominus y +_c u \ominus v = x + u \ominus y + v$.

4. The **value** of $x \ominus y$ is $[x \ominus y]_\cong = \{u \ominus v : u \ominus v \cong x \ominus y\}$. We define $\mathbb{Z} = \{[x \ominus y]_\cong : x, y \in \mathbb{N}^+\}$.

5. The operation $+_i$ corresponding to $+$ is defined as follows: If $\mathfrak{a} = [x \ominus y]_\cong$ and $\mathfrak{b} = [u \ominus v]_\cong$, define $\mathfrak{a} +_i \mathfrak{b} = [x \ominus y +_c u \ominus v]_\cong = [x + u \ominus y + v]_\cong$.

6. **Closure Law**. For all $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Z}$, $\mathfrak{a} +_i \mathfrak{b}$ is an element of $\mathbb{Z}$ uniquely determined by $\mathfrak{a}$ and $\mathfrak{b}$. See Theorem 3.4.18.

7. **Commutative Law**. For all $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Z}$, we have $\mathfrak{a} +_i \mathfrak{b} = \mathfrak{b} +_i \mathfrak{a}$. See Theorem 3.4.19.

8. **Associative Law**. For all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathbb{Z}$, we have $(\mathfrak{a} +_i \mathfrak{b}) +_i \mathfrak{c} = \mathfrak{a} +_i (\mathfrak{b} +_i \mathfrak{c})$. See Theorem 3.4.20.

9. **Cancellation law for $\cdot_r$ on $\mathbb{Z}$**. For all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathbb{Z}$, if $\mathfrak{a}+_i\mathfrak{c} = \mathfrak{b}+_i\mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$. See Exercise 3.14.

10. $(\mathbb{Z}, +_i)$ **is a commutative cancellation semigroup**. See Exercise 3.14.

11. $\mathbb{N}^+$ **is a subset of** $\mathbb{Z}$. For each x in $\mathbb{N}^+$, $x_i = [y + x \ominus y]_{\cong}$ is in $\mathbb{Z}$. See Definition 3.4.22 and Lemma 3.4.21.

12. **Identity for** $\mathbb{Z}$. Define 0_i to be $[x \ominus x]_{\cong}$ for some x in $\mathbb{N}^+$. For all $\mathfrak{a}$ in $\mathbb{Z}$, we have $0_i +_i \mathfrak{a} = \mathfrak{a} +_i 0_i = \mathfrak{a}$. See Theorem 3.4.26.

13. **Inverses for Dyads**. If $\mathfrak{a} = [x \ominus y]_{\cong}$, define $-\mathfrak{a}$ to be $[y \ominus x]_{\cong}$. We have $\mathfrak{a} +_i -\mathfrak{a} = 0_i = -\mathfrak{a} +_i \mathfrak{a}$. See Theorem 3.4.29.

We build $(\mathbb{Q}, +_{r'})$ from $(\mathbb{Q}^+, +_r)$. We have already built $(\mathbb{Q}^+, +_r)$ in Section 3.2 and there, we have also shown the closure, commutative, associative and cancellation laws for $+_r$. Hence, we know that $(\mathbb{Q}^+, +_r)$ is a commutative cancellation semigroup.

1. For $x, y \in \mathbb{Q}^+$, we write (x, y) as $x \ominus y$.

2. We build a congruence $\sim$ on $\mathbb{Q}^+ \times \mathbb{Q}^+$ based on $(\mathbb{Q}^+, +_r)$ as follows:

   $(x, y) \sim (u, v)$ iff $x +_r v = y +_r u$. In our notation, $x \ominus y \sim u \ominus v$ iff $x +_r v = y +_r u$.

   By Theorem 3.4.9 $\sim$ is an equivalence relation.

3. The operation $+_{c'}$ on $\mathbb{Q}^+ \times \mathbb{Q}^+$ inherited from $+_r$ is defined by $(x, y) +_{c'} (u, v) = (x +_r u, y +_r v)$. In our notation, $x \ominus y +_{c'} u \ominus v = x +_r u \ominus y +_r v$.

4. The **value** of $x \ominus y$ is $[x \ominus y]_{\sim} = \{u \ominus v : u \ominus v \sim x \ominus y\}$. We define $\mathbb{Q} = \{[x \ominus y]_{\sim} : x, y \in \mathbb{Q}^+\}$.

5. The operation $+_{r'}$ corresponding to $+_r$ is defined as follows: If $\mathfrak{a} = [x \ominus y]_{\sim}$ and $\mathfrak{b} = [u \ominus v]_{\sim}$, define $\mathfrak{a} +_{r'} \mathfrak{b} = [x \ominus y +_{c'} u \ominus v]_{\sim} = [x +_r u \ominus y +_r v]_{\sim}$.

6. **Closure Law**. For all $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Q}$, $\mathfrak{a}+_{r'}\mathfrak{b}$ is an element of $\mathbb{Q}$ uniquely determined by $\mathfrak{a}$ and $\mathfrak{b}$. See Theorem 3.4.18.

7. **Commutative Law**. For all $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Q}$, we have $\mathfrak{a} +_{r'} \mathfrak{b} = \mathfrak{b} +_{r'} \mathfrak{a}$. See Theorem 3.4.19.

8. **Associative Law**. For all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathbb{Z}$, we have $(\mathfrak{a} +_{r'} \mathfrak{b}) +_{r'} \mathfrak{c} = \mathfrak{a} +_{r'} (\mathfrak{b} +_{r'} \mathfrak{c})$. See Theorem 3.4.20.

9. **Cancellation law for $\cdot_{r'}$ on $\mathbb{Q}$**. For all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathbb{Q}$, if $\mathfrak{a} +_{r'} \mathfrak{c} = \mathfrak{b} +_{r'} \mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$. See Exercise 3.14.

10. $(\mathbb{Q}, +_{r'})$ **is a commutative cancellation semigroup**. See Exercise 3.14.

11. $\mathbb{Q}^+$ **is a subset of** $\mathbb{Q}$. For each x in $\mathbb{Q}^+$, $x_{r'} = [y +_r x \ominus y]_\sim$ is in $\mathbb{Q}$. See Definition 3.4.22 and Lemma 3.4.21.

12. **Identity for** $\mathbb{Q}$. Define $0_{r'}$ to be $[x \ominus x]_\sim$ for some x in $\mathbb{Q}^+$. For all $\mathfrak{a}$ in $\mathbb{Q}$, we have $0_{r'} +_{r'} \mathfrak{a} = \mathfrak{a} +_{r'} 0_{r'} = \mathfrak{a}$. See Theorem 3.4.26.

13. **Inverses for Dyads**. If $\mathfrak{a} = [x \ominus y]_\sim$, define $-\mathfrak{a}$ to be $[y \ominus x]_\sim$. We have $\mathfrak{a} +_{r'} -\mathfrak{a} = 0_{r'} = -\mathfrak{a} +_{r'} \mathfrak{a}$. See Theorem 3.4.29.

C.4 Solutions for Chapter 4

Solution C.45. [Of Exercise 4.1.]

1. This is because there is no element of $\emptyset$, and so every element of $\emptyset$ is an element of every other set. When you study logic in Chapter 6, you will learn that something that is false will imply anything you want. We also know that for all objects a, $a \notin \emptyset$ and hence, $a \in \emptyset$ is false which will imply anything including $a \in S$. Hence, for all objects a, if $a \in \emptyset$ then $a \in S$ and by the definition of subset, $\emptyset \subseteq S$.

2. By above, $\emptyset \subseteq S$. If also $S \neq \emptyset$ then by definition of proper subset, $\emptyset \subset S$.

3. Let S and T be sets.

   - Assume $S = T$. Then, by the principle of Extensionality of Page 86, for every x, ($x \in S$ if and only if $x \in T$). Hence, for every x, ((if $x \in S$ then $x \in T$) and (if $x \in T$ then $x \in S$)). Thus, for every x, (if $x \in S$ then $x \in T$) and for every x, (if $x \in T$ then $x \in S$). That is: $S \subseteq T$ and $T \subseteq S$.
   - Assume $S \subseteq T$ and $T \subseteq S$. Then, (for every x, if $x \in S$, then $x \in T$) and (for every x, if $x \in T$, then $x \in S$). Hence, for every x, ((if $x \in T$, then $x \in S$) and (if $x \in S$, then $x \in ST$)). Thus, for every x, ($x \in S$ if and only if $x \in T$) and by the principle of Extensionality of Page 86, $S = T$.

4. Left to the reader.

5. Left to the reader.

Solution C.46. [Of Exercise 4.2.]
Clearly $\emptyset \in \{\emptyset\}$.

Since $\emptyset \in \{\emptyset\}$ then $\{\emptyset\} \neq \emptyset$. Hence, By Lemma 4.1.2, $\emptyset \subset \{\emptyset\}$.

The proof of the remaining item is similar to the above.

Solution C.47. [Of Exercise 4.3.]
Since $S \subseteq \mathbb{N}$, $0 \in S$, and whenever $n \in S$ we also have $n + 1 = \{n\} \in S$, then by the induction axiom for $\mathbb{N}$, $S = \mathbb{N}$.

Solution C.48. [Of Exercise 4.4.]

1. One direction is clear by Lemma 4.1.4.1. We do the other direction, using the induction axiom for $\mathbb{N}$. Assume $T = \{n \in \mathbb{N} : \text{if } m \in \mathbb{N} \text{ and } \mathbb{N}_{<m} \text{ is in one-to-one correspondence with } \mathbb{N}_{<n} \text{ then } m = n\ \}$. We will show that $T = \mathbb{N}$ using the induction axiom for $\mathbb{N}$.
   Clearly, $0 \in T$ because $\emptyset$ is the only set in one-to-one correspondence with $\emptyset$.
   Assume $n \in T$. We will show $n+1 \in T$. Let $m \in \mathbb{N}$ be such that $\mathbb{N}_{<m}$ is in one-to-one correspondence (say f) with $\mathbb{N}_{<n+1}$. We need to show that $m = n+1$. First, note that $m \neq 0$ because $n+1 \neq 0$ and we have $0 \in T$.

   - If the correspondence f takes $m-1 \in \mathbb{N}_{<m}$ to $n \in \mathbb{N}_{<n+1}$ then f is also a one-to-one correspondence between $\mathbb{N}_{<m-1}$ and $\mathbb{N}_{<n}$ and since $n \in T$, we have $n = m-1$ and hence $m = n+1$.
   - If the correspondence f takes $m-1 \in \mathbb{N}_{<m}$ to $y \in \mathbb{N}_{<n+1}$ where $y < n$ and also a certain $x \in \mathbb{N}_{<m}$ to $n \in \mathbb{N}_{<n+1}$, then let g be the one-to-one correspondence between $\mathbb{N}_{<m-1}$ and $\mathbb{N}_{<n}$ which corresponds $m-1$ to y and every $j < m-1$ to its correspondence by f. Clearly g is a one-to-one correspondence between $\mathbb{N}_{<m-1}$ and $\mathbb{N}_{<n}$ and since $n \in T$, we have $m-1 = n$. Therefore, $m = n+1$.

   Hence by the the induction axiom for $\mathbb{N}$ we have $T = \mathbb{N}$.

2. Again, we will use the induction axiom for $\mathbb{N}$. Let us say that $\mathbb{N}_{<m}$ uniquely associates[1] to $\mathbb{N}_{<n}$ if every element of $\mathbb{N}_{<m}$ corresponds to a unique element of $\mathbb{N}_{<n}$ and no two different elements of $\mathbb{N}_{<m}$ correspond to the same element of $\mathbb{N}_{<n}$. Let $T = \{n \in \mathbb{N} : \text{if } m \in \mathbb{N} \text{ and } \mathbb{N}_{<m} \text{ uniquely associates to } \mathbb{N}_{<n} \text{ then } m \leq n\}$. We will show that $T = \mathbb{N}$ using the induction axiom for $\mathbb{N}$. Clearly, $0 \in T$. Assume $n \in T$. We will show $n+1 \in T$. Let $m \in \mathbb{N}$ be such that $\mathbb{N}_{<m}$ uniquely associates to $\mathbb{N}_{<n+1}$. We need to show that $m \leq n+1$. If $m = 0$, there is nothing to show. Assume $m \neq 0$.

   - If there is no $i \in \mathbb{N}_{<m}$ such that i associates to $n \in \mathbb{N}_{<n+1}$, then the same unique association from $\mathbb{N}_{<m}$ to $\mathbb{N}_{<n+1}$ is also a unique association from $\mathbb{N}_{<m}$ to $\mathbb{N}_{<n}$ and since $n \in T$, we have $m \leq n$. Hence, $m \leq n+1$.
   - Assume f is the unique association from $\mathbb{N}_{<m}$ to $\mathbb{N}_{<n+1}$. If there is $k \in \mathbb{N}_{<m}$ such that k associates by f to $n \in \mathbb{N}_{<n+1}$, then take the association g from $\mathbb{N}_{<m-1}$ to $\mathbb{N}_{<n}$ such that for every

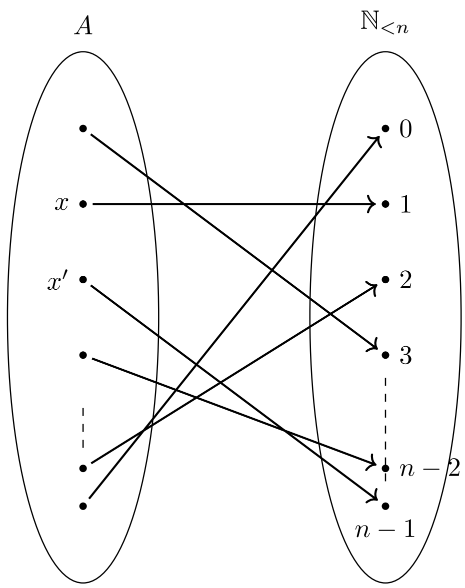
[1]Note that this is the definition of an injection which we study later on.

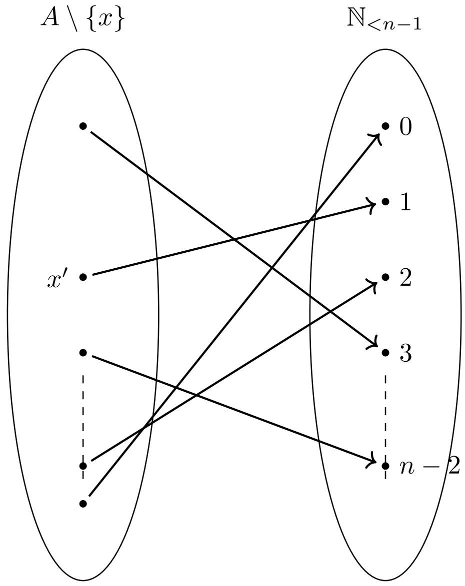
$i \in \mathbb{N}_{<m-1}$ where $i \neq k$, the association by g to i is the same as the association by f. For k, the association by g to k is the association by f to $m-1$. Clearly, $\mathbb{N}_{<m-1}$ uniquely associates to $\mathbb{N}_{<n}$ and since $n \in T$, we have $m-1 \leq n$. Hence, $m \leq n+1$.

Hence by the the induction axiom for $\mathbb{N}$ we have $T = \mathbb{N}$.

Solution C.49. [Of Exercise 4.5.]

1. We write $A \setminus \{x\}$ instead of B. Since A is a finite non empty set and $x \in A$, then let $|A| = n$ where $n > 0$ and let there be a one-to-one correspondence between A and $\mathbb{N}_{<n}$. If x corresponds to $n-1$ then the same one-to-one correspondence between A and $\mathbb{N}_{<n}$ is also a one-to-one correspondence between $A \setminus \{x\}$ and $\mathbb{N}_{<n-1}$ because we have removed the x and $n-1$ from A resp. $\mathbb{N}_{<n}$. If on the other hand x corresponds to $m < n-1$ and there is an $x' \in A$ which corresponds to $n-1$ then we take the correspondence between $A \setminus \{x\}$ and $\mathbb{N}_{<n-1}$ which takes any $y \neq x'$ to what it corresponded to earlier, but we take x' to what x corresponded to earlier. That is:





Clearly, this is a one-to-one correspondence between $A \setminus \{x\}$ and $\mathbb{N}_{<n-1}$ and hence $A \setminus \{x\}$ is finite and $|A \setminus \{x\}| = n - 1 = |A| - 1$.

2. We write $A \cup \{x\}$ instead of B. Since A is a finite set then by definition, there is a one-to-one correspondence between A and $\mathbb{N}_{<n}$ for some $n \in \mathbb{N}$ and $|A| = n$. Since $x \notin A$, then the above one-to-one correspondence between A and $\mathbb{N}_{<n}$ can be extended into a one-to-one correspondence between $A \cup \{x\}$ and $\mathbb{N}_{<n+1}$ by associating x to n. Hence, $A \cup \{x\}$ is finite and $|A \cup \{x\}| = n + 1 = |A| + 1$. That is:

   Since $\boxed{A \quad \leftrightarrow \quad \mathbb{N}_{<n}}$ then $\boxed{A \cup \{x\} \quad \leftrightarrow \quad \mathbb{N}_{<n} \cup \{n\} = \mathbb{N}_{<n+1}}$

3. We will use the induction axiom for $\mathbb{N}$. Let us say that A injectively associates[2] to $\mathbb{N}_{<n}$ if every element of A corresponds to a unique element of $\mathbb{N}_{<n}$ and no two different elements of A correspond to the same element of $\mathbb{N}_{<n}$. Let $T = \{n \in \mathbb{N} :$ if A injectively associates to $\mathbb{N}_{<n}$ then A is finite and $|A| \leq n\}$. We will show that $T = \mathbb{N}$ using the induction axiom for $\mathbb{N}$. Clearly, $0 \in T$ since if A injectively associates to $\mathbb{N}_{<0}$ then A is empty and hence A is in one-to-one correspondence with $\mathbb{N}_{<0}$ and A is finite and $|A| = 0 \leq 0$. Assume $n \in T$. We will show $n + 1 \in T$. Let A such that A injectively

[2]Note that this is the definition of an injection which we study later on.

associates to $\mathbb{N}_{<n+1}$. Call this association f. We need to show that A is finite and $|A| \leq n+1$. If $A = \emptyset$, there is nothing to show. Assume $A \neq \emptyset$,

- If there is no $x \in A$ that associates to $n \in \mathbb{N}_{<n+1}$, then all elements on A are associated to elements of $\mathbb{N}_{<n}$ and the same injective association f from A to $\mathbb{N}_{<n+1}$ is also a injective association from A to $\mathbb{N}_{<n}$ and since $n \in T$, we have that A is finite and $|A| \leq n$. Hence, A is finite and $|A| \leq n+1$. So, $n+1 \in T$.
- If there is $x \in A$ such that x associates to $n \in \mathbb{N}_{<n+1}$, then take the association from $A \setminus \{x\}$ to to $\mathbb{N}_{<n}$ which keeps to each element of $A \setminus \{x\}$ the same association in $\mathbb{N}_{<n+1}$. Obviously this is an injective association from $A \setminus \{x\}$ to $\mathbb{N}_{<n}$ and since $n \in T$, we deduce that $A \setminus \{x\}$ is finite and $|A \setminus \{x\}| \leq n$. By item 2 above, $A = (A \setminus \{x\}) \cup \{x\}$ is finite and $|A| = |A \setminus \{x\}| + 1 \leq n+1$. So, $n+1 \in T$.

Hence by the the induction axiom for $\mathbb{N}$ we have $T = \mathbb{N}$.

4. Since B is finite then by definition there is an $n \in \mathbb{N}$ such that B is in one-to-one correspondence with $\mathbb{N}_{<n}$ and $|B| = n$. We can easily show that this one-to-one correspondence between B and $\mathbb{N}_{<n}$ associates to every element of A a unique element of $\mathbb{N}_{<n}$ such that no two different elements of A correspond to the same element of $\mathbb{N}_{<n}$. Hence, by 3 A is finite and $|A| \leq n$.

5. If B is finite then by 4 above, A is also finite. Absurd since A is infinite by hypothesis.

Solution C.50. [Of Exercise 4.6.] Let $\frac{a_1}{a_2} \in \mathbb{Q}$ where $a_1, a_2 \in \mathbb{Z}$. Without loss of generality, we can assume that a_2 is a positive integer. Let $n \in \mathbb{N}^+$. Now, $\sqrt[n]{\frac{a_1}{a_2}}$ is an nth root for the equation $a_2 x^n + (-a_1) = 0$. Hence, $\sqrt[n]{\frac{a_1}{a_2}}$ is algebraic.

Solution C.51. [Of Exercise 4.7.]

1. The polynomials of height 4 are: x^3, $2x^2$, x^2+x, x^2-x, x^2+1, x^2-1, $x+2$, $x-2$, $2x+1$, $2x-1$.

2. The polynomials for height 5 which give all the new numbers listed under height 5 in the proof of Theorem 4.1.13 are as follows:
$x+3$ and $x-3$ which give resp. the numbers -3 and 3.
$3x+1$ and $3x-1$ which give resp. the numbers $-\frac{1}{3}$ and $\frac{1}{3}$.

x^2-2 which gives the numbers $-\sqrt{2}$ and $\sqrt{2}$.
$2x^2-1$ which gives the numbers $-\frac{1}{2}\sqrt{2}$ and $\frac{1}{2}\sqrt{2}$.
x^2-x-1 whose solutions are $\frac{1}{2}+\frac{1}{2}\sqrt{5}$ and $\frac{1}{2}-\frac{1}{2}\sqrt{5}$.
x^2+x-1 whose solutions are $-\frac{1}{2}-\frac{1}{2}\sqrt{5}$ and $-\frac{1}{2}+\frac{1}{2}\sqrt{5}$.
x^2-x+1 whose solutions are $\frac{1}{2}-\frac{i}{2}\sqrt{3}$ and $\frac{1}{2}+\frac{i}{2}\sqrt{3}$.
x^2+x+1 whose solutions are $-\frac{1}{2}-\frac{i}{2}\sqrt{3}$ and $-\frac{1}{2}+\frac{i}{2}\sqrt{3}$.

Solution C.52. [Of Exercise 4.8.]

1. The one-to-one correspondence below shows that E^+ the set of positive even integers is countable.

$$\begin{array}{ccl} 1 & \leftrightarrow & 2 = e_1 \\ 2 & \leftrightarrow & 4 = e_2 \\ 3 & \leftrightarrow & 6 = e_3 \\ 4 & \leftrightarrow & 8 = e_4 \\ & \vdots & \\ n & \leftrightarrow & 2n = e_n \\ & \vdots & \end{array}$$

   To show that E is countable, we write E as $0, e_1, -e_1, e_2, -e_2, \cdots$. We can then find a one-to-one correspondence with the nonzero natural numbers as follows:

$$\begin{array}{ccl} 1 & \leftrightarrow & 0 \\ 2 & \leftrightarrow & 2 \\ 3 & \leftrightarrow & -2 \\ 4 & \leftrightarrow & 4 \\ 5 & \leftrightarrow & -4 \\ & \vdots & \\ n & \leftrightarrow & 2n \text{ if } n \text{ is even} \\ n & \leftrightarrow & -n+1 \text{ if } n \text{ is odd} \\ & \vdots & \end{array}$$

2. In the same spirit as above, the one-to-one correspondence below shows

that O^+ the set of positive odd integers is countable.

$$\begin{array}{ccl} 1 & \leftrightarrow & 1 = o_1 \\ 2 & \leftrightarrow & 3 = o_2 \\ 3 & \leftrightarrow & 5 = o_3 \\ 4 & \leftrightarrow & 7 = o_4 \\ & \vdots & \\ n & \leftrightarrow & 2n - 1 = o_n \\ & \vdots & \end{array}$$

To show that O is countable, we write O as $0, o_1, -o_1, o_2, -o_2, \cdots$. We can then find a one-to-one correspondence with the nonzero natural numbers as follows:

$$\begin{array}{ccl} 1 & \leftrightarrow & 0 \\ 2 & \leftrightarrow & 1 \\ 3 & \leftrightarrow & -1 \\ 4 & \leftrightarrow & 3 \\ 5 & \leftrightarrow & -3 \\ & \vdots & \\ n & \leftrightarrow & \text{ if } n-1 \text{ is even} \\ n & \leftrightarrow & \text{ if } -n+2 \text{ is odd} \\ & \vdots & \end{array}$$

Solution C.53. [Of Exercise 4.9.]

1. We need to prove that for any x, $x \in S \cup S$ if and only if $x \in S$. But by the definition of union, $x \in S \cup S$ if and only if $x \in S$ or $x \in S$, and this is clearly the same as $x \in S$.

2. Since $x \notin \emptyset$ for every object x, we have

$$\begin{aligned} x \in \emptyset \cup S &\Leftrightarrow x \in \emptyset \text{ or } x \in S \\ &\Leftrightarrow x \in S \\ &\Leftrightarrow x \in S \text{ or } x \in \emptyset \\ &\Leftrightarrow x \in S \cup \emptyset. \end{aligned}$$

3. We need to prove that $x \in S \cup T$ if and only if $x \in T \cup S$. We have

$$\begin{aligned} x \in S \cup T &\Leftrightarrow x \in S \text{ or } x \in T \\ &\Leftrightarrow x \in T \text{ or } x \in S \\ &\Leftrightarrow x \in T \cup S. \end{aligned}$$

Solution C.54. [Of Exercise 4.10.]

1. We need to prove that for any x, $x \in S \cap S$ if and only if $x \in S$. But by the definition of intersection, $x \in S \cap S$ if and only if $x \in S$ and $x \in S$, and this is clearly the same as $x \in S$.

2. For any x, if $x \in \emptyset \cap S$ then $x \in \emptyset$, but this is impossible. Therefore, for no x, is $x \in \emptyset \cap S$. The proof is similar for $x \in S \cap \emptyset$.

3. We need to prove that $x \in S \cap T$ if and only if $x \in T \cap S$. We have

$$\begin{aligned} x \in S \cap T &\Leftrightarrow x \in S \text{ and } x \in T \\ &\Leftrightarrow x \in T \text{ and } x \in S \\ &\Leftrightarrow x \in T \cap S. \end{aligned}$$

Solution C.55. [Of Exercise 4.11.]

$$\begin{aligned} x \in S \cup (T \cap R) &\Leftrightarrow x \in S \text{ or } (x \in T \text{ and } x \in R) \\ &\Leftrightarrow (x \in S \text{ or } x \in T) \text{ and } (x \in S \text{ or } x \in R) \\ &\Leftrightarrow (x \in S \cup T) \text{ and } (x \in S \cup R) \\ &\Leftrightarrow x \in (S \cup T) \cap (S \cup R) \end{aligned}$$

Solution C.56. [Of Exercise 4.12.] We only prove the first item and leave the rest to the reader.

1. We prove $A \cap B = A$ as follows:

   - $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B \Rightarrow x \in A$.
   - $x \in A \Rightarrow x \in A$ and $x \in A \Rightarrow^{\text{since } A \subseteq B} x \in A$ and $x \in B \Rightarrow x \in A \cap B$.

   We prove $A \cup B = B$ as follows:

   - $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B \Rightarrow^{\text{since } A \subseteq B} x \in B$ or $x \in B \Rightarrow x \in B$.
   - $x \in B \Rightarrow x \in A$ or $x \in B \Rightarrow x \in A \cup B$.

Solution C.57. [Of Exercise 4.13.]

1. $\emptyset$ and $\mathbb{N}_{<n}$ are 2 different elements of $\mathcal{P}\mathbb{N}_{<n}$ which are finite sets. They are different because $n \neq 0$ and hence $\mathbb{N}_{<n} \neq \emptyset$. Clearly $\emptyset$ and $\mathbb{N}_{<n}$ are finite and $|\emptyset| = 0$ and $|\mathbb{N}_{<n}| = n$.

2. $\emptyset$ and $\{0\}$ are 2 different elements of $\mathcal{P}\mathbb{N}_{<n}$ which are finite sets. They are different because $\{0\} \neq \emptyset$. Clearly $\emptyset$ and $\{0\}$ are finite and $|\emptyset| = 0$ and $|\{0\}| = 1$.

3. $\mathbb{N}_{\geq n}$ and $\mathbb{N}_{\geq n+1}$ are 2 different elements of $\mathcal{P}\mathbb{N}_{\geq n}$ which are infinite sets. They are different because $n \in \mathbb{N}_{\geq n}$ but $n \notin \mathbb{N}_{\geq n+1}$. They are infinite because each can be put in one-to-one correspondence with $\mathbb{N}$ as follows:

$\mathbb{N}$	$\leftrightarrow$	$\mathbb{N}_{\geq n}$
m	$\leftrightarrow$	$m+n$

$\mathbb{N}$	$\leftrightarrow$	$\mathbb{N}_{\geq n+1}$
m	$\leftrightarrow$	$m+n+1$

4. If $S \in \mathcal{P}\mathbb{N}_{\geq n}$, then $S \subseteq \mathbb{N}_{\geq n}$. By Corollary 4.1.15, S is countable.

Solution C.58. [Of Exercise 4.14.] Since $(0, 1] \subseteq \mathbb{R} \setminus \mathbb{N}_{n\geq 2}$ and by Theorem 4.1.17, $(0, 1]$ is uncountable, then $\mathbb{R}\setminus\mathbb{N}_{n\geq 2}$ is uncountable. Otherwise, by Theorem 4.1.14, $(0, 1]$ would also be countable contradicting Theorem 4.1.17.

Solution C.59. [Of Exercise 4.15.]

1. - Assume f is bijective. Then, (for every $b \in T$, there is a unique $a \in S$ such that $f(a) = b$). Obviously, f is surjective. If for some $a, b \in S$, $f(a) = f(b)$ then since $f(a) \in T$, there can only be one member $c \in S$ such that $f(c) = f(a) = f(b)$. Hence, $a = b = c$ and f is injective.

- Assume f is injective and surjective. Let $b \in T$. By surjection, there is $a \in S$ such that $f(a) = b$. If there is also $a' \in S$ such that $f(a') = b$, then $f(a) = f(a')$ and by injection, $a = a'$. Hence, for every $b \in T$, there is a unique $a \in S$ such that $f(a) = b$.

2. Let $b \in T$. Since f is bijective, there is a unique $a \in S$ such that $f(a) = b$. Hence, there is a unique $a \in S$ such that $f^{-1}(b) = a$, and so, f^{-1} is a function. Moreover, if $b, b' \in T$ such that $a = f^{-1}(b) = f^{-1}(b') = a'$ then $b = f(a) = f(a') = b'$ and hence, f^{-1} is injective. Finally, for any $a \in S$, since f is a function, $f(a) \in T$ and hence, there is $b \in T$ such that $f^{-1}(b) \in S$. Moreover,

   - If $x \in S$ then since f is bijective, there is a unique $y \in T$ such that $f(x) = y$. But, $f(x) = y$ iff $f^{-1}(y) = x$. Hence, $1_S(x) = x = f^{-1}(y) = f^{-1}(f(x))$ and so, $f^{-1} \circ f(x) = 1_S(x)$ and so, $f^{-1} \circ f = 1_S$.
   - The proof that $f \circ f^{-1} = 1_T$ is similar to the above item.

3. (a) Since f and g are functions, let $a \in S$, then there is a unique b in T such that $b = f(a)$ and hence, there is a unique c in V such that $c = g(f(a))$. Hence, there is a unique c in V such that $c = g \circ f(a)$ and $g \circ f$ is a function from S to V.

   (b) Assume f and g are injective and $a, b \in S$.
   $g \circ f(a) = g \circ f(b) \Rightarrow^{g \text{ injective}} f(a) = f(b) \Rightarrow^{f \text{ injective}} a = b$. Hence $g \circ f$ is injective.

   (c) Assume f and g are surjective.
   $c \in V \Rightarrow^{g \text{ surjective}} \exists b \in T$ such that $g(b) = c \Rightarrow^{f \text{ surjective}} \exists a \in S$ such that $f(a) = b$ and $g(b) = c \Rightarrow \exists a \in S$ such that $g(f(a)) = c$. Hence $g \circ f$ is surjective.

   (d) Assume f and g are bijective. Hence, f and g are injective and surjective. Hence, by the above two items $g \circ f$ is both injective and surjective and so, it is bijective.

4. Easy. Left to the reader.

5. Define $g : \mathcal{P}S \mapsto \mathcal{P}T$ such that for any $S' \subseteq S$, we set $g(S') = f[S'] = \{f(a) : a \in S'\}$. Clearly $g(S') \subseteq T$. We invite the reader to show that g is a function. We will show that g is bijective.

   - Assume $g(S_1) = g(S_2)$. Let $a \in S_1$. Then $f(a) \in g(S_1) = g(S_2)$ and hence there is $a' \in S_2$ such that $f(a) = f(a')$. Since f is injective then $a = a' \in S_2$ and so, $S_1 \subseteq S_2$. Similarly we show $S_2 \subseteq S_1$ and so, g is injective.

- Let $T' \in \mathcal{P}T$ and $S' = \{a \in S : f(a) \in T'\}$. Clearly $S' \in \mathcal{P}S$. We show that $g(S') = T'$:
  - If $b \in g(S')$ then for some $a \in S'$, $b = f(a)$ and hence $b \in T'$.
  - If $b \in T'$ then since f is surjective there is $a \in S$ such that $b = f(a)$. Hence by definition $a \in S'$ and $b \in g(S')$.

  Hence g is surjective.

6. If $f_{|S'}(x) = f_{|S'}(y)$ then $f(x) = f(y)$ and hence since f is injective, we have $x = y$. Therefore, $f_{|S'}$ is injective.
7. If we take $g : S' \mapsto S$ such that $g(a) = a$, then it is easy to show that g is an injection.

Solution C.60. [Of Exercise 4.16.] Let $y \in f[A]$. Then, $y = f(a)$ for some $a \in A$. Then, $y = f(a)$ for some $a \in B$. Hence $y \in f[B]$.
Let $x \in f^{-1}[C]$. Then, $x \in S$ and $f(x) \in C$. Then, $x \in S$ and $f(x) \in D$. Then $x \in f^{-1}[D]$.

Solution C.61. [Of Exercise 4.17.] h is a function because if $x \in A_1 \cup A_2$, then since $A_1 \cap A_2 = \emptyset$, x is either exclusively in A_1 or exclusively in A_2 and in each case, $h(x)$ is a unique element in $B_1 \cup B_2$ since f and g are functions.

Let $x, y \in A_1 \cup A_2$ such that $h(x) = h(y)$. If $x, y \in A_1$ then $f(x) = h(x) = h(y) = f(y)$ and by injectivity of f, $x = y$. The same proof holds if $x, y \in A_2$. The cases that ($x \in A_1$ and $y \in A_2$) or ($x \in A_2$ and $y \in A_1$) cannot hold since otherwise, we would have ($h(x) \in B_1$ and $h(y) \in B_2$) or resp. ($h(x) \in B_2$ and $h(y) \in B_1$) which would contradict $B_1 \cap B_2 = \emptyset$. Therefore, whenever $x, y \in A_1 \cup A_2$ such that $h(x) = h(y)$, we have $x = y$ and h is injective.

Let $y \in B_1 \cup B_2$. Then, since $B_1 \cap B_2 = \emptyset$, y is either exclusively in B_1 or exclusively in B_2. In the first case, by surjectivity of f, there is $x \in A_1 \subseteq A_1 \cup A_2$ such that $h(x) = f(x) = y$. In the second case, by surjectivity of g, there is $x \in A_2 \subseteq A_1 \cup A_2$ such that $h(x) = g(x) = y$. Hence, h is a surjection.

Solution C.62. [Of Exercise 4.18.] h is a function because if $x \in \bigcup_{n\geq 1} A_n$, then since for all $n \neq m$, $A_n \cap A_m = \emptyset$, x is exclusively in one of the A_n's (say $x \in A_p$) and $h(x) = f_p(x)$ is a unique element in $\bigcup_{n\geq 1} B_n$ since f_p is a function.

Let $x, y \in \bigcup_{n\geq 1} A_n$ such that $h(x) = h(y)$. If for some $n \geq 1$, $x, y \in A_n$ then $f_n(x) = h(x) = h(y) = f_n(y)$ and by injectivity of f_n, $x = y$. The cases that $x \in A_n$ and $y \in A_m$ where $n \neq m$ cannot hold since otherwise, we would have ($h(x) = f_n(x) \in B_n$ and $h(y) = f_m(x) \in B_m$) which would contradict $B_n \cap B_m = \emptyset$. Therefore, whenever $x, y \in \bigcup_{n\geq 1} A_n$ such that $h(x) = h(y)$, we have $x = y$ and h is injective.

Let $y \in \bigcup_{n\geq 1} B_n$. Then, since for all $n \neq m$, $B_n \cap B_m = \emptyset$, y is exclusively in one of the B_i's (say B_n) and by surjectivity of f_n, there is $x \in A_n \subseteq \bigcup_{n\geq 1} A_n$ such that $h(x) = f_n(x) = y$. Hence, h is a surjection.

Solution C.63. [Of Exercise 4.19.]

1. • Case $S_1 \cup S_2$:

$$
\begin{aligned}
& b \in f[S_1 \cup S_2] \Leftrightarrow \\
& \text{for some } a,\ (a \in S_1 \cup S_2 \text{ and } b = f(a)) \Leftrightarrow \\
& \text{for some } a,\ ((a \in S_1 \text{ or } a \in S_2) \text{ and } b = f(a)) \Leftrightarrow \\
& \text{for some } a,\ ((a \in S_1 \text{ and } b = f(a)) \text{ or} \\
& \qquad\qquad (a \in S_2 \text{ and } b = f(a))) \Leftrightarrow \\
& \text{for some } a,\ (a \in S_1 \text{ and } b = f(a)) \text{ or} \\
& \text{for some } a,\ (a \in S_2 \text{ and } b = f(a)) \Leftrightarrow \\
& (b \in f[S_1] \text{ or } b \in f[S_2]) \Leftrightarrow \\
& b \in f[S_1] \cup f[S_2]
\end{aligned}
$$

   • Case $\bigcup_{i=1}^{\infty} S_i$:

$$
\begin{aligned}
& b \in f[\textstyle\bigcup_{i=1}^{\infty} S_i] \Leftrightarrow \\
& \text{for some } a,\ (a \in \textstyle\bigcup_{i=1}^{\infty} S_i \text{ and } b = f(a)) \Leftrightarrow \\
& \text{for some } a, \text{ for some } i \in \mathbb{N}^+,\ (a \in S_i \text{ and } b = f(a)) \Leftrightarrow \\
& \text{for some } i \in \mathbb{N}^+, \text{ for some } a,\ (a \in S_i \text{ and } b = f(a)) \Leftrightarrow \\
& \text{for some } i \in \mathbb{N}^+,\ b \in f[S_i] \Leftrightarrow \\
& b \in \textstyle\bigcup_{i=1}^{\infty} f[S_i]
\end{aligned}
$$

2. • Case $S_1 \cap S_2$:

$$
\begin{aligned}
& b \in f[S_1 \cap S_2] \Rightarrow \\
& \text{for some } a,\ (a \in S_1 \cap S_2 \text{ and } b = f(a)) \Rightarrow \\
& \text{for some } a,\ (a \in S_1 \text{ and } a \in S_2 \text{ and } b = f(a)) \Rightarrow \\
& \text{for some } a,\ (a \in S_1 \text{ and } \text{ and } b = f(a)) \text{ and} \\
& \qquad\qquad (a \in S_2 \text{ and } b = f(a)) \Rightarrow \\
& \text{for some } a,\ (a \in S_1 \text{ and } b = f(a)) \text{ and} \\
& \text{for some } a,\ (a \in S_2 \text{ and } b = f(a)) \Rightarrow \\
& (b \in f[S_1] \text{ and } b \in f[S_2]) \Rightarrow \\
& (b \in f[S_1] \cap f[S_2])
\end{aligned}
$$

$b \in f[S_1] \cap f[S_2] \Rightarrow$
$(b \in f[S_1]$ and $b \in f[S_2]) \Rightarrow$
for some a, $(a \in S_1$ and $b = f(a))$ and
for some a', $(a' \in S_2$ and $b = f(a')) \Rightarrow$
for some $a \in S_1$, $a' \in S_2$, $b = f(a) = f(a') \Rightarrow^{f \text{ inj.}}$
for some $a \in S_1$, $a' \in S_2$, $a = a'$ and $b = f(a) = f(a') \Rightarrow$
for some $a \in S_1 \cap S_2$, $b = f(a) \Rightarrow$
$b \in f[S_1 \cap S_2]$

Now, if f is not injective, $S_1 = \{a\}$, $S_2 = \{a'\}$, $a \neq a'$ and $f(a) = f(a') = b$ then $S_1 \cap S_2 = \emptyset$, $f[S_1 \cap S_2] = \emptyset$, and $f[S_1] \cap f[S_2] = \{b\}$.

- Case $S_1 \setminus S_2$:

  $b \in f[S_1] \setminus f[S_2] \Rightarrow$
  $(b \in f[S_1]$ and $b \notin f[S_2]) \Rightarrow$
  (for some $a \in S_1$, $b = f(a)$ and for all $a' \in S_2$, $b \neq f(a')) \Rightarrow$
  (for some $a \in S_1 \setminus S_2$, $b = f(a)) \Rightarrow$
  $b \in f[S_1 \setminus S_2]$

  $b \in f[S_1 \setminus S_2] \Rightarrow$
  for some $a \in S_1 \setminus S_2$, $b = f(a) \Rightarrow$
  for some $a \in S_1$, $b = f(a) \Rightarrow$
  $b \in f[S_1]$

  So far we have shown that $f[S_1 \setminus S_2] \subseteq f[S_1]$ and $f[S_1] \setminus f[S_2] \subseteq f[S_1 \setminus S_2]$.

  - If f is injective, $b \in f[S_1 \setminus S_2]$ and $b \in f[S_2]$ then for some $a \in S_1 \setminus S_2$, $a' \in S_2$, $b = f(a) = f(a')$. Since f is injective then $a = a'$ and hence $a \notin S_1 \setminus S_2$ contradiction. Hence, if $b \in f[S_1 \setminus S_2]$ then $b \in f[S_1] \setminus f[S_2]$.
  - To give an example that $f[S_1 \setminus S_2] = f[S_1] \setminus f[S_2]$ fails when f is not injective, take $S_1 = \{a\}$, $S_2 = \{a'\}$, $a \neq a'$, $f(a) = f(a')$. Then, $S_1 \setminus S_2 = \{a\}$, $f[S_1 \setminus S_2] = \{f(a)\}$ but $f[S_1] \setminus f[S_2] = \emptyset$.

- Case $\bigcap_{i=1}^{\infty} S_i$:

  $b \in f[\bigcap_{i=1}^{\infty} S_i] \Rightarrow$
  for some a, $(a \in \bigcap_{i=1}^{\infty} S_i$ and $b = f(a)) \Rightarrow$
  for some a, (for all $i \in \mathbb{N}^+$, $a \in S_i$ and $b = f(a)) \Rightarrow$
  (for all $i \in \mathbb{N}^+$, for some a, $a \in S_i$ and $b = f(a)) \Rightarrow$
  (for all $i \in \mathbb{N}^+$, $b \in f[S_i]) \Rightarrow$
  $b \in \bigcap_{i=1}^{\infty} f[S_i]$

$b \in \bigcap_{i=1}^{\infty} f[S_i] \Rightarrow$
for all $i \in \mathbb{N}^+$, $b \in f[S_i] \Rightarrow$
for all $i \in \mathbb{N}^+$, for some a_i, $a_i \in S_i$ and $b = f(a_i) \Rightarrow$
for all $i \in \mathbb{N}^+$, for some a_i, $a_i \in S_i$ and $b = f(a_i)$ and
for all $j \in \mathbb{N}^+$, for some a_j, $a_j \in S_j$ and $b = f(a_j) \Rightarrow$
for all $i \in \mathbb{N}^+$, for some a_i, $a_i \in S_i$ and $b = f(a_i)$ and
for some a_1, $a_1 \in S_1$ and $b = f(a_1) \Rightarrow$
for all $i \in \mathbb{N}^+$, for some a_i, for some a_1,
$a_i \in S_i, a_1 \in S_1,$ and $b = f(a_i) = f(a_1) \Rightarrow^{f \text{ inj.}}$
for all $i \in \mathbb{N}^+$, for some a_i, for some a_1,
$a_i \in S_i, a_1 \in S_1,$ $a_1 = a_i$ and $b = f(a_1) \Rightarrow$
for all $i \in \mathbb{N}^+$, for some a_1, $a_1 \in S_i$ and $b = f(a_1) \Rightarrow$
for some a_1, for all $i \in \mathbb{N}^+$, $a_1 \in S_i$ and $b = f(a_1) \Rightarrow$
for some a_1, $a_1 \in \bigcap_{i=1}^{\infty} S_i$ and $b = f(a_1) \Rightarrow$
$b \in f[\bigcap_{i=1}^{\infty} S_i]$

To give a counterexample that shows that injectivity is needed, let $S_i = \{i\}$ and $f(i) = 1$ for each $i \in \mathbb{N}^+$. Then, $\bigcap_{i=1}^{\infty} S_i = \emptyset$, $f[\bigcap_{i=1}^{\infty} S_i] = \emptyset$ and $\bigcap_{i=1}^{\infty} f[S_i] = \{1\}$.

3.

$$\begin{aligned} a \in f^{-1}[T_1 \cup T_2] &\Leftrightarrow f(a) \in T_1 \cup T_2 \\ &\Leftrightarrow f(a) \in T_1 \text{ or } f(a) \in T_2 \\ &\Leftrightarrow a \in f^{-1}[T_1] \text{ or } a \in f^{-1}[T_2] \\ &\Leftrightarrow a \in f^{-1}[T_1] \cup f^{-1}[T_2] \end{aligned}$$

$$\begin{aligned} a \in f^{-1}[\bigcup_{i=1}^{\infty} T_i] &\Leftrightarrow f(a) \in \bigcup_{i=1}^{\infty} T_i \\ &\Leftrightarrow \text{for some } i \in \mathbb{N}^+,\ f(a) \in T_i \\ &\Leftrightarrow \text{for some } i \in \mathbb{N}^+,\ a \in f^{-1}[T_i] \\ &\Leftrightarrow a \in \bigcup_{i=1}^{\infty} f^{-1}[T_i] \end{aligned}$$

4.

$$\begin{aligned} a \in f^{-1}[T_1 \cap T_2] &\Leftrightarrow f(a) \in T_1 \cap T_2 \\ &\Leftrightarrow f(a) \in T_1 \text{ and } f(a) \in T_2 \\ &\Leftrightarrow a \in f^{-1}[T_1] \text{ and } a \in f^{-1}[T_2] \\ &\Leftrightarrow a \in f^{-1}[T_1] \cap f^{-1}[T_2] \end{aligned}$$

$$\begin{aligned} a \in f^{-1}[\bigcap_{i=1}^{\infty} T_i] &\Leftrightarrow f(a) \in \bigcap_{i=1}^{\infty} T_i \\ &\Leftrightarrow \text{for all } i \in \mathbb{N}^+,\ f(a) \in T_i \\ &\Leftrightarrow \text{for all } i \in \mathbb{N}^+,\ a \in f^{-1}[T_i] \\ &\Leftrightarrow a \in \bigcap_{i=1}^{\infty} f^{-1}[T_i] \end{aligned}$$

Solution C.64. [Of Exercise 4.20.] Let $A = \{n \in \mathbb{N}_{n\geq 1} : S_{n+1} \subseteq g[T_n] \subseteq S_n \text{ and } T_{n+1} \subseteq f[S_n] \subseteq T_n\}$. We prove by induction on $\mathbb{N}_{n\geq 1}$ that $A = \mathbb{N}_{n\geq 1}$.

- Note that $f[S] \subseteq T$ and so, by Exercise 4.16 $g[f[S]] \subseteq g[T]$. Therefore, $S_2 = g[f[S_1]] = g[f[S]] \subseteq g[T] = g[T_1] \subseteq S = S_1$. Hence $S_2 \subseteq g[T_1] \subseteq S_1$. Similarly, we prove that $T_2 \subseteq f[S_1] \subseteq T_1$. Hence, $1 \in A$.

- Assume that for some $n > 1$, $S_n \subseteq g[T_{n-1}] \subseteq S_{n-1}$ and $T_n \subseteq f[S_{n-1}] \subseteq T_{n-1}$. Then by Exercise 4.16, $g[f[S_n]] \subseteq g[f[g[T_{n-1}]]] \subseteq g[f[S_{n-1}]]$ and so, $S_{n+1} \subseteq g[T_n] \subseteq S_n$. Similarly, we prove $T_{n+1} \subseteq g[S_n] \subseteq T_n$.

Hence by induction on $\mathbb{N}_{n\geq 1}$, we conclude that $A = \mathbb{N}_{n\geq 1}$. So, for all $n \geq 1$, $S_{n+1} \subseteq g[T_n] \subseteq S_n$ and $T_{n+1} \subseteq f[S_n] \subseteq T_n$.
Now,

1. Since for all $n \geq 1$, $S_{n+1} \subseteq g[T_n] \subseteq S_n$, then:

   $\cdots \subseteq S_{n+2} \subseteq g[T_{n+1}] \subseteq S_{n+1} \subseteq g[T_n] \subseteq S_n \cdots \subseteq g[T_3] \subseteq S_3 \subseteq g[T_2] \subseteq S_2 \subseteq g[T_1] \subseteq S_1$.

2. Also, since for all $n \geq 1$, $T_{n+1} \subseteq f[S_n] \subseteq T_n$, then:

   $\cdots \subseteq T_{n+2} \subseteq f[S_{n+1}] \subseteq T_{n+1} \subseteq f[S_n] \subseteq T_n \cdots \subseteq f[S_3] \subseteq T_3 \subseteq f[S_2] \subseteq T_2 \subseteq f[S_1] \subseteq T_1$.

3. Assume f is injective. By Exercise 4.19, $f[S^*] = f[\bigcap_{n=1}^{\infty} S_n] = \bigcap_{n=1}^{\infty} f[S_n]$. Since for all n, $T_n \subseteq T_1$ and $S_n \subseteq S_1$, then $T^* = \bigcap_{n=1}^{\infty} T_n = T_1 \cap \bigcap_{n=2}^{\infty} T_n =^{\text{Exercise 4.12}} \bigcap_{n=2}^{\infty} T_n = \bigcap_{n=1}^{\infty} T_{n+1}$. Now,
   - $T^* = \bigcap_{n=1}^{\infty} T_{n+1} \supseteq \bigcap_{n=1}^{\infty} f[S_{n+1}] = \bigcap_{n=2}^{\infty} f[S_n] = f[S_1] \cap \bigcap_{n=2}^{\infty} f[S_n] = \bigcap_{n=1}^{\infty} f[S_n] = f[\bigcap_{n=1}^{\infty} S_n] = f[S^*]$.
   - $T^* = \bigcap_{n=1}^{\infty} T_{n+1} \subseteq \bigcap_{n=1}^{\infty} f[S_n] = f[S^*]$.

   Hence, $T^* = f[S^*]$.

   Furthermore, by Lemma 4.2.23, we can show that $f_{|S^*} : S^* \mapsto T^*$ is bijective. This is done as follows: Since $f : S \mapsto T$ is injective and $S^* \subseteq S$, hence $f_{|S^*} : S^* \mapsto T$ is injective and so $f_{|S*} : S^* \mapsto f[S^*] = T^*$ is a bijection.

4. Assume f and g are injective.
   - – If $b \in f[S_n \setminus g[T_n]]$ then $b = f(a)$ where $a \in S_n \setminus g[T_n]$ and hence $b = f(a)$ where $a \in S_n$ and $a \notin g[T_n]$. So, $b \in f[S_n]$ and $b = f(a)$ where $a \notin g[T_n]$.
       If $b \in T_{n+1} = f[g[T_n]]$ then for some $a' \in g[T_n]$, $b = f(a') = f(a)$ and hence since f is injective, $a = a'$ and $a \in g[T_n]$ absurd. Hence, $b \notin T_{n+1}$ and $b \in f[S_n] \setminus T_{n+1}$.
     – If $b \in f[S_n] \setminus T_{n+1}$ then $b = f(a)$ for some $a \in S_n$ and $b \notin T_{n+1} = f[g[T_n]]$. If $a \in g[T_n]$ then $b = f(a) \in f[g[T_n]] = T_{n+1}$ absurd. Hence, $b \in f[S_n \setminus g[T_n]]$.

     Hence $f[S_n \setminus g[T_n]] = f[S_n] \setminus T_{n+1}$.
   - Similarly we show that $g[T_n \setminus f[S_n]] = g[T_n] \setminus S_{n+1}$.

Solution C.65. [Of Exercise 4.21.] $f : \mathbb{N}^+ \mapsto \mathbb{Z}$ is defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Solution C.66. [Of Exercise 4.22.] Clearly this is a function because for every $x \in (-1, 1)$, $f(x)$ is a unique value in R.
f is injective because if $f(x) = f(y)$ then by inspection on the cases, we can show $x = y$.
f is surjective because if we take $0 \in \mathbb{R}$, we have $0 \in (-1, 1)$ such that $f(0) = 0$. If we take $y > 0$ then for $x = \frac{1}{y+1} \in (0, 1)$ we have $f(x) = \frac{1}{x} - 1 = y$. Finally, if $y < 0$ then for $x = \frac{1}{y-1} \in (-1, 0)$ we have $f(x) = \frac{1}{x} + 1 = y$.

Solution C.67. [Of exercise 4.23.] Let $R = \{A \in \mathcal{P}S : A \subseteq f(A)\}$ and take $T = \bigcup_{A \in R} A$. It is easy to show that $f(T) = T$:

- Note that if $A \in R$ then $A \subseteq \bigcup_{A \in R} A = T$ and hence by hypothesis, $f(A) \subseteq f(T)$.
  If $A \in R$ then by definition $A \in \mathcal{P}S$ and $A \subseteq f(A) \subseteq f(T)$. Hence, if $A \in R$ then $A \subseteq f(T)$. Therefore, $T = \bigcup_{A \in R} A \subseteq f(T)$.
- Since $T, f(T) \in \mathcal{P}S$ and by above, $T \subseteq f(T)$ then by hypothesis $f(T) \subseteq f(f(T))$. Hence, $f(T) \in R$ and since $T = \bigcup_{A \in R} A$, we get $f(T) \subseteq T$.

Solution C.68. [Of Exercise 4.24.]

1. Since S is infinite, then by Lemma 4.3.1.5 there is $S' \subseteq S$ and a bijection $g : \mathbb{N} \mapsto S'$. By Lemma 4.2.23.6, $f_{|S'} : S' \mapsto T$ is injective. By Lemma 4.2.23.4, $f_{|S'} : S' \mapsto f_{|S'}[S']$ is bijective. By Lemma 4.2.23.3, $f_{|S'} \circ g : \mathbb{N} \mapsto f_{|S'}[S'] \subseteq T$ is bijective. Hence, T is infinite.
2. This is a corollary of the previous item.
3. Since T is infinite, by Lemma 4.3.1.5, there is $T' \subseteq T$ such that T' is in one-to-one correspondence with $\mathbb{N}$. Since f is surjective, for each $y \in T'$ there is at least one $x \in S$ such that $f(x) = y$. Let us pick for each $y \in S'$ exactly one $x \in S$ such that $f(x) = y$. We collect these x's into a set $S' \subseteq S$. That is, $S' \subseteq S$ is such that for each $y \in T'$, S' contains exactly one $x \in S$ for which $f(x) = y$. Clearly, there is a one-to-one correspondence between S' and T'. Hence, there is a one-to-one correspondence between S' and $\mathbb{N}$ and S is infinite.
4. This is a corollary of the previous item.

Solution C.69. [Of Exercise 4.25.] Since S is countable, let $g : \mathbb{N} \mapsto S$ be a bijection. Then, by Lemma 4.2.23, $f \circ g : \mathbb{N} \mapsto T$ is a surjection and by Lemma 4.3.1.4, T is countable.

Solution C.70. [Of Exercise 4.26.] Let $g : \mathbb{N} \mapsto S$ be defined as follows:

$$
\begin{array}{rcl}
g(0) & = & f(0) \\
g(1) & = & f(p_1) \text{ where } p_1 \text{ is the least } p > 0 \text{ such that } f(p) \notin \{g(0)\} \\
g(2) & = & f(p_2) \text{ where } p_2 \text{ is the least } p > 1 \text{ such that } f(p) \notin \{g(0), g(1)\} \\
\vdots & & \\
g(n) & = & f(p_n) \text{ where } p_n \text{ is the least } p > n-1 \\
 & & \text{such that } f(p) \notin \{g(0), \cdots, g(n-1)\}
\end{array}
$$

By construction, g is injective because at every stage, we built $g(n)$ to be different from all of $g(0), g(1), \cdots g(n-1)$. But, g is also surjective. To see this, let $b \in S$. Since f is surjective, then $b = f(n)$ for some n.

- If $n = 0$ then $g(0) = f(0) = b$.

- If $n > 0$ then if $b \in \{g(0), \cdots, g(n-1)\}$ then we are done, else if $b \notin \{g(0), \cdots, g(n-1)\}$ then $b = g(n)$.

Hence by Definition 4.1.6, S is infinitely countable.

Solution C.71. [Of Exercise 4.28.] By Corollary 4.1.15, every subset of a countable set is countable. If S is countable, then since $S \setminus T \subseteq S$, we have $S \setminus T$ is also countable.

If we take $S = T$ then $S \setminus T = \emptyset$ is always countable no matter what S was.

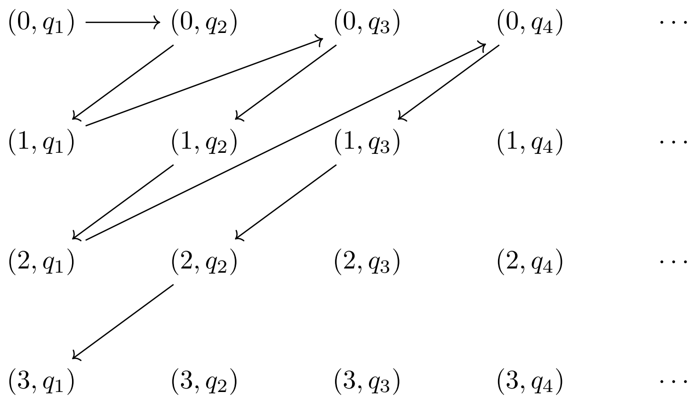
By Theorem 4.1.17, $(0, 1]$ is uncountable. Since $f : (0, 1] \mapsto [0, 1]$ such that $f(x) = x$ is injective, then by Lemma 4.3.1.1, $[0, 1]$ is uncountable. Let $S = [0, 1]$ and $T = (0, 1)$. Then, $S \setminus T = \{0\}$ is countable.

If on the other hand, we take $(0, 1]$ which is uncountable by Theorem 4.1.17, then since the functions $f : (0, 1] \mapsto (0, 2]$ such that $f(x) = x$ and $g : (0, 1] \mapsto (1, 2]$ such that $f(x) = 2x$ are injective, then by Lemma 4.3.1.1, $(0, 2]$ and $(1, 2]$ are both uncountable. Now, if $S = (0, 2]$ and $T = (0, 1]$ then $(1, 2] = S \setminus T = (0, 2] \setminus (0, 1]$ is uncountable.

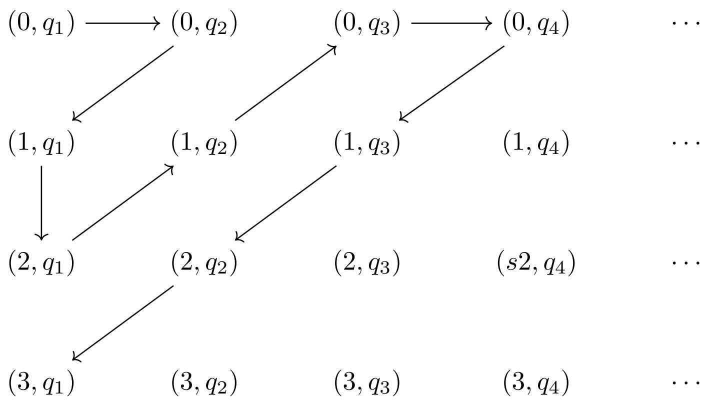
Solution C.72. [Of Exercise 4.29.] Let P_n be the set of polynomials of degree n and let $f : P_n \mapsto \mathbb{Z}^{n+1} = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n+1 \text{ times}}$ such that $f(a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0) = (a_n, a_{n-1}, \cdots, a_0)$. It is esy to show that f is bijective and hence P_n is infinitely countable. Hence, by Theorem 4.3.9, $P = \bigcup_{i=0}^{\infty} P_i$ is (infinitely) countable. Furthermore, if $p \in P$ and R_p is the set of roots of p then R_p is countable and has at most degree p elements. Hence again by Theorem 4.3.9, the set of algebraic numbers which is $\bigcup_{p \in P} R_p$ is countable. It is easy to show that the set of algebraic numbers is infinite and hence, it is infinitely countable.

Solution C.73. [Of Exercise 4.30.] Since $\mathbb{N}$ and $\mathbb{Q}$ are countable, let $0, 1, 2, \cdots$ respectively $q_1, q_2, \cdots$ be listings of $\mathbb{N}$ resp. $\mathbb{Q}$. Now, we give two listings of $\mathbb{N} \times \mathbb{Q}$.

The first listing is:



The second listing is:



Solution C.74. [Of Exercise 4.31.] If $S \cup T$ is finite then since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by Lemma 4.1.8.4, S and T are finite.
On the other hand, assume S and T are finite, then by definition, let $f : S \mapsto \mathbb{N}_{<n}$ and $g : T \mapsto \mathbb{N}_{<m}$ be bijections where $n, m \in \mathbb{N}$ and without loss of generality we can assume $n \geq m$. Now, $|S| = n$ and $|T| = m$. Let $h : S \cup T \mapsto \mathbb{N}_{<n+m}$ such that $h(x) = \begin{cases} f(x) & \text{if } x \in S \\ g(x) & \text{if } x \in T \setminus S \end{cases}$

It is easy to show that h is an injection and hence by Lemma 4.3.1.2, $S \cup T$ is finite and $|S \cup T| \leq n + m$.

Solution C.75. [Of Exercise 4.32.] We show f injective by induction on $\mathbb{N}$. Let $I = \{n \in \mathbb{N} : \text{ for all } m \in \mathbb{N}, \text{ if } f(n) = f(m) \text{ then } n = m\}$. We will show that $I = \mathbb{N}$.

- For all $m \in \mathbb{N}$, if $f(0) = f(m)$ then $f(0) = f(m) = (0,0)$ and by definition, $m = 0$. Hence $0 \in I$.
- Assume $n \in I$. For all $m \in \mathbb{N}$, if $f(n+1) = f(m)$ then
  - If $f(n+1) = f(m) = (0, k+1)$ then by definition, $m > 0$ and $f(n) = f(m-1) = (k-1, 0)$. Since $n \in I$, then $n = m-1$ and so, $n+1 = m$.
  - If $f(n+1) = f(m) = (k+1, l-1)$ then by definition, $m > 0$, $l > 0$ and $f(n) = f(m-1) = (k, l)$. Since $n \in I$, then $n = m-1$ and so, $n+1 = m$.

  Hence, $n+1 \in I$.

To show f surjective, we show by induction that for any $(x, y) \in \mathbb{N} \times \mathbb{N}$, there is $m \in \mathbb{N}$ such that $f(m) = (x, y)$.

- If $(x, y) = (0, 0)$, then take $m = 0$.
- Assume $x + y \neq 0$ and for any (x', y') such that either $(x' + y' = x + y$ and $x' < x)$ or $x' + y' < x + y$, we have an n where $f(n) = (x', y')$. We will show that there is also an m such that $f(m) = (x, y)$.
  If $x = 0$ then $y \neq 0$ and by Induction Hypothesis, there is n such that $f(n) = (y-1, 0)$ and hence, $f(n+1) = (0, y)$.
  If $x \neq 0$ then by IH, there is n such that $f(n) = (x-1, y+1)$ and hence $f(n+1) = (x, y)$.

Now, if we take $f(0), f(1), f(2), \cdots$ in this order we get:

$$\underbrace{(0,0)}_{1}, \underbrace{(0,1),(1,0)}_{2}, \underbrace{(0,2),(1,1),(2,0)}_{3}, \underbrace{(0,3),(1,2),(2,1),(3,0)}_{4}, \cdots$$

This is the listing we saw in the proof of Theorem 4.3.6 which is also the first listing we gave in Remark 4.3.7.

Solution C.76. [Of Exercise 4.33.] Let $f : \mathbb{N}_{<n} \times \mathbb{N}_{<m} \mapsto \mathbb{N}_{<n \times m}$ such that $f(i, k) = k + i(m-1)$. We leave it to the reader to show that this is a bijection. Hence, $\mathbb{N}_{<n} \times \mathbb{N}_{<m}$ is finite and $|\mathbb{N}_{<n} \times \mathbb{N}_{<m}| = n \times m$.

Solution C.77. [Of Exercise 4.34.] Since S and T are non empty, let $x \in S$ and $y \in T$. Clearly there is a bijection between S and $S \times \{y\}$ respectively T and $\{x\} \times T$ and $S \times \{y\} \subseteq S \times T$ and $\{x\} \times T \subseteq S \times T$. If $S \times T$ is finite then by Lemma 4.1.8.4, $S \times \{y\}$ and $\{x\} \times T$ are finite and hence S and T are finite.
On the other hand, assume S and T are finite, then by definition, let f :

$S \mapsto \mathbb{N}_{<n}$ and $g : T \mapsto \mathbb{N}_{<m}$ be bijections. Now, $|S| = n$ and $|T| = m$. Let $h : S \times T \mapsto \mathbb{N}_{<n} \times \mathbb{N}_{<m}$ such that $h((x, y)) = (f(x), g(y))$.

It is easy to show that h is a bijection. By Exercise 4.34, there is a bijection between $\mathbb{N}_{<n} \times \mathbb{N}_{<m}$ and $\mathbb{N}_{<n\times m}$ and hence between $S \times T$ and $\mathbb{N}_{<n\times m}$ and so, $S \times T$ is finite and $S \times T| = n \times m$.

Solution C.78. [Of Exercise 4.38.] Let $g : \mathbb{N} \mapsto \mathbb{Q} \times \mathbb{Q}$ such that:

$$\begin{aligned} g(0) &= (q_0, q_0) \\ g(n) &= \begin{cases} (q_0, q_{k+1}) & \text{if } n \neq 0 \text{ and } g(n-1) = (q_k, q_0) \\ (q_{k+1}, q_{l-1}) & \text{if } n \neq 0,\ l \neq 0 \text{ and } g(n-1) = (q_k, q_l) \end{cases} \end{aligned}$$

We now leave it to the reader to show that g is a bijection.

Solution C.79. [Of Exercise 4.39.]

1. Since T is finite and S is infinitely countable, there are $m \in \mathbb{N}$, and bijections $f : T \mapsto \mathbb{N}_{<m}$ and $g : S \mapsto \mathbb{N}$. Let $h : T \cup S \mapsto \mathbb{N}$ such that $h(x) = \begin{cases} f(x) & \text{if } x \in T \\ g(x) + m & \text{if } x \in S. \end{cases}$

   Since $S \cap T = \emptyset$, h is a function and since f and g are bijections, we can easily prove that h is also a bijection.

2. Since T is finite and S is infinitely countable, there are $m \in \mathbb{N}$, and bijections $f : T \mapsto \mathbb{N}_{<m}$ and $g : S \mapsto \mathbb{N}$. Let $h : T \cup S \mapsto \mathbb{N}$ such that $h(x) = \begin{cases} f(x) & \text{if } x \in T \\ g(x) + m & \text{if } x \in S \setminus T. \end{cases}$

   h is a function and since f and g are bijections, we can easily prove that h is also a bijection.

3. Since T and S are infinitely countable, there are bijections $f : T \mapsto \mathbb{N}$ and $g : S \mapsto \mathbb{N}$. Let $h : T \cup S \mapsto \mathbb{N}$ such that $h(x) = \begin{cases} 2f(x) & \text{if } x \in T \\ 2g(x) + 1 & \text{if } x \in S. \end{cases}$

   h is a function and since f and g are bijections, we can easily prove that h is also a bijection.

4. There are two cases:

   - $S \setminus T$ is finite: Since $S \setminus T$ is finite and T is infinitely countable, there are $m \in \mathbb{N}$, and bijections $f : S \setminus T \mapsto \mathbb{N}_{<m}$ and

$g : T \mapsto \mathbb{N}$. Let $h : S \cup T = (S \setminus T) \cup T \mapsto \mathbb{N}$ such that $h(x) = \begin{cases} f(x) & \text{if } x \in S \setminus T \\ g(x) + m & \text{if } x \in T. \end{cases}$

Since $(S \setminus T) \cap T = \emptyset$, h is a function and since f and g are bijections, we can easily prove that h is also a bijection.

- $S \setminus T$ is infinitely countable: Since $S \setminus T$ and T are infinitely countable, there are bijections $f : S \setminus T \mapsto \mathbb{N}$ and $g : T \mapsto \mathbb{N}$. Let $h : S \cup T = (S \setminus T) \cup T \mapsto \mathbb{N}$ such that $h(x) = \begin{cases} 2f(x) & \text{if } x \in S \setminus T \\ 2g(x) + 1 & \text{if } x \in T. \end{cases}$

  Since $(S \setminus T) \cap T = \emptyset$, h is a function and since f and g are bijections, we can easily prove that h is also a bijection.

C.5 Solutions for Chapter 5

Solution C.80. [Of Exercise 5.1.] We prove 1., first:

- **Reflexive:** For any set S, the identity correspondence 1_S which associates to any element, the element itself is a one-to-one correspondence between S and S.

- **Symmetric:** For any sets S and T, if f is a one-to-one correspondence from S to T then the inverse f^{-1} is a one-to-one correspondence from T to S.

- **Transitive:** For any sets S, T and V, if f is a one-to-one correspondence from S to T and g is a one-to-one correspondence from T to V then the composition $g \circ f$ is a one-to-one correspondence from S to V.

Now we prove 2.

- For any set S, $f : S \mapsto S$ such that $f(x) = x$ is injective and hence $S \preceq S$.

- For any set S, T and V, If $f : S \mapsto T$ and $g : T \mapsto V$ are injective, then $g \circ f : S \mapsto V$ is injective by Lemma 4.2.23.3. Hence if $S \preceq T$ and $T \preceq V$ then $S \preceq V$.

- If $S \preceq T$ and $T \preceq S$ then there is an injection from S to T and an injection from T to S, and by Theorem 4.2.24, $S \sim T$.
  On the other hand, if $S \sim T$, say $f : S \mapsto T$ is bijective then there is obviously an injection f from S to T and an injection f^{-1} from T to S, and hence $S \preceq T$ and $T \preceq S$.

- If $S = \emptyset$ then $S \sim \emptyset$.
  The case of a bijection $f : S \mapsto \emptyset$ when $S \neq \emptyset$ is impossible since otherwise, there is $x \in S$ which has no image in $\emptyset$.

Solution C.81. [Of Exercise 5.2.]

1. First note that $\emptyset = \mathbb{N}_{<0}$ and hence by Definition 4.1.6, $\#\emptyset = 0$. Now, by Definition 5.1.4, $\#S = \#\emptyset$ iff $S \sim \emptyset$ iff (by Lemma 5.1.3) $S = \emptyset$.

2. If $\#S = \#T$ then $S \sim T$ and hence by Lemma 4.3.1.1 either S and T are both finite or they are both infinite.

3. Since $\#S = \#T$ iff $S \sim T$, and by Theorem 5.1.3, $\sim$ is an equivalence relation, we can easily deduce that $=$ is an equivalence relation on cardinal numbers. We only show the transitive case:
   If $\#S = \#T$ and $\#T = \#U$ then $S \sim T$ and $T \sim U$ and hence $S \sim U$ and so, $\#S = \#U$.

4. Since $\#S \leq \#T$ iff $S \preceq T$, and by Theorem 5.1.3, $\preceq$ is reflexive and transitive, we can easily deduce that $\leq$ is reflexive and transitive on cardinal numbers. We only show the transitive case:
   If $\#S \leq \#T$ and $\#T \leq \#U$ then $S \preceq T$ and $T \preceq U$ and hence $S \preceq U$ and so, $\#S \leq \#U$.

5. Since by Theorem 5.1.3, $S \preceq T$ and $T \preceq S$ iff $S \sim T$, and since $\#S \leq \#T$ iff $S \preceq T$, and $\#S = \#T$ iff $S \sim T$, we can easily show that $\#S = \#T$ iff ($\#S \leq \#T$ and $\#T \leq \#S$).

6. $\#S \leq \#T$ iff $S \preceq T$ iff there is an injection $f : S \mapsto T$.

   - If there is an injection $f : S \mapsto T$ then there is a bijection $f : S \mapsto f[S] \subseteq T$ and hence there is a bijection from S to a subset of T and hence S is equivalent to a subset of T.
   - If S is equivalent to a subset of T' of T then let $f : S \mapsto T'$ be a bijection, then $f : S \mapsto T$ is an injection.

   Hence, there is an injection $f : S \mapsto T$ iff S is equivalent to a subset of T. So, $\#S \leq \#T$ iff S is equivalent to a subset of T.

7. By the third item above, $\#S = \#T$ iff ($\#S \leq \#T$ and $\#T \leq \#S$).
   Hence, $\#S \neq \#T$ iff ($\#S \not\leq \#T$ or $\#T \not\leq \#S$).
   Now, $\#S < \#T$ iff
   $\#S \leq \#T$ and $\#S \neq \#T$ iff
   $\#S \leq \#T$ and ($\#S \not\leq \#T$ or $\#T \not\leq \#S$) iff
   ($\#S \leq \#T$ and $\#S \not\leq \#T$) or ($\#S \leq \#T$ and $\#T \not\leq \#S$) iff
   ($\#S \leq \#T$ and $\#T \not\leq \#S$) iff by above item
   S is equivalent to a subset of T and T is not equivalent to a subset of S.

Solution C.82. [Of Exercise 5.3.] Since S and T are infinitely countable sets then there is a one-to-one correspondence between $\mathbb{N}$ and S and a one-to-one correspondence between $\mathbb{N}$ and T. Hence, there is a one-to-one correspondence between S and T and so, $\#S = \#T = \mathfrak{a}$.
Since $\mathbb{N}_{<n} \subset \mathbb{N} \subset \mathbb{R}$ then $\mathbb{N}_{<n} \preceq \mathbb{N} \preceq \mathbb{R}$ and hence $n = \#\mathbb{N}_{<n} \leq \#\mathbb{N} \leq \#\mathbb{R}$.
Since $\mathbb{R}$ is uncountable then $\mathbb{R} \not\sim \mathbb{N}$ and hence $\#\mathbb{R} \neq \#\mathbb{N}$ and so $\#\mathbb{N} < \#\mathbb{R}$.

Furthermore, since $\mathbb{N}$ is infinite and $\mathbb{N}_{<n}$ is finite, $\#\mathbb{N} \neq \#\mathbb{N}_{<n}$ and so, $n < \#\mathbb{N}$.
So, $n < \mathfrak{a} < \mathfrak{c}$.

Solution C.83. [Of Exercise 5.4.] Let $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$. Obviously $S \subseteq (0,1]$. Let $f : (0,1] \mapsto (0,1)$ such that $f(x) = \begin{cases} x & \text{if } x \in (0,1] \setminus S \\ \dfrac{1}{n+1} & \text{if } x = \dfrac{1}{n} \text{ where } n \in \mathbb{N}^+ \end{cases}$

It is easy to show that f is a one-to-one correspondence.
Now, to give a one-to-one correspondence between $[0,1]$ and $(0,1)$, let $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$. Obviously $S \subseteq [0,1]$. Let $f : [0,1] \mapsto (0,1)$ such that

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \setminus S \\ \frac{1}{2} & \text{if } x = 0 \\ \dfrac{1}{n+2} & \text{if } x = \dfrac{1}{n} \text{ where } n \in \mathbb{N}^+ \end{cases}$$

It is easy to show that g is a one-to-one correspondence.

Solution C.84. [Of Exercise 5.5.] You could also use a different proof as follows: By Theorem 4.1.10, $\mathbb{Q}$ is infinitely countable and hence $\mathbb{Q} \sim \mathbb{N}$.

By Lemma 4.2.23.5, if $S \sim T$ then $\mathcal{P}S \sim \mathcal{P}T$, and hence, $\mathcal{P}\mathbb{Q} \sim \mathcal{P}\mathbb{N}$.

By the proof of Theorem 5.1.11, we have an injection $f : \mathcal{P}\mathbb{N} \mapsto [0,1]$ and since $[0,1] \subset \mathbb{R}$ then we have an injection $f : \mathcal{P}\mathbb{N} \mapsto \mathbb{R}$.

Also, by the proof of Theorem 5.1.11, we have and an injection $g : \mathbb{R} \mapsto \mathcal{P}\mathbb{Q}$ and since $\mathcal{P}\mathbb{Q} \sim \mathcal{P}\mathbb{N}$ then by Lemma 4.2.23.3, we have an injection $h : \mathbb{R} \mapsto \mathcal{P}\mathbb{N}$.

Since $f : \mathcal{P}\mathbb{N} \mapsto \mathbb{R}$ and $h : \mathbb{R} \mapsto \mathcal{P}\mathbb{N}$ are injections, then by Theorem 4.2.24, $\mathbb{R} \sim \mathcal{P}\mathbb{N}$.

Solution C.85. [Of Exercise 5.6.] Since $\mathbb{Q}$ is countable then $\mathbb{Q} \cap [0,1]$ is countable. Since the infinite set $\{\frac{1}{n} : n \in \mathbb{N}^+\} \subseteq \mathbb{Q} \cap [0,1]$, we have $\mathbb{Q} \cap [0,1]$ is infinite. Hence, let $r : \mathbb{N}^+ \mapsto \mathbb{Q} \cap [0,1]$ be a one-to-one correspondence and denote $r(n)$ by r_n for ech $n \in \mathbb{N}^+$ ($r_1, r_2, \cdots$ is a listing of $\mathbb{Q} \cap [0,1]$). Take the infinite countable set $S = \{\frac{1}{\sqrt{n}} : n \in \mathbb{N}^+ \text{ and } \sqrt{n} \text{ is irrational}\}$. Similarly to $\mathbb{Q} \cap [0,1]$, let $s_1, s_2, \cdots$ be a listing of S. Now, let the one-to-one correspondence $g : S \cup (\mathbb{Q} \cap [0,1]) \mapsto S$ such that $g(x) = \begin{cases} s_{2n} & \text{if } x = r_n \\ s_{2n-1} & \text{if } x = s_n \end{cases}$

Obviously g is a one-to-one correspondence.

Let $f : [0,1] \mapsto [0,1] \setminus \mathbb{Q}$ such that $f(x) = \begin{cases} g(x) & \text{if } x \in S \cup (\mathbb{Q} \cap [0,1]) \\ x & \text{otherwise} \end{cases}$

It is easy to show that f is a bijection.

Solution C.86. [Of Exercise 5.7.] Let $f : [-1, 1] \mapsto [a, b]$ such that $f(x) = \frac{a(1-x)+b(1+x)}{2}$. We show that f is bijective.

- If $f(x) = f(y)$ then $b(1+x)+a(1-x) = b(1+y)+a(1-y)$ and hence $b(x-y) = a(x-y)$.
  If $x \neq y$ then $a = b$ which is a contradiction. Hence $x = y$ and f is injective.

- Let $y \in [a, b]$ and $x = \frac{2y-(b+a)}{b-a}$. Now, $x \in [-1, 1]$ because:

  $$\begin{array}{lllllll} a & \leq & y & \leq & b & \text{implies} \\ 2a & \leq & 2y & \leq & 2b & \text{implies} \\ 2a-(b+a) & \leq & 2y-(b+a) & \leq & 2b-(b+a) & \text{implies} \\ -(b-a) & \leq & 2y-(b+a) & \leq & b-a & \text{implies} \\ -1 & \leq & \frac{2y-(b+a)}{b-a} & \leq & 1 & \text{since } b-a \neq 0 \end{array}$$

  Furthermore, f is a surjection because $f(x) = y$ can be seen as follows:

  $$\begin{array}{lll} f(x) & = & \frac{a}{2}(1 - \frac{2y-(b+a)}{b-a}) + \frac{b}{2}(1 + \frac{2y-(b+a)}{b-a}) \\ & = & \frac{a}{2(b-a)}(b-a-2y+b+a) + \frac{b}{2(b-a)}(b-a+2y-b-a) \\ & = & \frac{1}{2(b-a)}(2ab-2ay) + \frac{1}{2(b-a)}(2by-2ab) \\ & = & \frac{1}{2(b-a)}(2y(b-a)) \\ & = & y. \end{array}$$

Solution C.87. [Of Exercise 5.8.] Since $S \sim T$, let $f : S \mapsto T$ a one-to-one correspondence between S and T. It is easy to show that $g : S \cup R \mapsto T \cup R$ defined by $g(x) = \begin{cases} f(x) & \text{if } x \in S \\ x & \text{if } x \in R \end{cases}$ is a one-to-one correspondence between $S \cup R$ and $T \cup R$. Hence, $S \cup R \sim T \cup R$.

Solution C.88. [Of Exercise 5.9.] Since $S \sim T$ and R and $S \cup T$ are disjoint, then by Exercise 5.8, $S \cup R \sim T \cup R$. Similarly, since $R \sim U$ and T and $R \cup U$ are disjoint, then by Exercise 5.8, $T \cup R = R \cup T \sim U \cup T = T \cup U$. Since $\sim$ is transitive and $S \cup R \sim T \cup R \sim T \cup U$, we get $S \cup R \sim T \cup U$.

Solution C.89. [Of Exercise 5.10.] If S is finite, nothing to prove. Assume S is infinite. Since $S \preceq \mathbb{N}$, then $\#S \leq \#\mathbb{N} = \mathfrak{a}$. But also by Lemma 4.3.1.5, there is $S' \subseteq S$ such that S' and $\mathbb{N}$ are in one-to-one correspondence. So, $\#S' = \#\mathbb{N} = \mathfrak{a}$. But, $\#S' \leq \#S$. Hence, $\mathfrak{a} \leq \#S \leq \mathfrak{a}$ and therefore, $\#S = \mathfrak{a}$.

Solution C.90. [Of Exercise 5.11.] Φ is a function because for every $f \in R^{S \cup T}$, there is a unique pair $(g, h) \in R^S \times R^T$ such that $g(x) = f(x)$ for

every $x \in S$ and $h(x) = f(x)$ for every $x \in T$.
Φ is an injection because if $\Phi(f) = \Phi(f')$ where

- $\Phi(f) = (g, h) \in R^S \times R^T$ where $g(x) = f(x)$ for every $x \in S$ and $h(x) = f(x)$ for every $x \in T$ and
- $\Phi(f') = (g', h') \in R^S \times R^T$ where $g'(x) = f(x)$ for every $x \in S$ and $h'(x) = f(x)$ for every $x \in T$,

then since for all $x \in S$, $g(x) = g'(x)$ we have $g = g'$ and also since for all $x \in T$, $h(x) = h'(x)$ we have $h = h'$ and so, $(g, h) = (g', h')$.
Φ is a surjection because for any $(g, h) \in R^S \times R^T$, we can define $f \in R^{S \cup T}$ such that $f(x) = \begin{cases} g(x) & \text{if } x \in S \\ h(x) & \text{if } x \in T. \end{cases}$
Since $S \cap T = \emptyset$, we can show that f is a function and $f \in R^{S \cup T}$. Hence, $\Phi(f) = (g, h)$.

Solution C.91. [Of Exercise 5.12.] Assume $\rho(f) = \rho(f')$ where $f, f' \in S^T$. Let $x \in T$. Then, $\rho(f)(\psi(x)) = \rho(f')(\psi(x))$ and so, $\phi(f(\psi^{-1}(\psi(x)))) = \phi(f'(\psi^{-1}(\psi(x))))$. But, since ψ is bijective, then by Lemma 4.2.23.2, $\psi^{-1}(\psi(x)) = x$ and so, $\phi(f(x)) = \phi(f'(x))$. But, ϕ is injective and hence $f(x) = f'(x)$. Therefore, $f = f'$ and ρ is injective.
Now, let $g \in S'^{T'}$ and let $f = \phi^{-1} \circ g \circ \psi$. Recall that $\phi \circ \phi^{-1} = 1_{S'}$ and $\psi \circ \psi^{-1} = 1_{T'}$. Then, $f \in S^T$ and for any $x \in T'$, $\rho(f)(x) = \rho(\phi^{-1} \circ g \circ \psi)(x) = \phi((\phi^{-1} \circ g \circ \psi)(\psi^{-1}(x))) = \phi(\phi^{-1} \circ g(\psi(\psi^{-1}(x))) = \phi(\phi^{-1} \circ g(x)) = \phi(\phi^{-1}(g(x))) = g(x)$. Hence, $\rho(f) = g$ and ρ is surjective.

Solution C.92. [Of Exercise 5.13.] Assume $\Phi(f) = \Phi(f')$ for $f, f' \in (R \times S)^T$ and let $x \in T$. Then, $\Phi(f) = (f_1, f_2) = (f'_1, f'_2) = \Phi(f')$ and $f(x) = (f_1(x), f_2(x)) = (f'_1(x), f'_2(x)) = f'(x)$. Hence, $f = f'$ and Φ is injective.
On the other hand, let $(g, h) \in R^T \times S^T$. We construct $f \in (R \times S)^T$ as follows: for any $x \in T$, we let $f(x) = (g(x), h(x))$. Clearly, $\Phi(f) = (g, h)$ and hence Φ is surjective.

Solution C.93. [Of Exercise 5.14.] Assume $\Phi(f) = \Phi(f') = h$ for $f, f' \in (R^S)^T$. We need to show that $f = f'$. We will show that for any $x \in T$, $f(x) = f'(x)$. Assume $x \in T$. For any $y \in S$, $(y, x) \in S \times T$ and $h((y, x)) = f(x)(y) = f'(x)(y)$. Since for any $y \in S$, $f(x)(y) = f'(x)(y)$, we conclude by function extensionality that $f(x) = f'(x)$. This is for any $x \in T$ and hence, by function extensionality $f = f'$ and Φ is injective.
On the other hand, let $h \in R^{S \times T}$. We construct $f \in (R^S)^T$ as follows: for any $x \in T$, we let $f(x) \in R^S$ such that for any $y \in S$, $f(x)(y) = h((y, x))$. Clearly, $\Phi(f) = h$ and hence Φ is surjective.

Solution C.94. [Of Exercise 5.15.] Let $\Phi(T) = \Phi(T')$. Then $\xi_T = \xi_{T'}$. Now,
$x \in T$ iff $\xi_T(x) = 1$ iff $\xi_{T'}(x) = 1$ iff $x \in T'$. Hence $T = T'$ and Φ is injective. On the other hand, let $g \in \{0,1\}^S$. Take $T = \{x \in S : g(x) = 1\}$. Clearly $g = \xi_T = \Phi(T)$ and hence, Φ is surjective.

Solution C.95. [Of Exercise 5.16.] Let $X \in S$. Then, by hypothesis there is $Y \in S$ such that $\#X < \#Y$. But, since $Y \subseteq \bigcup S$, we have $\#Y \leq \#\bigcup S$. Hence, $\#X < \#\bigcup S$.

Solution C.96. [Of Exercise 5.17.] Recall that $\#\mathbb{N} = \mathfrak{a}$ and by Theorems 5.1.14 and 5.2.21, for any set S, $\#\mathcal{P}S > \#S$, and $\#\mathcal{P}S = 2^{\#S}$. Hence, $2^{\mathfrak{a}} > \mathfrak{a}$ and hence $\mathfrak{a} \leq 2^{\mathfrak{a}}$.

Now, since $2 \leq \mathfrak{a}$ and $\mathfrak{a} \leq 2^{\mathfrak{a}}$, then by Theorem 5.2.23, $2^{\mathfrak{a}} \leq \mathfrak{a}^{\mathfrak{a}}$ and $\mathfrak{a}^{\mathfrak{a}} \leq (2^{\mathfrak{a}})^{\mathfrak{a}} =^{Cor\ 5.2.20} 2^{\mathfrak{a}\mathfrak{a}} =^{The\ 5.2.9.1} 2^{\mathfrak{a}}$.
Hence, $\mathfrak{a}^{\mathfrak{a}} = 2^{\mathfrak{a}}$.

Solution C.97. [Of Exercise 5.18.]

1.
   - For $n = 0$, $U_0 = \{\emptyset\}$ and $\#U_0 = 1$.
   - For $n = 1$, $U_1 = \{S \subseteq \mathbb{N} : S \sim \mathbb{N}_{<1}\} = \{S \subseteq \mathbb{N} : |S| = 1\} = \{S \subseteq \mathbb{N} : S = \{m\}$ where $m \in \mathbb{N}\}$. Let $f : U_1 \mapsto \mathbb{N}$ such that $f(\{m\}) = m$. It is easy to prove that f is a bijection and hence $\#U_1 = \mathfrak{a}$.
   - For $n = 2$, $U_2 = \{S \subseteq \mathbb{N} : S \sim \mathbb{N}_{<2}\} = \{S \subseteq \mathbb{N} : |S| = 2\} = \{S \subseteq \mathbb{N} : S = \{m, p\}$ where $m, p \in \mathbb{N}\}$.
     Let $f : U_2 \mapsto \mathbb{N} \times \mathbb{N}$ such that $f(\{m,p\}) = (m,p)$ where $m \leq p$. It is easy to prove that f is an injection and hence $\#U_2 \leq \#(\mathbb{N} \times \mathbb{N}) = \mathfrak{a}$.
     Let $g : \mathbb{N} \mapsto U_2$ such that $g(n) = \{n, n+1\}$. Then, g is injective. Hence, $\mathfrak{a} = \#\mathbb{N} \leq \#U_2$.

     That is: $\mathfrak{a} \leq \#U_2 \leq \mathfrak{a}$ and so, $\#U_2 = \mathfrak{a}$.
   - Assume that for $n \geq 1$ that $\#U_n = \mathfrak{a}$, we will show that $\#U_{n+1} = \mathfrak{a}$.

     Let $f : U_{n+1} \mapsto \mathbb{N} \times U_n$ such that for $S \in U_{n+1}$, $f(S) = (k, S \setminus \{k\})$ where k is the smallest natural in S. It is easy to prove that f is an injection and hence $\#U_{n+1} \leq \#(\mathbb{N} \times U_n) = \mathfrak{a}\mathfrak{a} = \mathfrak{a}$.
     Let $g : \mathbb{N} \mapsto U_{n+1}$ such that $g(k) = \{k, k+1, \cdots, k+n\}$. Then, g is injective. Hence, $\mathfrak{a} = \#\mathbb{N} \leq \#U_{n+1}$.

     That is: $\mathfrak{a} \leq \#U_{n+1} \leq \mathfrak{a}$ and so, $\#U_{n+1} = \mathfrak{a}$.

2. Clearly, $U = \bigcup_{n\in\mathbb{N}} U_n$. By Theorem 4.3.9 and the first item above, U is countable. Since U contains an infinite number of elements like $\{n\}$ where $n \in \mathbb{N}$, U is infinite. Hence, U is countably infinite and $\#U = \mathfrak{a}$.

3. Clearly $\mathcal{P}\mathbb{N} = U \cup V$. Hence, $\#\mathcal{P}\mathbb{N} = \#U + \#V$. By Theorem 5.2.21 and Corollary 5.2.22, $\#\mathcal{P}\mathbb{N} = 2^{\#\mathbb{N}} = \mathfrak{c}$. By the above item, $\#U = \mathfrak{a}$. Hence, $\mathfrak{c} = \mathfrak{a} + \#V$. Now:

   - By Theorem 5.2.2, $\#V = 0 + \#V \leq \mathfrak{a} + \#V = \mathfrak{c}$.
   - Note that V is infinite because for any $n \in \mathbb{N}$, $n\mathbb{N} = \{nk : k \in \mathbb{N}\} \in V$ and for any $n, m \in \mathbb{N}$ where $n \neq m$ we have $n\mathbb{N} \neq m\mathbb{N}$. Hence, $\mathfrak{a} \leq \#V$.
   - $\mathfrak{a} < \#V$, because if $\mathfrak{a} = \#V$ then $\mathfrak{c} = \mathfrak{a} + \#V = \mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ absurd.
   - Hence, $\mathfrak{a} < \#V \leq \mathfrak{c}$ and so, $\#V = \mathfrak{c}$.

Solution C.98. [Of Exercise 5.19.]

1. Let $S = \mathbb{N}$, $T = \mathbb{N}^+$, $S' = \mathbb{N}$, $T' = \{n \in \mathbb{N} : n \text{ is even}\}$. Then $S \setminus T = \{0\}$ and $S' \setminus T' = \{n \in \mathbb{N} : n \text{ is odd}\}$, $T \subseteq S$, $\#S = \#T = \#S' = \#T' = \#(S' \setminus T') = \mathfrak{a}$ but $\#(S \setminus T) = 1$. So $\#(S \setminus T) \neq \#(S' \setminus T')$.

2. Assume $S \sim S'$, $T \sim T'$, $T \subseteq S$, $T' \subseteq S'$ and $\#S > \#T$ then since $T \cap (S \setminus T) = T' \cap (S' \setminus T') = \emptyset$ and $S = (S \setminus T) \cup T$ and $S' = (S' \setminus T') \cup T'$, we have $\#S = \#(S \setminus T) \cup \#T$ and $\#S' = \#(S' \setminus T') \cup \#T'$. Since $\#S = \#S'$ and $\#T = \#T'$, then $\#(S \setminus T) = \#(S' \setminus T')$.

3. Left to the reader.

4. Left to the reader.

Solution C.99. [Of Exercise 5.20.] For all $x \in S$, let $X_x = \{y \in T : f(y) = x\}$ and let $U = \{X_x : x \in S\}$. Since $S \neq \emptyset$ and f is a surjection, by the axiom of choice, we can choose for each $X_x \in U$, a unique $y_x \in X_x$ such that $f(y_x) = x$.

Let $g : S \mapsto T$ such that $g(x) = y_x$. It is easy to show that g is injective.

Solution C.100. [Of Exercise 5.21.] Since $S \neq \emptyset$, let $a \in S$. Let $y \in T$. If $y = f(x)$ for $x \in S$, then this x is unique because f is injective. Hence, let $g : T \mapsto S$ such that $g(y) = \begin{cases} x & \text{if there is } x \in S \text{ such that } y = f(x) \\ a & \text{otherwise} \end{cases}$

It is easy to show that g is surjective.

Solution C.101. [Of Exercise 5.22.] Let $I = \{n \in \mathbb{N} : \#\overline{n} = n\}$. Use the induction axiom to show that $I = \mathbb{N}$. Hence, $\overline{n}$, $\#\overline{n} = n$.

This is not a good definition since for each $n \geq 1$, it defines $\overline{n}$ in terms of $\overline{n}$. This is not well founded. Instead, we can change the definition as follows:

$\overline{0} = \{\}$, $\overline{1} = \{\overline{0}\}$, $\overline{2} = \{\overline{0}, \overline{1}\}$, $\overline{3} = \{\overline{0}, \overline{1}, \overline{2}\}$, $\cdots$, for $n \geq 1$, $\overline{n} = \{\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{n-1}\}$.

C.6 Solutions for Chapter 6

Solution C.102. [Of Exercise 6.1.]

1. A_{100} is a wff and an atomic wff.
2. A_1 is a wff and an atomic wff.
3. $A_3 \vee A_5$ is a wff but not an atomic wff.
4. $(A_3 \vee A_5)$ is a wff but not an atomic wff.
5. $A_3 \vee A_5)$ is neither a wff nor an atomic wff.
6. $(A_3 \vee A_5$ is neither a wff nor an atomic wff.
7. $A_0\wedge$ is neither a wff nor an atomic wff.
8. $\neg A_0 \wedge A_1 \vee A_2$ is neither a wff nor an atomic wff since it is ambiguous as to which is grouped with which.
9. $\neg A_0 \wedge (A_1 \vee A_2)$ is a wff but not an atomic wff.
10. $(\neg A_0 \wedge A_1) \vee A_2$ is a wff but not an atomic wff.
11. $\neg A_0 \backsim (A_1 \vee A_2)$ is a wff but not an atomic wff.

If we take the case where all of A_0, A_1, A_2, A_3, A_5 and A_{100} are false then all the wffs in this exercise are false.
If we take the case where all of A_1, A_2, A_3, A_5 and A_{100} are true but A_0 is false then all the wffs in this exercise are true.

Solution C.103. [Of Exercise 6.2] Let $A \supset B$ be Φ and $B \supset A$ be Ψ. Here is a truth table for the desired formulas.

A	B	Φ	Ψ	$\Phi \wedge \Psi$	$\Phi \vee \Psi$	$\neg\Phi$	$\neg\Psi$	$\neg\Phi \wedge \neg\Psi$
T	T	T	T	T	T	F	F	F
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	F	T	F
F	F	T	T	T	T	F	F	F

$(A \supset B) \vee (B \supset A)$ is a tautology and $\neg(A \supset B) \wedge \neg(B \supset A)$ is a contradiction, whereas $(A \supset B) \wedge (B \supset A)$ is neither.

Solution C.104. [Of Exercise 6.3] Let $A \supset B$ be Φ and $\neg B \supset \neg A$ be Ψ. Here is a truth table for the desired formulas:

A	B	Φ	$\neg A$	$\neg B$	Ψ	$\Phi \backsim \Psi$	$\neg A \backsim \Phi$	$B \backsim \Phi$
T	F	F	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	T	T	T	T
F	F	T	T	F	T	T	T	T

All the formulas given in this exercise are tautologies.

Solution C.105. [Of Exercise 6.4] The proofs are all truth tables.

1.

A	$A \wedge A$	$A \wedge A \backsim A$
T	T	T
F	F	T

2.

A	B	$A \wedge B$	$B \wedge A$	$A \wedge B \backsim B \wedge A$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

3. Let Φ be $A \wedge (B \wedge C) \backsim (A \wedge B) \wedge C$.

A	B	C	$B \wedge C$	$A \wedge (B \wedge C)$	$A \wedge B$	$(A \wedge B) \wedge C$	Φ
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	T	F	F	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T

Solution C.106. [Of Exercise 6.5] The proofs are all truth tables.

1. Let Φ be $A \wedge (B \vee C) \backsim (A \wedge B) \vee (A \wedge C)$.

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee$ $(A \wedge C)$	Φ
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	T
T	F	T	T	T	F	T	T	T
T	F	F	F	F	F	F	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	F	F	F	T
F	F	T	T	F	F	F	F	T
F	F	F	F	F	F	F	F	T

2. Let Φ be $A \vee (B \wedge C) \backsim (A \vee B) \wedge (A \vee C)$. Then, Φ is a tautology.

A	B	C	$B \wedge C$	$A \vee (B \wedge C)$	$A \vee B$	$A \vee C$	$(A \vee B) \wedge$ $(A \vee C)$	Φ
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

Solution C.107. [Of Exercise 6.6.] Let A be $x \in T$ and B be $x \in R$. Below we will use the tautologies $x \in S \backsim x \in S \wedge x \in S$, $\neg(A \vee B) \backsim \neg A \wedge \neg B$, Then

$$
\begin{aligned}
x \in S \setminus (T \cup R) &\Leftrightarrow x \in S \text{ and } x \notin (T \cup R) \\
&\Leftrightarrow x \in S \text{ and not } x \in (T \cup R) \\
&\Leftrightarrow x \in S \text{ and not } ((x \in T) \text{ or } (x \in R)) \\
&\Leftrightarrow x \in S \text{ and not } (A \text{ or } B) \\
&\Leftrightarrow x \in S \text{ and } \neg(A \vee B) \\
&\Leftrightarrow x \in S \text{ and } \neg A \wedge \neg B \\
&\Leftrightarrow (x \in S \wedge x \in S) \wedge \neg A \wedge \neg B \\
&\Leftrightarrow (x \in S \wedge \neg A) \wedge (x \in S \wedge \neg B) \\
&\Leftrightarrow (x \in S \wedge x \notin T) \wedge (x \in S \wedge x \notin R) \\
&\Leftrightarrow (x \in S \setminus T) \wedge (x \in S \setminus R) \\
&\Leftrightarrow x \in (S \setminus T) \cap (S \setminus R).
\end{aligned}
$$

Solution C.108. [Solution of Exercise 6.7.]

1. $\Sigma_{k=0}^{k=0} 0 \times 0! = 0 = (0+1)! - 1$. Hence the property holds for 0.

2. Assume IH which is that the property holds for $n \in \mathbb{N}$. We will show the property for $n+1$.
   $\Sigma_{k=0}^{k=n+1} k \times k! = \Sigma_{k=0}^{k=n} k \times k! + (n+1) \times (n+1)! =^{IH} (n+1)! - 1 + (n+1) \times (n+1)! = (n+1)! \times (n+2) - 1 = (n+2)! - 1$.

Hence by induction, for every $n \in \mathbb{N}$, $\Sigma_{k=0}^{k=n} k \times k! = (n+1)! - 1$.

Solution C.109. [Solution of Exercise 6.8.]

1. If $n = 0$ then $a_2 = a_1 + a_0 = 1 = 1 + 0 = 1 + \Sigma_{k=0}^{k=0} a_k$.

2. Assume IH which is that the property holds for all $i \leq n \in \mathbb{N}$. We will show the property for $n+1$.
   $a_{n+1+2} = a_{n+1} + a_{n+2} =^{IH} a_{n+1} + 1 + \Sigma_{k=0}^{k=n} a_k = 1 + \Sigma_{k=0}^{k=n+1} a_k$.

Hence, by strong induction, for all $n \in \mathbb{N}$,

$$a_{n+2} = 1 + \Sigma_{k=0}^{k=n} a_k.$$

C.7 Solutions for Chapter 7

Solution C.110. [Of Exercise 7.2.]

1. Applying repeated/alternate subtraction (anthyphairesis) to 7 and 5 gives:

	r_i	$=$	q_i	$\times$	r_{i+1}	$+$	r_{i+2}.
	7	$=$	1	$\times$	5	$+$	2.
$i=0$	r_0	$=$	q_0	$\times$	r_1	$+$	r_2.
	5	$=$	2	$\times$	2	$+$	1.
$i=1$	r_1	$=$	q_1	$\times$	r_2	$+$	r_3.
	2	$=$	2	$\times$	1	$+$	0.
$i=2$	r_2	$=$	q_2	$\times$	r_3	$+$	r_4.

Since the one before the final r in the series is 1 ($r_3 = 1$), 7 and 5 are relatively prime.

2. Applying repeated/alternate subtraction (anthyphairesis) to 212 and 24 gives:

	r_i	$=$	q_i	$\times$	r_{i+1}	$+$	r_{i+2}.
	212	$=$	8	$\times$	24	$+$	20.
$i=0$	r_0	$=$	q_0	$\times$	r_1	$+$	r_2.
	24	$=$	1	$\times$	20	$+$	4.
$i=1$	r_1	$=$	q_1	$\times$	r_2	$+$	r_3.
	20	$=$	5	$\times$	4	$+$	0.
$i=2$	r_2	$=$	q_2	$\times$	r_3	$+$	r_4.

Since the one before the final r in the series is 1 ($r_3 = 4 \neq 1$), it is the GCD of 212 and 24.

Solution C.111. [Of Exercise 7.3.] Assume $b < a$ and let $r_0 = a$ and $r_1 = b$. Using anthyphairesis to calculate the GCD of a, b goes as follows:

	r_i	$=$	q_i	$\times$	r_{i+1}	$+$	r_{i+2}.	$[q_0, \ldots, q_i]$
$i=0$	r_0	$=$	q_0	$\times$	r_1	$+$	r_2.	$[q_0]$
$i=1$	r_1	$=$	q_1	$\times$	r_2	$+$	r_3.	$[q_0, q_1]$
::	::	::	::	::	::	::	::	::
$i=n$	r_n	$=$	q_n	$\times$	r_{n+1}	$+$	r_{n+2}.	$[q_0, q_1, \ldots, q_n]$

The process stops when $r_{n+2} = 0$. If $r_{n+1} = 1$ then a and b are relatively prime to one another. Else, r_{n+1} is the GCD of a, b.

Using anthyphairesis to calculate the GCD of ma, mb goes as follows:

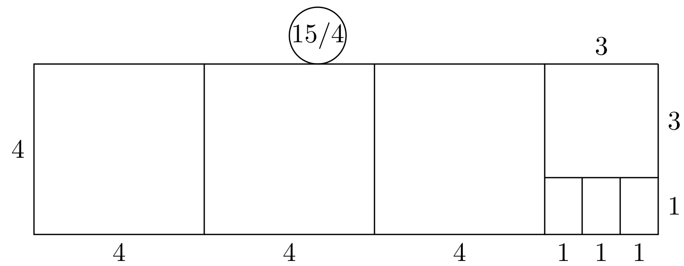
	mr_i	$=$	q_i	$\times$	mr_{i+1}	$+$	$mr_{i+2}.$	$[q_0, \ldots, q_i]$
$i = 0$	mr_0	$=$	q_0	$\times$	mr_1	$+$	$mr_2.$	$[q_0]$
$i = 1$	mr_1	$=$	q_1	$\times$	mr_2	$+$	$mr_3.$	$[q_0, q_1]$
::	::	::	::	::	::	::	::	::
$i = n$	mr_n	$=$	q_n	$\times$	mr_{n+1}	$+$	$mr_{n+2}.$	$[q_0, q_1, \ldots, q_n]$

Again here, the process stops when $mr_{n+2} = 0$. Since $mr_{n+1} \neq 1$, a and b are not relatively prime to one another and mr_{n+1} is the GCD of ma, mb.

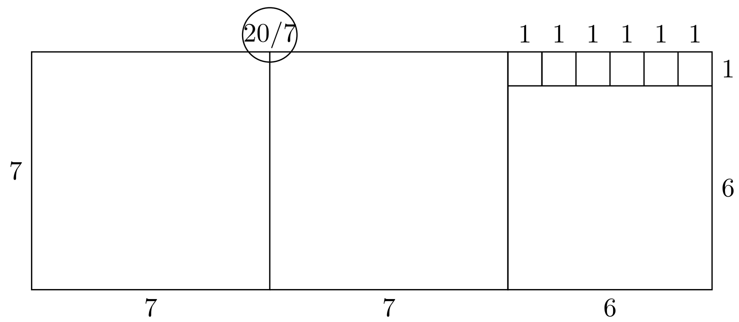
Hence we see that the characterising sequences for a/b and ma/mb are the same, but at every stage, the remainders of ma/mb are m times the remainders of a/b. The GCD of ma/mb is m times that of a/b if a, b have a GCD. Else it is m.

Solution C.112. [Of Exercise 7.4.]

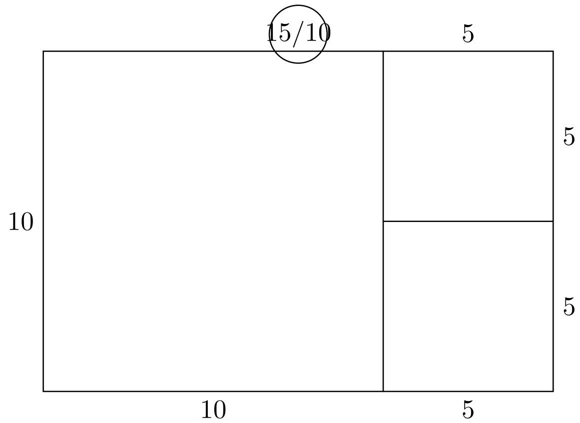
1) The ratio of 15 to 4 is characterised by the sequence $[3, 1, 3]$ as is shown in the geometric diagram below:



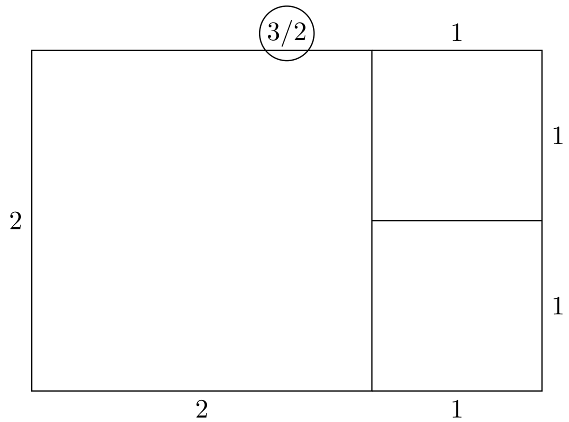
2) The ratio of 20 to 7 is characterised by the sequence $[2, 1, 6]$ as is shown in the geometric diagram below:



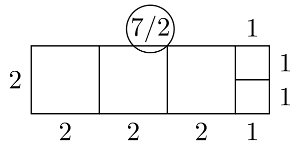
3) The ratio of 15 to 10 is characterised by the sequence [1, 2] as is shown in the geometric diagram below:



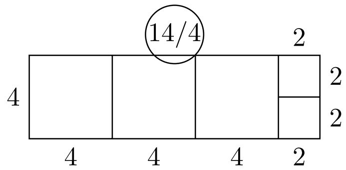
4) The ratio of 3 to 2 is characterised by the sequence [1, 2] as is shown in the geometric diagram below:



5) The ratio of 7 to 2 is characterised by the sequence [3, 2] as is shown in the geometric diagram below:



6) The ratio of 14 to 4 is characterised by the sequence [3, 2] as is shown in the geometric diagram below:



Solution C.113. [Of Exercise 7.5.]

1. The ratio of $\sqrt{3}$ to 1 is calculated as follows:

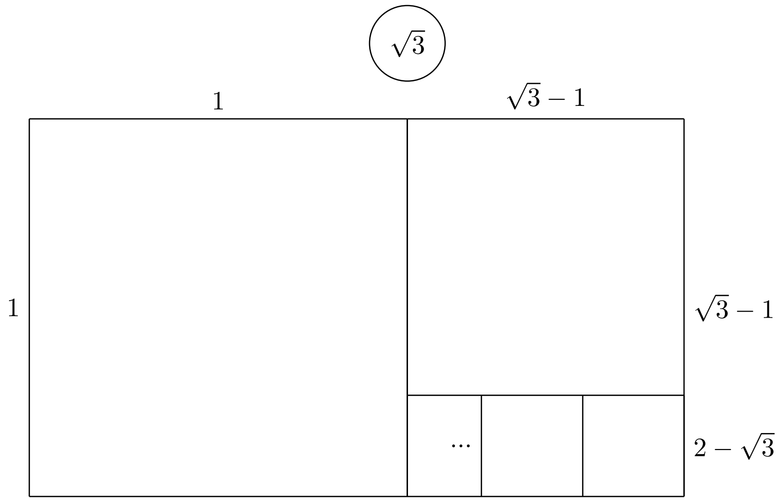


Figure C.1: Ratio of $\sqrt{3}$ to 1

- Let $r_0 = \sqrt{3}$ and $r_1 = 1$. Since $1 < \sqrt{3} < 2$ then $0 < \sqrt{3} - 1 < 1$. Let $q_0 = 1$ and $r_2 = \sqrt{3} - 1$. Note $0 < r_2 < r_1$. We have

$$r_0 = q_0 \times r_1 + r_2 \quad \text{or} \quad \frac{r_0}{r_1} = q_0 + \frac{r_2}{r_1}.$$

$$\sqrt{3} = 1 + (\sqrt{3} - 1) \quad \text{or} \quad \frac{\sqrt{3}}{1} = 1 + \frac{\sqrt{3} - 1}{1} = 1 + \cfrac{1}{\cfrac{1}{\sqrt{3} - 1}}$$

- Recall that we need to find r_{i+2}, q_i for $i \geq 0$ such that $0 < \ldots < r_{i+2} < r_{i+1} < r_i < \ldots < r_2 < r_1 < r_0$ and $r_i = q_i \times r_{i+1} + r_{i+2}$. I.e.,
$$\frac{r_i}{r_{i+1}} = q_i + \frac{1}{\frac{r_{i+1}}{r_{i+2}}}.$$
Let us calculate $\frac{r_{i+1}}{r_{i+2}}$ and q_i for $i \geq 1$.

- Let $i = 1$. Then $\frac{r_1}{r_2} = \frac{1}{\sqrt{3}-1} = 1 + (\frac{1}{\sqrt{3}-1} - 1) = 1 + \frac{2-\sqrt{3}}{\sqrt{3}-1} = 1 + \frac{1}{\frac{\sqrt{3}-1}{2-\sqrt{3}}} = 1 + \frac{1}{\frac{(\sqrt{3}-1)(\sqrt{3}+1)}{(2-\sqrt{3})(\sqrt{3}+1)}} = 1 + \frac{1}{\frac{2}{\sqrt{3}-1}}$
Let $q_1 = 1$, $r_3 = 2 - \sqrt{3}$ and note that $0 < r_3 < r_2$. We have:
$$r_1 = q_1 \times r_2 + r_3 \quad \text{or} \quad \frac{r_1}{r_2} = q_1 + \frac{1}{\frac{r_2}{r_3}}.$$
$$1 = 1 \times (\sqrt{3}-1) + (2-\sqrt{3}) \quad \text{or} \quad \frac{1}{\sqrt{3}-1} = 1 + \frac{1}{\frac{2}{\sqrt{3}-1}}.$$
Hence so far:
$$\frac{r_1}{r_2} = \frac{1}{\sqrt{3}-1} = 1 + \frac{1}{\frac{2}{\sqrt{3}-1}} = 1 + \frac{1}{\frac{r_2}{r_3}}.$$

- $$\frac{r_2}{r_3} = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \sqrt{3}+1 = 2 + \frac{1}{\frac{1}{\sqrt{3}-1}}.$$
Hence
$$\frac{r_2}{r_3} = 2 + \frac{1}{\frac{r_1}{r_2}}.$$
Now we see that the process is infinite as follows:
$$\frac{r_1}{r_2} = 1 + \frac{1}{\frac{r_2}{r_3}} = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{r_2}{r_3}}}}.$$

So the ratio of $\sqrt{3}$ to 1 is characterised by the repeating (after 1) infinite sequence $[1, 1, 2, 1, 2, 1, 2, ...]$ which we also write as $[1, \overline{1, 2}]$.

Figure C.1 shows the diagram version.

2. The ratio of $\sqrt{5}$ to 1 is calculated as follows:

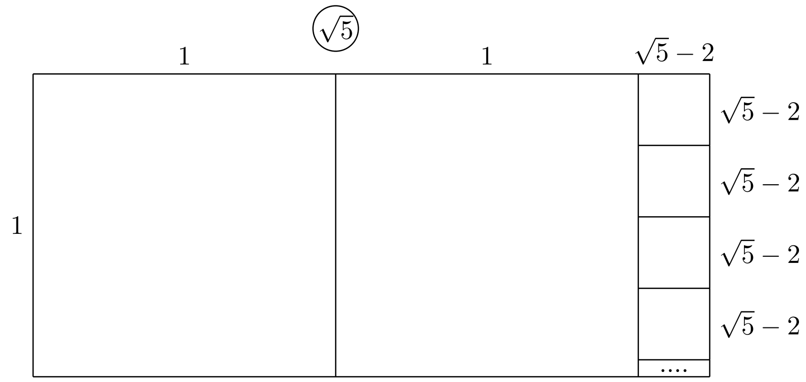


Figure C.2: Ratio of $\sqrt{5}$ to 1

- Let $r_0 = \sqrt{5}$ and $r_1 = 1$ and $q_0 = 2$. Note $0 < r_2 < r_1$. We have

$$r_0 = q_0 \times r_1 + r_2 \quad \text{or} \quad \frac{r_0}{r_1} = q_0 + \frac{r_2}{r_1}.$$

$$\sqrt{5} = 2 + (\sqrt{5} - 2) \quad \text{or} \quad \frac{\sqrt{5}}{1} = 2 + \frac{\sqrt{5} - 2}{1} = 2 + \cfrac{1}{\cfrac{1}{\sqrt{5} - 2}}$$

- Recall that we need to find r_{i+2}, q_i for $i \geq 0$ such that $0 < \ldots < r_{i+2} < r_{i+1} < r_i < \ldots < r_2 < r_1 < r_0$ and $r_i = q_i \times r_{i+1} + r_{i+2}$. I.e.,

$$\frac{r_i}{r_{i+1}} = q_i + \cfrac{1}{\cfrac{r_{i+1}}{r_{i+2}}}.$$

Let us calculate $\frac{r_{i+1}}{r_{i+2}}$ and q_i for $i \geq 1$.

- Let $i = 1$. Then $\frac{r_1}{r_2} = \frac{1}{\sqrt{5} - 2} = 4 + \frac{9 - 4\sqrt{5}}{\sqrt{5} - 2} = 4 + \frac{(9 - 4\sqrt{5})(\sqrt{5} + 2)}{(\sqrt{5} - 2)(\sqrt{5} + 2)} = 4 + (\sqrt{5} - 2) = 4 + \cfrac{1}{\cfrac{1}{\sqrt{5} - 2}} = 4 + \cfrac{1}{\cfrac{r_1}{r_2}} =$

$$4+\cfrac{1}{4+\cfrac{1}{\cfrac{r_1}{r_2}}} = ...$$

Now we see that the process is infinite.

So the ratio of $\sqrt{5}$ to 1 is characterised by the repeating (after 2) infinite sequence $[2, 4, 4, 4, ...]$ which we also write as $[2, \overline{4}]$.

Figure C.2 shows the diagram version.

3. The ratio of $\sqrt{7}$ to 1 is calculated as follows:

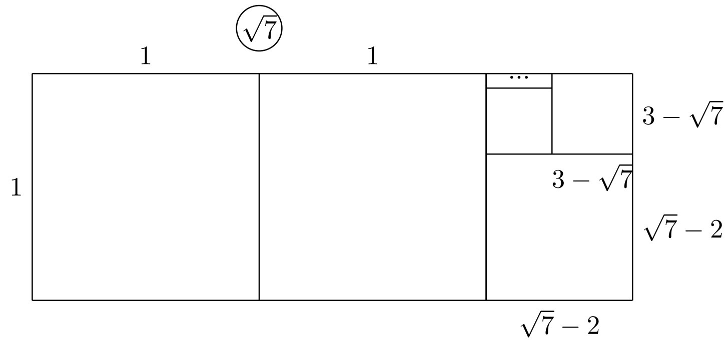


Figure C.3: Ratio of $\sqrt{7}$ to 1

- Let $r_0 = \sqrt{7}$ and $r_1 = 1$. Recall that we need to find r_{i+2}, q_i for $i \geq 0$ such that $0 < \ldots < r_{i+2} < r_{i+1} < r_i < \ldots < r_2 < r_1 < r_0$ and $r_i = q_i \times r_{i+1} + r_{i+2}$. I.e.,

$$\frac{r_i}{r_{i+1}} = q_i + \cfrac{1}{\cfrac{r_{i+1}}{r_{i+2}}}.$$

  Let us calculate $\frac{r_{i+1}}{r_{i+2}}$ and q_i for $i \geq 1$.

- Let $q_0 = 2$ and $r_2 = \sqrt{7}-2$. Note $0 < r_2 < r_1$ and $r_0 = q_0 \times r_1 + r_2$. Now,

$$\sqrt{7} = 2 + (\sqrt{7} - 2) \quad \text{or} \quad \frac{\sqrt{7}}{1} = 2 + \frac{\sqrt{7}-2}{1} = 2 + \cfrac{1}{\cfrac{1}{\sqrt{7}-2}}$$

- Note that $\frac{r_1}{r_2} = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{(\sqrt{7}-2)(\sqrt{7}+2)} = \frac{\sqrt{7}+2}{3}$.
  Let $q_1 = 1$ and $r_3 = 3 - \sqrt{7}$. Let $q_2 = 1$ and $r_4 = 2\sqrt{7} - 5$. Let $q_3 = 1$ and $r_5 = 8 - 3\sqrt{7}$. Let $q_4 = 4$ and $r_6 = 14\sqrt{7} - 37$.
  We have for $0 \leq i \leq 4$:
  $$r_i = q_i \times r_{i+1} + r_{i+2} \quad \text{i.e.,} \quad \frac{r_i}{r_{i+1}} = q_i + \frac{1}{\frac{r_{i+1}}{r_{i+2}}}.$$
  Now, it is easy to show that $\frac{\sqrt{7}+2}{3} = \frac{8-3\sqrt{7}}{14\sqrt{7}-37}$. Hence
  $$\frac{r_5}{r_6} = \frac{r_1}{r_2}$$
  Hence we see that this process is infinite with a chracterizing sequence of the ration $\sqrt{7}$ to 1 being $[2, 1, 1, 1, 4, 1, 1, 1, 4, ...] = [2, \overline{1, 1, 1, 4}]$.

Figure C.3 shows the diagram version.

Solution C.114. [Of Exercise 7.6] The solutions are respectively; 1, 2, 3 and 4.

Solution C.115. [Of Exercise 7.7.] We only do the first two cases since the remaining cases are similar to above.

1. For ACF and AFD: Since AF is the bisector of $\angle CAD$ and $CA = AD$, then the two triangles are similar. The angles are as follows: $\angle CAF = \angle FAD = 22.5°$, $\angle FDA = \angle FCA = 90°$. $\angle CFA = \angle DFA = 67.5°$.

2. For BFD and BCA: Since ACF and AFD are similar, then $\angle FDB$ is a right angle. Hence, $\angle FDB = 90°$. Since $\angle FBD = 45°$ then $\angle BFD = 45°$ and the triangles BFD and BCA have the same angles. Hence, they are similar.

Solution C.116. [Of Exercise 7.8.]

1. Since Theorem 7.2.5 already dealt with the case of a square of area 3, we assume $n > 0$. Let p be a number of the form $4n + 3$. Note that $\frac{p-1}{2} = 2n + 1$ and $\frac{p+1}{2} = 2n + 2$. The square of area p units has a side of length $\sqrt{p}$. Now consider a right triangle whose legs have lengths $\sqrt{p}$ and $\frac{p-1}{2}$, which will mean by the Pythagorean Theorem, that the hypotenuse has length $\frac{p+1}{2}$. We will use this triangle to prove

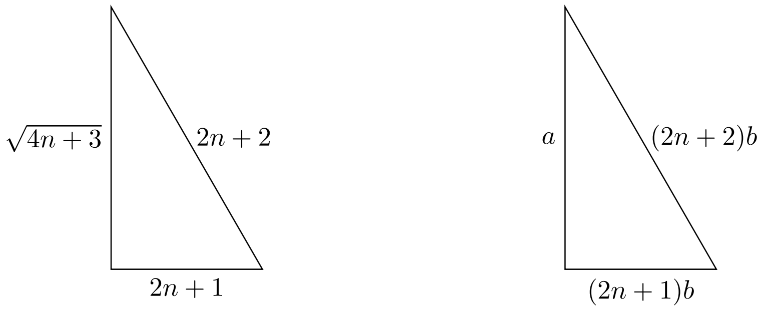


Figure C.4: Diagram for the proof of irrationality of $\sqrt{4n+3}$

the irrationality of $\sqrt{p}$. Assuming commensurability of $\sqrt{p}$ with the unit, there must be a right triangle whose legs have lengths a and $(2n+1)b$ and whose hypotenuse has length $(2n+2)b$ where a and b are positive integers; see Figure C.4. Now since the hypotenuse has length $(2n+2)b$, it is even, and so by Theorem 2.5.5, both legs are even. This means that a and $(2n+1)b$ (and hence b) are all even. Thus, we can get a smaller triangle of the same form whose linear dimensions are half of those of the triangle we started with. But then we have a triangle whose legs have lengths $a/2$ and $(2n+1)b/2$ and whose hypotenuse has length $(2n+2)(b/2)$, or a', $(2n+1)b'$ and $(2n+2)b'$, where $a = a/2$ and $b' = b/2$. Assuming that a' and b' are positive integers, this is, again, a right triangle whose hypotenuse is even, and the above argument can be repeated. Clearly, we cannot indefinitely repeat this argument. Hence, there is no right triangle whose legs have lengths a and $(2n+1)b$ and whose hypotenuse has length $(2n+2)b$ where a and b are integers. It follows that $\sqrt{p}$ and 1 are incommunserable.

2. Since Theorem 7.2.6 already dealt with the case of a square of area 5, we assume $n > 0$. Let q be a number of the form $8n+5$. Note that $\frac{q-1}{2} = 4n+2$ and $\frac{q+1}{2} = 4n+3$. The square of area q units has a side of length $\sqrt{q}$. Now consider a right triangle whose legs have lengths $\sqrt{q}$ and $\frac{q-1}{2}$ and whose hypotenuse therefore has length $\frac{q+1}{2}$ by the Pythagorean Theorem. We will use this triangle to prove that $\sqrt{q}$ is irrational. Assuming commensurability of $\sqrt{q}$, there must be a right triangle whose legs have lengths a and $(4n+2)b$ and whose hypotenuse has length $(4n+3)b$ where a and b are positive integers; see Figure C.5. Now b is either even or odd.

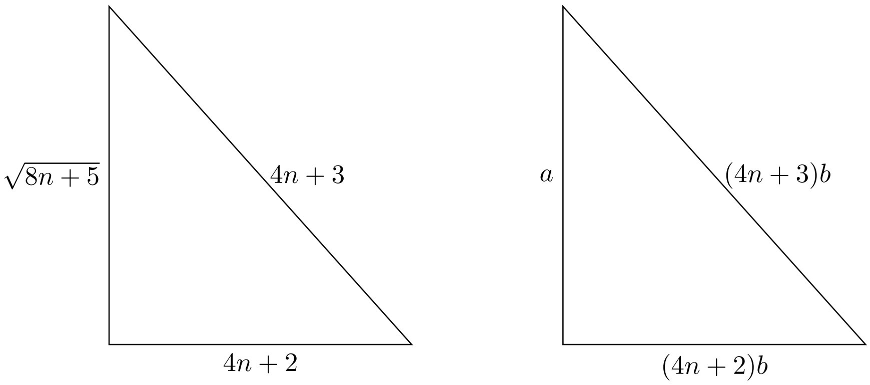


Figure C.5: Diagram for the proof of irrationality of $\sqrt{8n+5}$

If b is even, then $(4n + 3)b$ is even, so the hypotenuse is even, and hence, by Theorem 2.5.5, both a and $(4n + 2)b$ are even. Then we can construct another triangle of the same form by halving each dimension. This cannot be repeated indefinitely, so there must be a triangle of this form in which b is odd. It follows that $(4n + 3)b$ is odd. Now we may assume that a and b have no common factors, since otherwise we can divide out these common factors, and we cannot keep doing this indefinitely. Now, since $(4n + 3)b$ is odd, it is not a multiple of 4. By Theorem 7.2.1, only one of a and $(4n + 2)b$ is a multiple of 4. If a is a multiple of 4, then it is even, so $(4n+3)b$ must be the sum of two even squares and cannot be odd. Hence, a is not a multiple of 4, and so $(4n + 2)b$ must be a multiple of 4, from which it follows that b is even, contradicting its being odd. Hence, there is no right triangle whose legs have length a and $(4n + 2)b$, where a and b are positive integers, and $\sqrt{q}$ is incommensurable with 1.

3. Let r be a number of the form $2(2n+1)$ (i.e., an even number which is not a multiple of 4.) Consider a right triangle whose legs have lengths $2\sqrt{r}$ and $r - 1$ and whose hypotenuse therefore has length $r + 1$. We will use this right triangle to prove that $\sqrt{r}$ is irrational. Assuming commensurability of $\sqrt{r}$, there must be a right triangle whose legs legs have lengths $2a$ and $(r-1)b$ and whose hypotenuse has length $(r+1)b$ where a and b are positive integers; see Figure C.6. Now if a and b are both even, we could divide all the linear dimensions by 2, and we cannot do this indefinitely. So we may assume that a and b are not

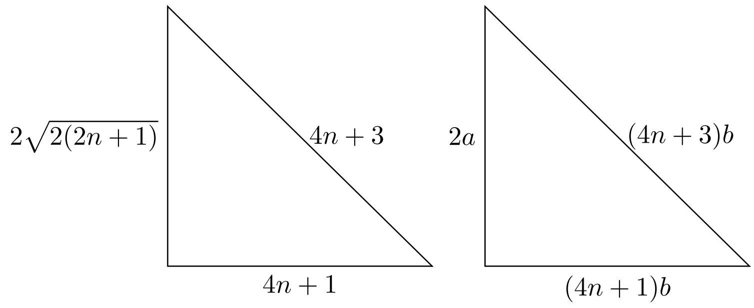


Figure C.6: Diagram for the proof of irrationality of $\sqrt{2(2n+1)}$

both even. Now suppose b is even, so that a must be odd. Then b cannot be a multiple of 4, since then the hypotenuse would also be divisible by 4, and so by Theorem 7.2.1, $2a$ would also be divisible by 4, contradicting the oddness of a. Hence, b is not divisible by 4. Then neither are $(4n+3)b$ (the hypotenuse) and $(4n+1)b$ divisible by 4, from which it follows by Theorem 7.2.1 that $2a$ is divisible by 4, again contradicting the oddness of a. Hence, b cannot be even, so it must be odd. Then, as before, $(4n+3)b$ and $(4n+1)b$ are odd, and hence not divisible by 4, so by Theorem 7.2.1, $2a$ is divisible by 4, and so a is even. Say $a = 2c$. Now the Pythagorean condition implies that

$$(2a)^2 = ((4n+3)b)^2 - ((4n+1)b)^2,$$

and substituting $2c$ for a, this gives us

$$16c^2 = 8(2n+1)b^2.$$

This is equivalent to

$$2c^2 = (2n+1)b^2.$$

Now since b is odd, $(2n+1)b^2$ is odd, so it cannot equal $2c^2$. It follows that there is no right triangle whose legs are $2a$ and $(4n+1)b$ and whose hypotenuse is $(4n+3)b$ where a and b are positive integers.

Solution C.117. [Of Exercise 7.9.] We will prove that a positive integer has a rational nth root if and only if it is a n-th power of a positive integer. It is clear that if a positive integer is an nth power then it has a rational nth root, namely the positive integer itself, so we need only prove the converse.

Suppose k is a positive integer with a rational nth root, say $\frac{p}{q}$. Then p and q are positive integers, and

$$\frac{p^n}{q^n} = k,$$

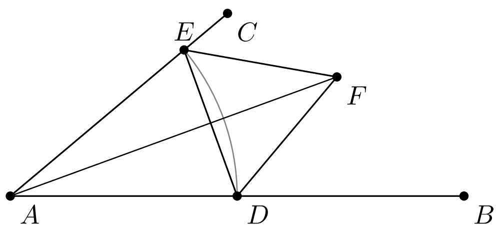
or

$$p^n = kq^n.$$

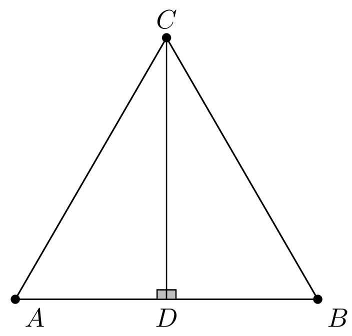
Now consider the prime factorisations of the two sides. Since p^n is an nth power, the number of times each prime number occurs in its prime factorisation is a multiple of n. Similarly, the number of times each prime number occurs in q^n is a multiple of n. Now since the number of times each prime factor occurs in kq^n is the same as the number of times it occurs in p^n, the number of times it occurs in k is the number of times it occurs in p^n minus the number of times it occurs in q^n. But this means that the number of times each prime number occurs in the prime factorisation of k is the difference of two multiples of n and is therefore a multiple of n. It follows that k is an nth power of a positive integer.

Solution C.118. [Of Exercise 7.10.]

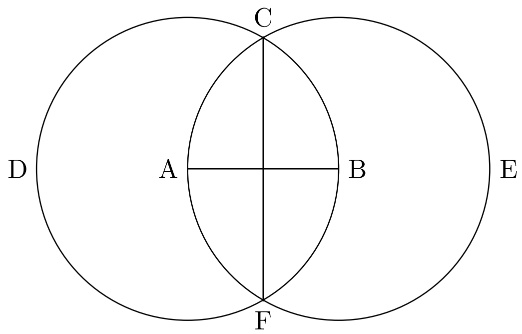
1. Let the angle be $\angle BAC$. From A draw the arc of a circle to bisect both AB and AC resp. at D and E. By Proposition 1, Book I, we can construct the equilateral triangle DEF as seen in the following figure. Since $AD = AE$ and $DF = EF$ then the two triangles ADF and AFE are similar and hence $\angle BAF = \angle FAC$. Hence AF is the bisector of the angle $\angle BAC$.



2. Let the line be AB. By Proposition 1, Book I, we can construct the equilateral triangle ABC as in the following picture. By 1 above, let CD bisect the angle $\angle ACB$. Since $AC = BC$, $\angle ACD = \angle DCB$ and since CD is common between the triangles ADC and CDB, we get that $AD = DB$. Furthermore, since $\angle DAC = \angle DBC$ and $\angle ACD = \angle DCB$, we get $\angle ADC = \angle BDC$ and hence by Definition 10, book 1, each of $\angle ADC$ and $\angle BDC$ is right angle. Therefore, CD is the perpendicular bisector of AB.



Here is another way of doing the proof which we leave to the student to fill in the remaining details.
Let AB be the given finite straight line. It is required to construct the perpendicular bisector of AB.

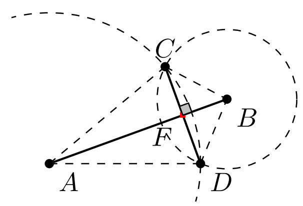


Describe the circle BCD with centre A and radius AB. Again describe the circle AFE with centre B and radius BA. [Post.3]
Join the straight lines CF. [Post.1]

Now, CF is the perpendicular bisector of AB. The proof will need some other propositions along the way from Proposition 1 that was proved in Section 7.3. We leave this to the reader.

3. Let AB be the line and C the point. From B, draw the circle with radius BC. From A draw the circle with radius AC. Let D be the

other intersection of these circles. Let F be the intersection of CD and AB. Since the triangles ACB and ADB are similar then $\angle ABC = \angle ABD$ and hence the triangles BCF and BDF are similar. Hence $\angle BFC = \angle BFD$ and by Definition 10, book 1, $\angle BFC$ is right angle. See the picture below:



C.8 Solutions for Chapter 8

Solution C.119. [Of Exercise 8.1.] We need to prove that if c is a value or ∞ or $-\infty$, and if

$$\lim_{x\to c} f(x) = l \text{ and } \lim_{x\to c} g(x) = m,$$

then

1. $$\lim_{x\to c}[f(x) - g(x)] = l - m.$$

2. If $m \neq 0$ then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{l}{m}.$$

3. If n is any positive integer, then
$$\lim_{x\to c}[f(x)]^n = l^n.$$

4. If p, q are positive integers and $l \geq 0$ whenever q is even,
$$\lim_{x\to c}[f(x)]^{\frac{p}{q}} = l^{\frac{p}{q}}.$$

1. We have
$$\begin{array}{ll} \lim_{x\to c}[f(x) - g(x)] & = \\ \lim_{x\to c}[f(x) + (-1)g(x)] & =^{\text{by LF3}} \\ [\lim_{x\to c} f(x)] + [\lim_{x\to c}(-1)g(x)] & =^{\text{by LF4}} \\ l + [\lim_{x\to c}(-1)][\lim_{x\to c} g(x)] & =^{\text{by LF2}} \\ l + (-1)m & = \\ l - m. & \end{array}$$

2. We have
$$\begin{array}{rcll} \lim_{x\to c} \frac{f(x)}{g(x)} & = & \lim_{x\to c} \left(f(x).\frac{1}{g(x)} \right) & \\ & = & (\lim_{x\to c} f(x)) \left(\lim_{x\to c} \frac{1}{g(x)} \right) & \text{by LF4} \\ & = & l\frac{1}{m}. & \text{by LF5} \\ & = & \frac{l}{m}. & \end{array}$$

3. By induction on n.
Basis: If $n = 1$, we have $[f(x)]^n = [fx)]^1 = f(x)$ and $l^n = l^1 = l$, and the result follows immediately by hypothesis.
Induction step: Assume that

$$\lim_{x \to c} [f(x)]^k = l^{\cdot}$$

Then we have

$$\begin{aligned} \lim_{x\to c}[f(x)]^{k+1} &= \lim_{x\to c}[f(x)]^k f(x) \\ &= (\lim_{x\to c}[f(x)]^k)(\lim_{x\to c} f(x)) \quad \text{by LF4} \\ &= l^k l \quad \text{by the induction hypothesis} \\ &= l^{k+1}. \end{aligned}$$

4. We have

$$\begin{aligned} \lim_{x\to c}[f(x)]^{\frac{p}{q}} &= \lim_{x\to c} \left(\sqrt[q]{f(x)}\right)^p \\ &= \left(\lim_{x\to c} \sqrt[q]{f(x)}\right)^p \quad \text{by the above item 3.} \\ &= (\sqrt[q]{l})^p \quad \text{by LF6} \\ &= l^{\frac{p}{q}}. \end{aligned}$$

Solution C.120. [Of Exercise 8.2]

c^-: We prove that: If c is a value and if

$$\lim_{x \to c^-} f(x) = l \text{ and } \lim_{x \to c^-} g(x) = m,$$

then

1.
$$\lim_{x \to c^-} [f(x) - g(x)] = l - m.$$

2. If $m \neq 0$ then
$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = \frac{l}{m}.$$

3. If n is any positive integer, then
$$\lim_{x \to c^-} [f(x)]^n = l^n.$$

4. If p, q are positive integers and $l \geq 0$ whenever q even, then
$$\lim_{x \to c^-} [f(x)]^{\frac{p}{q}} = l^{\frac{p}{q}}.$$

The proof is as follows:

1. We have $\lim_{x\to c^-}[f(x) - g(x)]$

$$\begin{array}{lll} = & \lim_{x\to c^-}[f(x) + (-1)g(x)] & \\ = & [\lim_{x\to c^-} f(x)] + [\lim_{x\to c}(-1)g(x)] & \text{by LF3} \\ = & l + [\lim_{x\to c^-}(-1)][\lim_{x\to c} g(x)] & \text{by LF4} \\ = & l + (-1)m & \text{by LF2} \\ = & l - m. & \end{array}$$

2. We have $\lim_{x\to c^-} \dfrac{f(x)}{g(x)}$

$$\begin{array}{lll} = & \lim_{x\to c^-} f(x).\dfrac{1}{g(x)} & \\ = & (\lim_{x\to c^-} f(x))\left(\lim_{x\to c^-} \dfrac{1}{g(x)}\right) & \text{by LF4} \\ = & l\dfrac{1}{m}. & \text{by LF5} \\ = & \dfrac{l}{m}. & \end{array}$$

3. By induction on n.
Basis: If $n = 1$, we have $[f(x)]^n = [fx)]^1 = f(x)$ and $l^n = l^1 = l$, and the result follows immediately by hypothesis.
Induction step: Assume that

$$\lim_{x\to c^-}[f(x)]^k = l^{\cdot}$$

Then we have $\lim_{x\to c^-}[f(x)]^{k+1}$

$$\begin{array}{lll} = & \lim_{x\to c^-}[f(x)]^k f(x) & \\ = & (\lim_{x\to c^-}[f(x)]^k)(\lim_{x\to c^-} f(x)) & \text{by LF4} \\ = & l^k l & \text{by the induction hypothesis} \\ = & l^{k+1}. & \end{array}$$

4. We have $\lim_{x\to c^-}[f(x)]^{\frac{p}{q}}$

$$\begin{array}{lll} = & \lim_{x\to c^-}\left(\sqrt[q]{f(x)}\right)^p & \\ = & \left(\lim_{x\to c^-}\sqrt[q]{f(x)}\right)^p & \text{by the above item 3.} \\ = & (\sqrt[q]{l})^p & \text{by LF6} \\ = & l^{\frac{p}{q}}. & \end{array}$$

c^+: Exactly like that of c^-, just replace every c^- by c^+.

Note that throughout this exercise, we only used properties LF2..LF6 of Definition 8.1.1.

Solution C.121. [Of Exercise 8.3.]

LF13 Let $f(x) = x$. Since n is a positive integer, and by LF1 $\lim_{x\to c} f(x) = c$, we get by LF12.3 that $\lim_{x\to c} x^n = c^n$.

LF14 By LF2, $\lim_{x\to c} a = a$. By LF13, $\lim_{x\to c} x^n = c^n$. Hence, by LF4, $\lim_{x\to c} ax^n = (\lim_{x\to c} a)(\lim_{x\to c} x^n) = ac^n$.

LF15 Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where for all $0 \leq i \leq n$, a_i is a constant (and of course for all $1 \leq i \leq n$, i is a positive integer). By LF14, $\lim_{x\to c} a_i x^i = a_i c^i$ for all $0 < i \leq n$. By LF2, $\lim_{x\to c} a_0 = a_0$. Hence by n repetitions of LF3,

$$\begin{aligned}&\lim_{x\to c} p(x)\\ &= \lim_{x\to c} a_n x^n + a_{n-1}x^{n-1} + \ldots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0\\ &= a_n c^n + a_{n-1}c^{n-1} + \ldots + a_1 c + a_0 = p(c).\end{aligned}$$

LF16 First we need to prove that if for all x, $q(x) \neq 0$ then $\lim_{x\to c} q(x) \neq 0$. By LF15, we have $\lim_{x\to c} q(x) = q(c)$. Since for all x, $q(x) \neq 0$, we have $q(c) \neq 0$ and $\lim_{x\to c} q(x) \neq 0$. Then:

$$\lim_{x\to c} r(x) = \lim_{x\to c} \frac{p(x)}{q(x)} =^{LF12.2} \frac{\lim\limits_{x\to c} p(x)}{\lim\limits_{x\to c} q(x)} =^{LF15} \frac{p(c)}{q(c)} = r(c).$$

Solution C.122. [Of Exercise 8.4.] Recall that the area of a triangle is $\frac{1}{2}bh$ where b is the base of the triangle and h is its height. Hence, the area of the triangle OCD is $\frac{1}{2}\sin x$ and the area of the triangle OAB is $\frac{1}{2}\tan x$. Furthermore, the area of the slice of the square OBC is $\frac{x}{2}$. Note that $\frac{1}{2}\sin x \leq \frac{x}{2} \leq \frac{1}{2}\tan x$. Recall that $\tan x = \frac{\sin x}{\cos x}$. Hence $\sin x \leq x \leq \frac{\sin x}{\cos x}$. Note that we can add absolute values to all sides without changing anything. Hence $|\sin x| \leq |x| \leq \frac{|\sin x|}{|\cos x|}$. Hence $\frac{|\sin x|}{|\sin x|} \leq \frac{|x|}{|\sin x|} \leq \frac{1}{|\cos x|}$. Hence $1 \leq \frac{|x|}{|\sin x|} \leq \frac{1}{|\cos x|}$. I.e., $|\cos x| \leq \frac{|\sin x|}{|x|} \leq 1$. Whether x is positive or negative, as x approaches 0, $\cos x$ is positive and $\sin x$ and x have the same sign. Hence, $\cos x \leq \frac{\sin x}{x} \leq 1$. Since $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$, by LF7, $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Solution C.123. [Of Exercise 8.5.]

LF18 $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{1}{x}\frac{\sin x}{\cos x} = \lim_{x\to 0}(\frac{1}{\cos x}\frac{\sin x}{x}) =^{LF4}$ $(\lim_{x\to 0}\frac{1}{\cos x})(\lim_{x\to 0}\frac{\sin x}{x}) =^{LF5} (\frac{1}{\lim\limits_{x\to 0}\cos x})(\lim_{x\to 0}\frac{\sin x}{x}) =^{LF17}$ $\frac{1}{\lim\limits_{x\to 0}\cos x} = \frac{1}{1} = 1.$

LF19 $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} =$ $\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} =$ $\lim_{x\to 0}(\frac{\sin x}{x})\left(\frac{\sin x}{(1+\cos x)}\right) =^{\text{LF4}}$ $\lim_{x\to 0}(\frac{\sin x}{x})\lim_{x\to 0}\left(\frac{\sin x}{(1+\cos x)}\right) =^{LF17}$ $\lim_{x\to 0}\left(\frac{\sin x}{1+\cos x}\right) =^{LF12.2} \frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}(1+\cos x)} =^{\text{LF12,LF3}}$ $\frac{0}{2} = 0.$

LF20 $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} =$ $\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} =^{\text{LF4}}$ $\lim_{x\to 0}(\frac{\sin x}{x})\lim_{x\to 0}(\frac{\sin x}{x})\left(\frac{1}{1+\cos x}\right) =^{\text{LF4}}$ $\lim_{x\to 0}(\frac{\sin x}{x})\lim_{x\to 0}(\frac{\sin x}{x})\lim_{x\to 0}\left(\frac{1}{1+\cos x}\right) =^{LF17}$ $\lim_{x\to 0}\left(\frac{1}{1+\cos x}\right) =^{\text{LF5}} \frac{1}{\lim\limits_{x\to 0}(1+\cos x)} =^{\text{LF3}} \frac{1}{2}.$

Solution C.124. [Of Exercise 8.6.] We need to prove that if $k > 0$ and if, for each $1 \le i \le k$,

$$\lim_{n\to\infty} a_{i,n} = a_i,$$

then

$$\lim_{n\to\infty}\left(\sum_{i=1}^{k} a_{i,n}\right) = \sum_{i=1}^{k} a_i.$$

The proof is by mathematical induction on k. *Basis*: $k = 1$. Then

$$\lim_{n\to\infty}\left(\sum_{i=1}^{k} a_{i,n}\right) = \lim_{n\to\infty} a_{1,n}$$

and

$$\sum_{i=1}^{k} a_i = a_1,$$

and so the conclusion follows by the hypothesis.

Inductive step. $k = m + 1$. By the induction hypothesis,

$$\lim_{n\to\infty}\left(\sum_{i=1}^{m} a_{i,n}\right) = \sum_{i=1}^{m} a_i.$$

Then

$$\begin{array}{lll} \lim_{n\to\infty}\left(\sum_{i=1}^{m+1} a_{i,n}\right) & = & \lim_{n\to\infty}(\sum_{i=1}^{m} a_{i,n} + a_{m+1,n}) \\ & =^{\text{LS2}} & \lim_{n\to\infty}\sum_{i=1}^{m} a_{i,n} + \lim_{n\to\infty} a_{m+1,n} \\ & =^{\text{IH \& Hyp.}} & \sum_{i=1}^{m} a_i + a_{m+1} \\ & = & \sum_{i=1}^{m+1} a_i. \end{array}$$

Solution C.125. [Of Exercise 8.7.] We need to prove that

1. If k is any constant value and if

$$\lim_{n\to\infty} a_n = a,$$

then

$$\lim_{n\to\infty} ka_n = ka.$$

2. If

$$\lim_{n\to\infty} a_n = a, \text{ and } \lim_{n\to\infty} b_n = b,$$

then

$$\lim_{n\to\infty}(a_n - b_n) = a - b.$$

We do this as follows:

1. We have

$$\begin{array}{llll} \lim_{n\to\infty} ka_n & = & (\lim_{n\to\infty} k)(\lim_{n\to\infty} a_n) & \text{by LS3} \\ & = & k(\lim_{n\to\infty} a_n) & \text{by LS1} \\ & = & ka. & \end{array}$$

2. We have

$$\begin{array}{llll} \lim_{n\to\infty}(a_n - b_n) & = & \lim_{n\to\infty}[a_n + (-1)b_n] & \\ & = & \lim_{n\to\infty} a_n + \lim_{n\to\infty}(-1)b_n & \text{by LS2} \\ & = & \lim_{n\to\infty} a_n + (-1)(\lim_{n\to\infty} b_n) & \text{by above} \\ & = & a + (-1)b & \\ & = & a - b. & \end{array}$$

Solution C.126. [Of Exercise 8.8.] Since $b \neq 0$, we have by LS4

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Hence,

$$\begin{array}{rcll} \lim_{n\to\infty} \frac{a_n}{b_n} & = & \lim_{n\to\infty} a_n \frac{1}{b_n} & \\ & = & (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} \frac{1}{b_n}) & \text{by LS3} \\ & = & a\frac{1}{b} & \\ & = & \frac{a}{b}. & \end{array}$$

Solution C.127. [Of Exercise 8.9.]

LS21. By LS17, $\lim_{n\to\infty} -n = -\infty$ iff $\lim_{n\to\infty} n = +\infty$. Since by LS11, $\lim_{n\to\infty} n = +\infty$, we have $\lim_{n\to\infty} -n = -\infty$.

LS22. Let $c = -b$. We have $\lim_{n\to\infty} -b_n =^{\text{LS3}} (\lim_{n\to\infty} -1)(\lim_{n\to\infty} b_n) =^{LS1} -(\lim_{n\to\infty} b_n) = -b = c$. Hence by LS12, $\lim_{n\to\infty}(a_n - b_n) = \pm\infty$.

LS23. By LS17, $\lim_{n\to\infty} -b_n = \mp\infty$. Hence by LS12, $\lim_{n\to\infty} a_n - b_n = \mp\infty$.

LS24. By LS17, $\lim_{n\to\infty} -b_n = \pm\infty$. Hence by LS12, $\lim_{n\to\infty}(a_n - b_n) = \pm\infty$.

LS25. The proof is by mathematical induction on k. *Basis*: $k = 1$. Then

$$\lim_{n\to\infty} \left(\prod_{i=1}^{k} a_{i,n} \right) = \lim_{n\to\infty} a_{1,n}$$

and

$$\prod_{i=1}^{k} a_i = a_1,$$

and so the conclusion follows by the hypothesis.

Inductive step. $k = m + 1$. By the induction hypothesis,

$$\lim_{n\to\infty} \left(\prod_{i=1}^{m} a_{i,n} \right) = \prod_{i=1}^{m} a_i.$$

Then $\lim_{n\to\infty}\left(\prod_{i=1}^{m+1} a_{i,n}\right)$

$$
\begin{array}{lll}
= & \lim_{n\to\infty}\left(\prod_{i=1}^{m} a_{i,n}\right)(a_{m+1,n}) & \\
= & \lim_{n\to\infty}\left(\prod_{i=1}^{m} a_{i,n}\right)(\lim_{n\to\infty} a_{m+1,n}) & \text{by LS3} \\
= & \left(\prod_{i=1}^{m} a_i\right)(a_{m+1}) & \text{by IH and Hyp.} \\
= & \prod_{i=1}^{m+1} a_i. &
\end{array}
$$

LS26. The proof is by mathematical induction on k.

Basis: $k = 1$. Then $\lim_{n\to\infty} a_n = a$ by hypothesis.

Inductive step. $k = m+1$. By the induction hypothesis, $\lim_{n\to\infty} a_n^m = a^m$. Then $\lim_{n\to\infty} a_n^{m+1} =^{\text{LS3}} (\lim_{n\to\infty} a_n^m)(\lim_{n\to\infty} a_n) =^{\text{IH, Hyp.}} a^m a = a^{m+1}$.

LS27. Assume $\lim_{n\to\infty} a_n = \pm\infty$ and $\lim_{n\to\infty} b_n = b < 0$. Then $\lim_{n\to\infty} -b_n =^{\text{LS3}} \lim_{n\to\infty}(-1)\lim_{n\to\infty} b_n =^{\text{LS1}} -1 \times \lim_{n\to\infty} b_n = -b > 0$. Hence by LS13, $\lim_{n\to\infty} -(a_n b_n) = \lim_{n\to\infty} a_n(-b_n) = \pm\infty$. By LS17, $\lim_{n\to\infty}(a_n b_n) = \mp\infty$. The case $\lim_{n\to\infty} a_n = a < 0$ and $\lim_{n\to\infty} b_n = \pm\infty$, is similar.

LS28. By LS17, $\lim_{n\to\infty} -b_n = \pm\infty$. Hence $\lim_{n\to\infty} -(a_n b_n) = \lim_{n\to\infty} a_n(-b_n) =^{\text{LS14}} \infty$ and by LS17 $\lim_{n\to\infty}(a_n b_n) = -\infty$.

LS29. By LS4, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b} > 0$. Hence by LS13, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \frac{1}{b_n} = \pm\infty$.

LS30. By LS4, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b} < 0$.

Now $\lim_{n\to\infty} -\frac{1}{b_n} =^{\text{LS3}} (\lim_{n\to\infty} -1)\left(\lim_{n\to\infty} \frac{1}{b_n}\right) =^{\text{LS1}} -\frac{1}{b} > 0$.

Hence by LS13, $\lim_{n\to\infty} -\frac{a_n}{b_n} = \lim_{n\to\infty} a_n\left(-\frac{1}{b_n}\right) = \pm\infty$. Finally, by LS17, $\lim_{n\to\infty} \frac{a_n}{b_n} = \mp\infty$.

LS31. Since $a_n \leq b_n$ for all $n > N$, we have $a_n - b_n \leq 0$ for all $n > N$. By LS22, $\lim_{n\to\infty}(a_n - b_n) = a - b$ and by LS5, $a - b \leq 0$. Hence $a \leq b$.

LS32. By LS19, $\lim_{n\to\infty} -a_n = 0$. Since $|b_n| \leq a_n$ for all $n > N$, then for all $n > N$, $-a_n \leq b_n \leq a_n$. Hence by LS9, $\lim_{n\to\infty} b_n$ exists. By LS31, $\lim_{n\to\infty} b_n \leq 0$ and $0 \leq \lim_{n\to\infty} b_n$. Hence, $\lim_{n\to\infty} b_n = 0$. By LS6, $\lim_{n\to\infty} |b_n| = 0$.

LS33. By LS9, $\lim_{n\to\infty} b_n$ exists. By LS31, $\lim_{n\to\infty} b_n \leq l$ and $l \leq \lim_{n\to\infty} b_n$. Hence, $\lim_{n\to\infty} b_n = l$.

LS34. Let $b_n = a$ for all n. By LS1, $\lim_{n\to\infty} b_n = a$.

- If $\lim_{n\to\infty} a_n = a$ then by LS20, $\lim_{n\to\infty}(a_n - b_n) = a - a = 0$.
- If $\lim_{n\to\infty}(a_n - b_n) = 0$ then $\lim_{n\to\infty} a_n$ exists for otherwise if $\lim_{n\to\infty} a_n = \pm\infty$ then by LS22, $\lim_{n\to\infty}(a_n - b_n) = \pm\infty$ contradiction. By LS20, $\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - a = 0$ and hence $\lim_{n\to\infty} a_n = a$.

Solution C.128. [Of Exercise 8.10.] $\angle ABD = \angle BCA = 60°$. $\angle ADB = \angle ADC = 90°$. $\angle BAD = \angle DAC = 30°$.
Moreover, $BD = DC = \frac{l}{2}$. Since $AB = BC = AC = l$, then $AD^2 = l^2 - (\frac{l}{2})^2 = \frac{3l^2}{4}$. Hence, $AD = \frac{l\sqrt{3}}{2}$.
The area of $ABC = \frac{1}{2}(AD)(BC) = \frac{1}{2}\frac{l\sqrt{3}}{2}l = \frac{\sqrt{3}}{4}l^2$.
The area of the square $ADEF = (\frac{l\sqrt{3}}{2})^2 = \frac{3l^2}{4}$.
Similarly, $\angle A'B'D' = \angle B'C'A' = 60°$. $\angle A'D'B' = \angle A'D'C' = 90°$. $\angle B'A'D' = \angle D'A'C' = 30°$.
Moreover, $B'D' = D'C' = \frac{2l}{2} = l$. Since $A'B' = B'C' = A'C' = 2l$, then $A'D'^2 = 4l^2 - l^2 = 3l^2$. Hence, $A'D' = \sqrt{3}l$.
The area of $A'B'C' = \frac{1}{2}(A'D')(B'C') = \frac{1}{2}\sqrt{3}l(2l) = \sqrt{3}l^2$.
The area of the square $A'D'E'F' = (\sqrt{3}l)^2 = 3l^2$.
Since the corresponding angles of ABC and $A'B'C'$ are equal, then these triangles are similar.
Also, since all angles of both squares $ADEF$ and $A'D'E'F'$ are right angles, and since each side of $ADEF$ is equal to half of any side of $A'D'E'F'$, the two squares are similar.
Finally, the area ratio $\frac{\text{area } ABC}{\text{area } A'B'C'}$ of the triangles ABC to $A'B'C'$ is equal $(\frac{AB}{A'B'})^2 = (\frac{1}{2})^2 = (\frac{AD}{A'D'})^2$.
And also, the area ratio $\frac{\text{area } ADEF}{\text{area } A'D'E'F'}$ of the squares $ADEF$ to $A'D'E'F'$ is equal $(\frac{AD}{A'D'})^2 = (\frac{1}{2})^2$.

C.9 Solutions for Chapter 9

Solution C.129. [Of Exercise 9.1.] We will show that for any $\varepsilon > 0$, $|l_1 - l_2| < \varepsilon$. Let $\varepsilon > 0$. By definition, there are M_1 and M_2 such that if $n > M_1$ then $|x_n - l_1| < \varepsilon/2$ and if $n > M_2$ then $|x_n - l_2| < \varepsilon/2$. Let $M = \max\{M_1, M_2\}$. If $n > M$ then $|x_n - l_1| < \varepsilon/2$ and $|x_n - l_2| < \varepsilon/2$. Let $n > M$. Now, $|l_1 - l_2| = |l_1 - x_n + x_n - l_2| \leq |x_n - l_1| + |x_n - l_2| < \varepsilon$.

Solution C.130. [Of Exercise 9.2.]

LS6.

$\lim_{n\to\infty} a_n = 0 \Leftrightarrow$
for every $\varepsilon > 0$, there is an N such that for each $n > N$, $|a_n| < \varepsilon \Leftrightarrow$
for every $\varepsilon > 0$, there is an N such that for each $n > N$, $||a_n|| < \varepsilon \Leftrightarrow$
$\lim_{n\to\infty} |a_n| = 0$.

LS7. Suppose $\lim_{n\to\infty} a_n = 0$. Let b_n be defined for $n > k$ (for some $k > 0$), and let there be a g, independent of n, such that $|b_n| \leq g$ for $n > k$.

Let $\varepsilon > 0$. By definition, there is N' such that such that for each $n > N'$, $|a_n| < \frac{\varepsilon}{g}$. Let $N = \max\{N', k\}$. For each $n > N$, we have $|a_n b_n| = |a_n||b_n| < \frac{\varepsilon}{g} g = \varepsilon$. Hence $\lim_{n\to\infty}(a_n b_n) = 0$.

LS16. Assume $\lim_{n\to\infty} a_n = \infty$ and for all $n > N$ for some N, we have $a_n \leq b_n$. Let $M > 0$. By definition, there is an N such that for all $n > N$, $a_n > M$. Hence for all $n > N$, $b_n \geq a_n > M$ and by definition, $\lim_{n\to\infty} b_n = \infty$.

LS18. We need to prove that if $k > 0$ and if, for each $1 \leq i \leq k$,

$$\lim_{n\to\infty} a_{i,n} = a_i,$$

then

$$\lim_{n\to\infty} \left(\sum_{i=1}^{k} a_{i,n} \right) = \sum_{i=1}^{k} a_i.$$

Let $k > 0$ and $\varepsilon > 0$. By definition, for each $1 \leq i \leq k$, there is N_i such that such that for each $n > N_i$, $|a_{i,n} - a_i| < \frac{\varepsilon}{k}$. Let N be the maximum of $N_1, N_2, \cdots, N_k$. Then, for each $n > N$, $|\sum_{i=1}^{k} a_{i,n} - \sum_{i=1}^{k} a_i| <= |\sum_{i=1}^{k}(a_{i,n} - a_i)| \leq \sum_{i=1}^{k} |a_{i,n} - a_i| < k\frac{\varepsilon}{k} = \varepsilon$.

LS19. We need to prove that:
If k is any constant value and if

$$\lim_{n\to\infty} a_n = a,$$

then

$$\lim_{n\to\infty} ka_n = ka.$$

Let $\varepsilon > 0$. By definition, for each there is N such that such that for each $n > N_i$, $|a_n - a| < \frac{\varepsilon}{k}$. Hence, for each $n > N_i$, $|ka_n - ka| = k|a_n - a| < k\frac{\varepsilon}{k} = \varepsilon$. Hence, $\lim_{n\to\infty} ka_n = ka$.

LS20. We need to prove that:
If

$$\lim_{n\to\infty} a_n = a, \text{ and } \lim_{n\to\infty} b_n = b,$$

then

$$\lim_{n\to\infty} (a_n - b_n) = a - b.$$

Let $\varepsilon > 0$. By definition, there are N_1 and N_2 such that such that for each $n > N_1$, $|a_n - a| < \frac{\varepsilon}{2}$ and for each $n > N_2$, $|b_n - b| < \frac{\varepsilon}{2}$. Let N be the maximum of N_1, N_2. Then, for each $n > N$, $|a_n - b_n - (a - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence, $\lim_{n\to\infty}(a_n - b_n) = a - b$.

LS21. Let $M < 0$ and a natural number N such that $N \geq -M$. Then, for each $n > N$, $a_n = -n < -N \leq M$. Hence, $\lim_{n \to \infty} -n = -\infty$

LS22. We need to prove that:
If $\lim_{n \to \infty} a_n = \pm\infty$ and $\lim_{n \to \infty} b_n = b$, then

$$\lim_{n \to \infty} (a_n - b_n) = \pm\infty.$$

1. The hypotheses are $\lim_{n \to \infty} a_n = \infty$ and $\lim_{n \to \infty} b_n = b$. By the hypotheses,
   (a) for every $M > 0$, there is N such that for $n > N$, $a_n > M$,
   (b) for every $\epsilon > 0$, there is N such that for $n > N$, $|b_n - b| < \epsilon$.

   Let $M > 0$ be given. Then there are N_1 and N_2 such that
   (a) For $n > N_1$, $a_n > M + |b| + 1$; and
   (b) For $n > N_2$, $|b_n - b| < \epsilon$, where if $b = 0$, $\epsilon = \frac{1}{2}$ and if $b \neq 0$, ϵ is the minimum of $\frac{1}{2}$ and $|b|$.

Let N be the maximum of N_1 and N_2. Then for $n > N$,

$$a_n - b_n > M + |b| + 1 - b_n.$$

This is clearly greater than M if b_n is 0 or negative, so suppose b_n is positive. Then b is either 0 or positive. If b is 0, then $|b_n - b| = |b_n| < \frac{1}{2}$, so, since b_n is positive, $b_n < \frac{1}{2}$, then $-b_n > -\frac{1}{2}$, and $1 - b_n > \frac{1}{2}$. Therefore,

$$M + |b| + 1 - b_n = M + 1 - b_n > M + \frac{1}{2} > M.$$

If b is positive, then $|b| = b$ and $|b - b_n| < \frac{1}{2}$. Hence, $b - b_n > -\frac{1}{2}$ and $|b| + 1 - b_n > 1 - \frac{1}{2} = \frac{1}{2}$. Hence,

$$M + |b| + 1 + b_n > M + \frac{1}{2} > M.$$

In either case, $a_n - b_n > M$.

2. The hypotheses are $\lim_{n \to \infty} a_n = -\infty$ and $\lim_{n \to \infty} b_n = b$. By the hypotheses,
   (a) for every $M < 0$, there is N such that for $n > N$, $a_n < M$,
   (b) for every $\epsilon > 0$, there is N such that for $n > N$, $|b_n - b| < \epsilon$.

   Let $M < 0$ be given. Then there are N_1 and N_2 such that
   (a) For $n > N_1$, $a_n < M - |b| - 1$; and
   (b) For $n > N_2$, $|b_n - b| < \epsilon$, where if $b = 0$, $\epsilon = \frac{1}{2}$ and if $b \neq 0$, ϵ is the minimum of $\frac{1}{2}$ and $|b|$.

   Let N be the maximum of N_1 and N_2. Then for $n > N$,

$$a_n - b_n < M - |b| - 1 - b_n.$$

This is clearly less than M if b_n is positive or 0, so suppose b_n is negative. Then b is 0 or negative. If b is 0, then $|b_n - b| = |b_n| < \frac{1}{2}$, so, since b_n is negative, $-b_n < \frac{1}{2}$, and

$$M - |b| - 1 - b_n = M - 1 - b_n < M - 1 - \frac{1}{2} < M.$$

If b is negative, then $|b| = -b$ and $|b - b_n| < \frac{1}{2}$. Hence,

$$-\frac{1}{2} < b - b_n < \frac{1}{2},$$

and so,

$$b - 1 - b_n < -\frac{1}{2}.$$

Hence,

$$M - |b| - 1 - b_n = M + b - 1 - b_n < M - \frac{1}{2} < M.$$

In either case, $a_n - b_n < M$.

LS23. If $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = \pm\infty$, then

$$\lim_{n \to \infty} (a_n - b_n) = \mp\infty.$$

1. The hypotheses are $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = \infty$. This is like the first case of LS22 above with the a_n and b_n interchanged.
2. The hypotheses are $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = -\infty$. This is like the second case of LS22 above with the roles of a_n and b_n interchanged.

LS24. If $\lim_{n \to \infty} a_n = \pm\infty$ and $\lim_{n \to \infty} b_n = \mp\infty$, then

$$\lim_{n \to \infty} (a_n - b_n) = \pm\infty.$$

1. The hypotheses are $\lim_{n \to \infty} a_n = \infty$ and $\lim_{n \to \infty} b_n = -\infty$. From the hypotheses,
   (a) For every $M > 0$, there is $N > 0$ such that if $n > N$, $a_n > M$,
   (b) For every $M < 0$, there is $N > 0$ such that if $n > N$, $b_n < M$.

   Let $M > 0$ be given. Then there are N_1 and N_2 such that
   (a) if $n > N_1$, $a_n > \frac{M}{2}$, and
   (b) if $n > N_2$, $b_n < -\frac{M}{2}$.

   Let N be the maximum of N_1 and N_2. Then for $n > N$,

   $$a_n - b_n > \frac{M}{2} + \frac{M}{2} = M.$$

2. The hypotheses are $\lim_{n \to \infty} a_n = -\infty$ and $\lim_{n \to \infty} b_n = \infty$. From the hypotheses,
   (a) For every $M < 0$, there is $N > 0$ such that if $n > N$, $a_n < M$,
   (b) For every $M < 0$, there is $N > 0$ such that if $n > N$, $b_n > M$.

Let $M < 0$ be given. Then there are N_1 and N_2 such that

(a) if $n > N_1$, $a_n < \frac{M}{2}$, and

(b) if $n > N_2$, $b_n > -\frac{M}{2}$.

Let N be the maximum of N_1 and N_2. Then for $n > N$,

$$a_n - b_n < \frac{M}{2} + \frac{M}{2} = M.$$

LS25. We need to prove that:
If $k > 0$ and for each i, $0 < i < k$, $\lim_{n \to \infty} a_{i,n} = a_i$, then

$$\lim_{n \to \infty} \left(\prod_{i=1}^{k} a_{i,n} \right) = \prod_{i=1}^{k} a_i.$$

Let $I = \{k \in \mathbb{N}^* : \text{ if } \forall 0 < i < k, \lim_{n \to \infty} a_{i,n} = a_i, \text{ then } \lim_{n \to \infty} \left(\prod_{i=1}^{k} a_{i,n} \right) = \prod_{i=1}^{k} a_i \}$.

We prove by induction that $I = \mathbb{N}^*$.

Clearly $1 \in I$. Assume $k \in I$. We have if $\forall 0 < i < k$, if $\lim_{n \to \infty} a_{i,n} = a_i$, then $\lim_{n \to \infty} \left(\prod_{i=1}^{k} a_{i,n} \right) = \prod_{i=1}^{k} a_i$. Repeating the proof we gave for LS3 on Page 247, on $\lim_{n \to \infty} \left(\prod_{i=1}^{k} a_{i,n} \right) = \prod_{i=1}^{k} a_i$ and $\lim_{n \to \infty} a_{k+1,n} = a_{k+1}$, we can show that $\lim_{n \to \infty} \left(\prod_{i=1}^{k+1} a_{i,n} \right) = \prod_{i=1}^{k+1} a_i$. Hence, $k + 1 \in I$. Therefore, by induction, $I = \mathbb{N}^*$ and we are done.

LS26. If $\lim_{n \to \infty} a_n = a$ and if k is a positive integer, then

$$\lim_{n \to \infty} a_n^k = a^k.$$

The proof of LS26 is similar to the proof of LS25 where for each i, $0 < i < k$, $a_{i,n} = a_n$, and each $a_i = a$.

LS27. Left to the reader.

LS28. If $\lim_{n \to \infty} a_n = \pm\infty$ and $\lim_{n \to \infty} b_n = \mp\infty$, then

$$\lim_{n \to \infty} (a_n b_n) = -\infty.$$

This is two results.

1. The hypotheses are $\lim_{n \to \infty} a_n = \infty$ and $\lim_{n \to \infty} b_n = -\infty$. By the hypotheses,

   (a) For every $M > 0$, there is N such that for $n > N$, $a_n > M$.

   (b) For every $M < 0$, there is N such that for $n > N$, $b_n < M$.

   Let $M < 0$ be given. Then there is N_1 such that for $n > N_1$, $a_n > M^2$ and there is N_2 such that for $n > N_2$, $b_n < \frac{1}{M}$. Let N be the maximum of N_1 and N_2. Then for all $n > N$, $b_n < \frac{1}{M}$ and $a_n > M^2$. Hence, $-a_n < -M^2$ and $-a_n b_n > -M$ and so, $a_n b_n < M$.

2. If the hypotheses are $\lim_{n \to \infty} a_n = -\infty$ and $\lim_{n \to \infty} b_n = \infty$ is similar.

LS29. If $\lim_{n \to \infty} a_n = \pm\infty$ and $\lim_{n \to \infty} b_n = b > 0$, then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \pm\infty.$$

This is two results.

1. The first hypothesis is $\lim_{n \to \infty} a_n = \infty$. By the hypotheses,

   (a) For every $\epsilon > 0$, there is N such that for $n > N_1$, $|b_n - b| < \epsilon$, and

   (b) For every $M > 0$, there is N such that for $n > N$, $a_n > M$.

   Let $M > 0$ be given. Then

   (a) There is N_1 such that for all $n > N_1$, $|b_n - b| < \frac{b}{2}$, and

   (b) There is N_2 such that for all $n > N_2$, $a_n > \frac{3bM}{2}$.

   Then for $n > N_1$, $|b_n| = |b_n - b + b| \leq |b_n - b| + |b| < b + \frac{b}{2} = \frac{3b}{2}$, so $\frac{1}{|b_n|} > \frac{2}{3b}$. Let N be the maximum of N_1 and N_2. Then for $n > N$, $\left|\frac{a_n}{b_n}\right| = \frac{|a_n|}{|b_n|} > \frac{3bM}{2}\frac{2}{3b} = M$.

2. The first hypothesis is $\lim_{n \to \infty} a_n = -\infty$. By the hypotheses,

   (a) For every $\epsilon > 0$, there is N such that for $n > N$, $|b_n - b| < \epsilon$, and

   (b) For every $M < 0$, there is N such that for $n > N$, $a_n < M$.

   Let $M < 0$. Then

   (a) There is N_1 such that for all $n > N$, $|b_n - b| < \frac{b}{2}$, and

(b) There is N_2 such that for all $n > N_2$, $a_n < \frac{Mb}{2}$.

Then for $n > N_1$, $-\frac{b}{2} < b_n - b < \frac{b}{2}$ and so, $b_n > b - \frac{b}{2} = \frac{b}{2} > 0$. so $\frac{1}{|b_n|} < \frac{2}{b}$. Let N be the maximum of N_1 and N_2. Then for $n > N$, $\left|\frac{a_n}{b_n}\right| = \frac{|a_n|}{|b_n|} < \frac{bM}{2}\frac{2}{b} = M$.

LS30. Left to the reader.

LS31. Left to the reader.

LS32. Left to the reader.

LS33. Left to the reader.

LS34. Left to the reader.

Solution C.131. [Of Exercise 9.3.]

1. Let $\varepsilon > 0$, and let N be the smallest integer such that $N > \frac{1}{\varepsilon} > 0$. For all $n > N$ we have $\frac{1}{\varepsilon} < N < n$ and hence $\frac{1}{n} < \varepsilon$. Now, for each $n > N$, $\left|\frac{(-1)^n}{n} - 0\right| = \frac{|(-1)^n|}{|n|} = \frac{1}{n} < \varepsilon$. Hence $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

2. Given $\varepsilon > 0$, we want, for large values of n:

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon.$$

But $\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \left|\frac{-7}{3(3n+2)}\right| = \frac{7}{3(3n+2)}$. Hence we want

$$\frac{7}{3(3n+2)} < \varepsilon.$$

Let N be the smallest positive integer such that $N > \frac{7}{9\varepsilon} - \frac{2}{3}$. Hence for all $n > N$ we have $n + \frac{6}{9} > \frac{7}{9\varepsilon}$ and hence $\frac{7}{3(3n+2)} < \varepsilon$.

Now, $\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| =$
$\left|\frac{-7}{3(3n+2)}\right| = \frac{7}{3(3n+2)} < \varepsilon$. Hence by definition

$$\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}.$$

3. Given $\varepsilon > 0$, we want, for large values of n:

$$\left|\frac{n+6}{n^2-6}\right| < \varepsilon.$$

But $\left|\frac{n+6}{n^2-6}\right| = \left|\frac{1}{n-6}\right|$. Hence (for $n > 6$) we want

$$\frac{1}{n-6} < \varepsilon.$$

Let N be the smallest integer such that $N > \frac{1}{\varepsilon} + 6 > 6$. Hence for all $n > N$ we have $\frac{1}{n-6} < \varepsilon$. Now $\left|\frac{n+6}{n^2-6}\right| = \left|\frac{1}{n-6}\right| < \varepsilon$. Hence by definition,

$$\lim_{n\to\infty} \frac{n+6}{n^2-6} = 0.$$

4. Given $\varepsilon > 0$, we want, for large values of n:

$$\left|\sqrt{n^2+1} - n\right| < \varepsilon.$$

But $\left|\sqrt{n^2+1} - n\right| = \left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\right| = \left|\frac{1}{\sqrt{n^2+1}+n}\right|$. Hence we want

$$\frac{1}{\sqrt{n^2+1}+n} < \varepsilon.$$

But, for $n > 1$ we have $\frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n}$. Hence, we need $n > \frac{1}{2\varepsilon}$. Let N be the smallest integer greater than 1 such that $N > \frac{1}{2\varepsilon}$. Then, for any $n > N$ we have $n > \frac{1}{2\varepsilon}$ and hence $\frac{1}{2n} < \varepsilon$.

Now, $\left|\sqrt{n^2+1} - n\right| = \left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\right| = \left|\frac{1}{\sqrt{n^2+1}+n}\right| = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n} < \varepsilon$. Hence by definition,

$$\lim_{n\to\infty} \left[\sqrt{n^2+1} - n\right] = 0.$$

5. Given $\varepsilon > 0$, we want, for large values of n:

$$\left|\sqrt{n^2+n} - n - 1/2\right| < \varepsilon.$$

But $\left|\sqrt{n^2+n} - n - 1/2\right| = \left|\frac{(\sqrt{n^2+n} - (n+1/2))(\sqrt{n^2+n} + (n+1/2))}{\sqrt{n^2+n} + n + 1/2}\right| = \left|\frac{-1/4}{\sqrt{n^2+n}+n+1/2}\right| = \frac{1}{4\sqrt{n^2+n}+4n+2}$. Hence we want

$$\frac{1}{4\sqrt{n^2+n}+4n+2} < \varepsilon.$$

But, for $n > 1$ we have $\frac{1}{4\sqrt{n^2+n}+4n+2} < \frac{1}{4n+4n+2} < \frac{1}{4n+4n} = \frac{1}{8n}$. Hence, we need $\frac{1}{8n} < \varepsilon$ or $n > \frac{1}{10\varepsilon}$. Let N be the smallest integer greater than 1 such that $N > \frac{1}{8\varepsilon}$. Then, for any $n > N$ we have $n > \frac{1}{10\varepsilon}$ and hence $\frac{1}{8n} < \varepsilon$.
Now, $\left|\sqrt{n^2+n} - n - 1/2\right| = \left|\frac{(\sqrt{n^2+n} - (n+1/2))(\sqrt{n^2+n} + (n+1/2))}{\sqrt{n^2+n} + (n+1/2)}\right| = \left|\frac{-1/4}{\sqrt{n^2+n}+n+1/2}\right| = \frac{1}{4\sqrt{n^2+n}+4n+2} < \frac{1}{8n} < \varepsilon$. Hence by definition,

$$\lim_{n\to\infty}\left[\sqrt{n^2+n} - n\right] = \frac{1}{2}.$$

Solution C.132. [Of Exercise 9.4.]

1. We will show that

$$\lim_{n\to\infty} |n/(n^2+1)| = 0.$$

Given $\varepsilon > 0$, we want, for large values of n (note that $n > 0$):

$$\left|n/(n^2+1)\right| < \varepsilon.$$

But $\left|n/(n^2+1)\right| < n/n^2 = \frac{1}{n}$. Hence we want

$$\frac{1}{n} < \varepsilon.$$

Let N be the smallest integer such that $N > \frac{1}{\varepsilon} > 0$. Then, for any $n > N$ we have $n > \frac{1}{\varepsilon}$ and hence $\frac{1}{n} < \varepsilon$.

Now, $\left|n/(n^2+1)\right| < n/n^2 = \frac{1}{n} < \varepsilon$. Hence by definition,

$$\lim_{n\to\infty} n/(n^2+1) = 0.$$

2. We will show that

$$\lim_{n\to\infty} (7n-19)/(3n+7) = 7/3.$$

Given $\varepsilon > 0$, we want, for large values of n (note that $n > 0$):

$$|(7n-19)/(3n+7) - 7/3| < \varepsilon.$$

But $|(7n-19)/(3n+7) - 7/3| = \left|\frac{21n-57-21n-49}{3(3n+7)}\right| = \left|\frac{-106}{3(3n+7)}\right| = \frac{106}{3(3n+7)} < \frac{12\times 9}{9n} = \frac{12}{n}$. Hence we want

$$\frac{12}{n} < \varepsilon.$$

Let N be the smallest integer such that $N > \frac{12}{\varepsilon} > 0$. Then, for any $n > N$ we have $n > \frac{12}{\varepsilon}$ and hence $\frac{12}{n} < \varepsilon$.

Now, $|(7n-19)/(3n+7) - 7/3| = \left|\frac{21n-57-21n-49}{3(3n+7)}\right| = \left|\frac{-106}{3(3n+7)}\right| = \frac{106}{3(3n+7)} < \frac{12\times 9}{9n} = \frac{12}{n} < \varepsilon$.

Hence by definition,

$$\lim_{n\to\infty} (7n-19)/(3n+7) = 7/3.$$

3. We will show that

$$\lim_{n\to\infty} (4n+3)/(7n-5) = 4/7.$$

Given $\varepsilon > 0$, we want, for large values of n (note that $n > 0$):

$$|(4n+3)/(7n-5) - 4/7| < \varepsilon.$$

But for $n > 1$, we have $|(4n+3)/(7n-5) - 4/7| = \left|\frac{28n+21-28n+20}{7(7n-5)}\right| = \left|\frac{41}{7(7n-5)}\right| = \frac{41}{7(7n-5)} < \frac{42}{7(7n-5)} = \frac{6}{7n-5} < \frac{6}{7n-5n} = \frac{3}{n}$. Hence we want

$$\frac{3}{n} < \varepsilon.$$

Let N be the smallest integer greater than 1 such that $N > \frac{3}{\varepsilon} > 0$. Then, for any $n > N$ we have $n > \frac{3}{\varepsilon}$ and hence $\frac{3}{n} < \varepsilon$.

Now, $|(4n+3)/(7n-5) - 4/7| = \left|\frac{28n+21-28n+20}{7(7n-5)}\right| = \left|\frac{41}{7(7n-5)}\right| = \frac{41}{7(7n-5)} < \frac{42}{7(7n-5)} = \frac{6}{7n-5} < \frac{6}{7n-5n} = \frac{3}{n} < \varepsilon$.

Hence by definition,

$$\lim_{n\to\infty} (4n+3)/(7n-5) = 4/7.$$

4. We will show that

$$\lim_{n\to\infty} (2n+4)/(5n+2) = 2/5.$$

Given $\varepsilon > 0$, we want, for large values of n (note that $n > 0$):

$$|(2n+4)/(5n+2) - 2/5| < \varepsilon.$$

But for $n > 1$, we have $|(2n+4)/(5n+2) - 2/5| = \left|\frac{10n+20-10n-4}{5(5n+2)}\right| = \left|\frac{16}{5(5n+2)}\right| < \frac{16}{5(7n)} < \frac{16}{32n} = \frac{1}{2n}$. Hence we want

$$\frac{1}{2n} < \varepsilon.$$

Let N be the smallest integer greater than 1 such that $N > \frac{1}{2\varepsilon} > 0$. Then, for any $n > N$ we have $n > \frac{1}{2\varepsilon}$ and hence $\frac{1}{2n} < \varepsilon$.

Now, $|(2n+4)/(5n+2) - 2/5| = \left|\frac{10n+20-10n-4}{5(5n+2)}\right| = \left|\frac{16}{5(5n+2)}\right| < \frac{16}{5(7n)} < \frac{16}{32n} = \frac{1}{2n} < \varepsilon$.

Hence by definition,

$$\lim_{n\to\infty} (2n+4)/(5n+2) = 2/5.$$

5. We will show that
$$\lim_{n\to\infty}(1/n)\sin n = 0.$$
Given $\varepsilon > 0$, we want, for large values of n (note that $n > 0$):
$$|(1/n)\sin n| < \varepsilon.$$
But $|(1/n)\sin n| = \dfrac{|\sin n|}{n} < \dfrac{2}{n}$. Hence let N be the smallest integer such that $N > \dfrac{2}{\varepsilon}$. Then for any $n > N$ we have $n > \dfrac{2}{\varepsilon}$ and so, $\dfrac{2}{n} < \varepsilon$. Now, for any $n > N$ we have $|(1/n)\sin n| = \dfrac{|\sin n|}{n} < \dfrac{2}{n} < \varepsilon$.

Solution C.133. [Of Exercise 9.5.]

LF1. We need to prove
$$\lim_{x\to c^-} x = c \text{ and } \lim_{x\to c^+} x = c \text{ and } \lim_{x\to\infty} x = \infty \text{ and } \lim_{x\to-\infty} x = -\infty$$

- Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Then, since $f(x) = x$ for all x, we have that if $0 < c - x < \varepsilon$, then $|f(x) - c| = |x - c| < \varepsilon$. Hence by definition $\lim_{x\to c^-} x = c$.
- Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Then, since $f(x) = x$ for all x, we have that if $0 < x - c < \varepsilon$, then $|f(x) - c| = |x - c| < \varepsilon$. Hence by definition $\lim_{x\to c^+} x = c$.
- Let $M_1 > 0$ and let $M_2 = M_1$. If $x > M_2$ then $f(x) = x > M_1$. Hence by definition $\lim_{x\to\infty} x = \infty$.
- Let $M_1 < 0$ and let $M_2 = M_1$. If $x < M_2$ then $f(x) = x < M_1$. Hence by definition $\lim_{x\to-\infty} x = -\infty$.

LF2. We need to prove
$$\lim_{x\to c^-} k = k \text{ and } \lim_{x\to c^+} k = k \text{ and } \lim_{x\to\infty} k = k \text{ and } \lim_{x\to-\infty} k = k$$

- Let $\varepsilon > 0$ be given. Let δ be no matter what. Then, since $f(x) = k$ for all x, we have that if $0 < c-x < \delta$, then $|f(x)-k| = |k - k| = 0 < \varepsilon$. Hence by definition $\lim_{x\to c^-} k = k$.
- Let $\varepsilon > 0$ be given. Let δ be no matter what. Then, since $f(x) = k$ for all x, we have that if $0 < x-c < \delta$, then $|f(x)-k| = |k - k| = 0 < \varepsilon$. Hence by definition $\lim_{x\to c^+} k = k$.
- Let $\varepsilon > 0$ be given. Let $M > 0$ be anything. If $x > M$ then $|f(x) - k| = |k - k| = 0 < \varepsilon$. Hence by definition $\lim_{x\to\infty} k = k$.

– Let $\varepsilon > 0$ be given. Let $M < 0$ be anything. If $x < M$ then $|f(x) - k| = |k - k| = 0 < \varepsilon$. Hence by definition $\lim_{x\to-\infty} k = k$.

LF3. We need to prove the following:

1. If $\lim_{x\to\pm\infty} f(x) = l$ and $\lim_{x\to\pm\infty} g(x) = m$ then $\lim_{x\to\pm\infty}[f(x) + g(x)] = l + m$.
2. If $\lim_{x\to c^-} f(x) = l$ and $\lim_{x\to c^-} g(x) = m$ then $\lim_{x\to c^-}[f(x) + g(x)] = l + m$.
3. If $\lim_{x\to c^+} f(x) = l$ and $\lim_{x\to c^+} g(x) = m$ then $\lim_{x\to c^+}[f(x) + g(x)] = l + m$.

The proof is as follows:

1. Let $\varepsilon > 0$ be given. Then there are $M_1 > 0$ and $M_2 > 0$ (resp. $M_1 < 0$ and $M_2 < 0$) such that
   (a) if $x > M_1$ (resp. $x < M_1$) then $|f(x) - l| < \frac{\varepsilon}{2}$, and
   (b) if $x > M_2$ (resp. $x < M_2$) then $|g(x) - m| < \frac{\varepsilon}{2}$.

   Let M be the larger of M_1 and M_2 (resp. the smaller of M_1 and M_2). Then for $x > M$ (resp. $x < M$) we have

   $$\begin{aligned} & |(f(x) + g(x)) - (l + m)| \\ = & |(f(x) - l) + (g(x) - m)| \\ \leq & |f(x) - l| + |g(x) - m| \\ < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ = & \varepsilon. \end{aligned}$$

   Hence by definition $\lim_{x\to\pm\infty}[f(x) + g(x)] = l + m$.
2. Let $\varepsilon > 0$ be given. Then there are δ_1 and δ_2 such that
   (a) if $0 < c - x < \delta_1$, then $|f(x) - l| < \frac{\varepsilon}{2}$, and
   (b) if $0 < c - x < \delta_2$, then $|g(x) - m| < \frac{\varepsilon}{2}$.

   Let δ be the smaller of δ_1 and δ_2. Then for $0 < c - x < \delta$, we have

   $$\begin{aligned} & |(f(x) + g(x)) - (l + m)| \\ = & |(f(x) - l) + (g(x) - m)| \\ \leq & |f(x) - l| + |g(x) - m| \\ < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ = & \varepsilon. \end{aligned}$$

   Hence by definition $\lim_{x\to c^-}[f(x) + g(x)] = l + m$.
3. Let $\varepsilon > 0$ be given. Then there are δ_1 and δ_2 such that

(a) if $0 < x - c < \delta_1$, then $|f(x) - l| < \frac{\varepsilon}{2}$, and

(b) if $0 < x - c < \delta_2$, then $|g(x) - m| < \frac{\varepsilon}{2}$.

Let δ be the smaller of δ_1 and δ_2. Then for $0 < x - c < \delta$, we have

$$\begin{aligned} |(f(x) + g(x)) - (l + m)| &= |(f(x) - l) + (g(x) - m)| \\ &\leq |f(x) - l| + |g(x) - m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence by definition $\lim_{x \to c^+} [f(x) + g(x)] = l + m$.

LF4. We need to prove the following:

1. If $\lim_{x \to \pm\infty} f(x) = l$ and $\lim_{x \to \pm\infty} g(x) = m$ then $\lim_{x \to \pm\infty} [f(x)g(x)] = lm$.
2. If $\lim_{x \to c^-} f(x) = l$ and $\lim_{x \to c^-} g(x) = m$ then $\lim_{x \to c^-} [f(x)g(x)] = lm$.
3. If $\lim_{x \to c^+} f(x) = l$ and $\lim_{x \to c^+} g(x) = m$ then $\lim_{x \to c^+} [f(x)g(x)] = lm$.

The proof is as follows:

1. Let $\varepsilon > 0$ be given. Then by hypothesis there are $M_1 > 0$ and $M_2 > 0$ (resp. $M_1 < 0$ and $M_2 < 0$) such that

   (a) if $x > M_1$ (resp. $x < M_1$), then $|f(x) - l| < \dfrac{\varepsilon}{2\left(\dfrac{\varepsilon}{2(|l| + 1)} + |m|\right)}$, and

   (b) if $x > M_2$ (resp. $x < M_2$), then $|g(x) - m| < \dfrac{\varepsilon}{2(|l| + 1)}$.

   (Here, $|l| + 1$ is used instead of $|l|$ since l might be 0.) Since $|g(x) - m| < \dfrac{\varepsilon}{2(|l| + 1)}$, then $|g(x)| = |g(x) - m + m| \leq |g(x) - m| + |m| < \dfrac{\varepsilon}{2(|l| + 1)} + |m|$. Let M be the minimum (resp. maximum) of M_1 and M_2. Then for $x > M$ (resp. $x < M$):

$$\begin{aligned}
& |f(x)g(x) - lm| \\
= \; & |f(x)g(x) - lg(x) + lg(x) - lm| \\
= \; & |(f(x) - l)g(x) + l(g(x) - m)| \\
\leq \; & |f(x) - l||g(x)| + |l||g(x) - m| \\
< \; & \frac{\varepsilon}{2\left(\frac{\varepsilon}{2(|l|+1)} + |m|\right)}\left(\frac{\varepsilon}{2(|l|+1)} + |m|\right) + |l|\frac{\varepsilon}{2(|l|+1)} \\
< \; & \frac{\varepsilon}{2} + (|l|+1)\frac{\varepsilon}{2(|l|+1)} \\
= \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \; & \varepsilon.
\end{aligned}$$

Hence by definition $\lim_{x\to\pm\infty}[f(x)g(x)] = lm$.

2. Let $\varepsilon > 0$ be given. Then by hypothesis there are δ_1 and δ_2 such that

   (a) if $0 < c - x < \delta_1$, then $|f(x) - l| < \dfrac{\varepsilon}{2\left(\frac{\varepsilon}{2(|l|+1)} + |m|\right)}$, and

   (b) if $0 < c - x < \delta_2$, then $|g(x) - m| < \dfrac{\varepsilon}{2(|l|+1)}$.

   (Here, $|l| + 1$ is used instead of $|l|$ since l might be 0.) Note that the second of these implies that $|g(x)| < \frac{\varepsilon}{2(|l|+1)} + |m|$ (this is similar to what we did on Page 97). Let δ be the minimum of δ_1 and δ_2. Then for $0 < c - x < \delta$,

$$\begin{aligned}
& |f(x)g(x) - lm| \\
= \; & |f(x)g(x) - lg(x) + lg(x) - lm| \\
= \; & |(f(x) - l)g(x) + l(g(x) - m)| \\
\leq \; & |f(x) - l||g(x)| + |l||g(x) - m| \\
< \; & \frac{\varepsilon}{2\left(\frac{\varepsilon}{2(|l|+1)} + |m|\right)}\left(\frac{\varepsilon}{2(|l|+1)} + |m|\right) + |l|\frac{\varepsilon}{2(|l|+1)} \\
< \; & \frac{\varepsilon}{2} + (|l|+1)\frac{\varepsilon}{2(|l|+1)} \\
= \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \; & \varepsilon.
\end{aligned}$$

   Hence by definition $\lim_{x\to c^-}[f(x)g(x)] = lm$.

3. Let $\varepsilon > 0$ be given. Then by hypothesis there are δ_1 and δ_2 such that

(a) if $0 < x - c < \delta_1$, then $|f(x) - l| < \dfrac{\varepsilon}{2\left(\dfrac{\varepsilon}{2(|l|+1)} + |m|\right)}$, and

(b) if $0 < x - c < \delta_2$, then $|g(x) - m| < \dfrac{\varepsilon}{2(|l|+1)}$.

(Here, $|l| + 1$ is used instead of $|l|$ since l might be 0.) Note that the second of these implies that $|g(x)| < \dfrac{\varepsilon}{2(|l|+1)} + |m|$ (this is similar to what we did on Page 97). Let δ be the minimum of δ_1 and δ_2. Then for $0 < x - c < \delta$:

$$\begin{aligned}
& |f(x)g(x) - lm| \\
= \; & |f(x)g(x) - lg(x) + lg(x) - lm| \\
= \; & |(f(x) - l)g(x) + l(g(x) - m)| \\
\leq \; & |f(x) - l||g(x)| + |l||g(x) - m| \\
< \; & \frac{\varepsilon}{2\left(\dfrac{\varepsilon}{2(|l|+1)} + |m|\right)}\left(\frac{\varepsilon}{2(|l|+1)} + |m|\right) + |l|\frac{\varepsilon}{2(|l|+1)} \\
< \; & \frac{\varepsilon}{2} + (|l|+1)\frac{\varepsilon}{2(|l|+1)} \\
= \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \; & \varepsilon.
\end{aligned}$$

Hence by definition $\lim_{x\to c^+}[f(x)g(x)] = lm$.

LF5. We need to prove the following:

1. If $\lim_{x\to\pm\infty} f(x) = l \neq 0$ then $\lim_{x\to\pm\infty} \dfrac{1}{f(x)} = \dfrac{1}{l}$.
2. If $\lim_{x\to c^-} f(x) = l \neq 0$ then $\lim_{x\to c^-} \dfrac{1}{f(x)} = \dfrac{1}{l}$.
3. If $\lim_{x\to c^+} f(x) = l \neq 0$ then $\lim_{x\to c^+} \dfrac{1}{f(x)} = \dfrac{1}{l}$.

The proof is as follows:

1. Let $\varepsilon > 0$ be given. Let $\varepsilon' > 0$ be the minimum of $|l|/2$ and $(l^2\varepsilon)/2$. By hypothesis, there is $M > 0$ (resp. $M < 0$) such that if $x > M$ (resp. $x < M$) then

$$|f(x) - l| < \frac{l^2\varepsilon}{2}$$

and

$$|f(x) - l| < \frac{|l|}{2}.$$

We also have $f(x) \geq |l|/2$, since otherwise we would have

$$|l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)| < \frac{|l|}{2} + \frac{|l|}{2} = |l|,$$

a contradiction. It follows that

$$\frac{1}{|f(x)|} \leq \frac{2}{|l|}.$$

Hence if $x > M$ (resp. $x < M$) then

$$\begin{aligned} \left| \frac{1}{f(x)} - \frac{1}{l} \right| &= \frac{|f(x) - l|}{|l||f(x)|} \\ &= \frac{|f(x) - l|}{|l||f(x)|} \\ &\leq \frac{2}{|l|} \frac{|f(x) - l|}{|l|} \\ &< \frac{2}{l^2} |f(x) - l| \\ &< \frac{2}{l^2} \frac{l^2 \varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence by definition, $\lim_{x \to \pm\infty} \frac{1}{f(x)} = \frac{1}{l}$.

2. Let $\varepsilon > 0$ be given. Let $\varepsilon' > 0$ be the minimum of $|l|/2$ and $(l^2 \varepsilon)/2$. By hypothesis, there is $\delta > 0$ such that if $0 < c - x < \delta$, then

$$|f(x) - l| < \frac{l^2 \varepsilon}{2}$$

and

$$|f(x) - l| < \frac{|l|}{2}.$$

We also have $f(x) \geq |l|/2$, since otherwise we would have

$$|l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)| < \frac{|l|}{2} + \frac{|l|}{2} = |l|,$$

a contradiction. It follows that

$$\frac{1}{|f(x)|} \leq \frac{2}{|l|}.$$

Hence if $0 < c - x < \delta$, then

$$\begin{aligned}\left|\frac{1}{f(x)} - \frac{1}{l}\right| &= \frac{|f(x) - l|}{|l||f(x)|} \\ &= \frac{|f(x) - l|}{|l||f(x)|} \\ &\leq \frac{2}{|l|}\frac{|f(x) - l|}{|l|} \\ &< \frac{2}{l^2}|f(x) - l| \\ &< \frac{2}{l^2}\frac{l^2\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence by definition, $\lim_{x\to c^-} \frac{1}{f(x)} = \frac{1}{l}$.

3. Let $\varepsilon > 0$ be given. Let $\varepsilon' > 0$ be the minimum of $|l|/2$ and $(l^2\varepsilon)/2$. By hypothesis, there is $\delta > 0$ such that if $0 < x - c < \delta$, then

$$|f(x) - l| < \frac{l^2\varepsilon}{2}$$

and

$$|f(x) - l| < \frac{|l|}{2}.$$

We also have $f(x) \geq |l|/2$, since otherwise we would have

$$|l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)| < \frac{|l|}{2} + \frac{|l|}{2} = |l|,$$

a contradiction. It follows that

$$\frac{1}{|f(x)|} \leq \frac{2}{|l|}.$$

Hence if $0 < x - c < \delta$, then

$$\begin{aligned}\left|\frac{1}{f(x)} - \frac{1}{l}\right| &= \frac{|f(x) - l|}{|l||f(x)|} \\ &= \frac{|f(x) - l|}{|l||f(x)|} \\ &\leq \frac{2}{|l|}\frac{|f(x) - l|}{|l|} \\ &< \frac{2}{l^2}|f(x) - l| \\ &< \frac{2}{l^2}\frac{l^2\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence by definition, $\lim_{x\to c^+} \frac{1}{f(x)} = \frac{1}{l}$.

LF6. We need to prove the following:

1. If $\lim_{x\to\pm\infty} f(x) = l$ and if $l \geq 0$ whenever n is even, then $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{l}$.
2. If $\lim_{x\to c^-} f(x) = l$ and if $l \geq 0$ whenever n is even, then $\lim_{x\to c^-} \sqrt[n]{f(x)} = \sqrt[n]{l}$.
3. If $\lim_{x\to c^+} f(x) = l$ and if $l \geq 0$ whenever n is even, then $\lim_{x\to c^+} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

The proof is as follows:

1. There are three cases.

Case 1. $l = 0$. Let $\varepsilon > 0$ be given. Then there is $M > 0$ (resp. $M < 0$) such that if $x > M$ (resp. $x < M$), we have $|f(x)| < \varepsilon^n$. Hence, if $x > M$ (resp. $x < M$), we have $|\sqrt[n]{f(x)}| = \sqrt[n]{|f(x)|} < \sqrt[n]{\varepsilon^n} = \varepsilon$. Hence by definition, $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

Case2. $l > 0$. Let $\varepsilon > 0$ be given. Then there is $M > 0$ (resp. $M < 0$) such that if $x > M$ (resp. $x < M$), we have

$$|f(x) - l| < \varepsilon|\sqrt[n]{l^{n-1}}|$$

$$|f(x) - l| < \frac{|l|}{2}.$$

From the second of these, it follows that $f(x)$ and l have the same sign. For any $n \in \mathbb{N}$, let P_q stand for $\sqrt[n]{f(x)^q}$. It follows that for these values of x,

$$\begin{aligned}
&|\sqrt[n]{f(x)} - \sqrt[n]{l}| \\
&= \left|\frac{(P_1 - \sqrt[n]{l})(P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}})}{(P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}})}\right| \\
&= \frac{|f(x) - l|}{|P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}}|} \\
&< \frac{\varepsilon|\sqrt[n]{l^{n-1}}|}{|\sqrt[n]{l^{n-1}}|} \\
&= \varepsilon.
\end{aligned}$$

Hence by definition, $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

Case 3. $l < 0$. Then n is odd. Let $\varepsilon > 0$ be given. Then there is $M > 0$ (resp. $M < 0$) such that if $x > M$ (resp. $x < M$), we have

$$|f(x) - l| < \varepsilon|\sqrt[n]{l^{n-1}}|$$
$$|f(x) - l| < \frac{|l|}{2}.$$

From the second of these, it follows that $f(x)$ and l have the same sign, both negative and $|f(x)| = -f(x)$ and $|l| = -l$. It follows that if $0 < |x - c| < \delta$,

$$\begin{aligned} |\sqrt[n]{f(x)} - \sqrt[n]{l}| &= |\sqrt[n]{-|f(x)|} - \sqrt[n]{-|l|}| \\ &= |-\sqrt[n]{|f(x)|} + \sqrt[n]{|l|}| \\ &= |\sqrt[n]{|f(x)|} - \sqrt[n]{|l|}| \end{aligned}$$

and we can prove this less than ε by Case 2 above. Hence by definition, $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

2. There are three cases.

Case 1. $l = 0$. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $0 < c-x < \delta$, we have $|f(x)| < \varepsilon^n$. Hence, if $0 < c-x < \delta$, we have $|\sqrt[n]{f(x)}| = \sqrt[n]{|f(x)|} < \sqrt[n]{\varepsilon^n} = \varepsilon$. Hence by definition, $\lim_{x\to c^-} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

Case2. $l > 0$. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $0 < c - x < \delta$, we have

$$|f(x) - l| < \varepsilon|\sqrt[n]{l^{n-1}}|$$
$$|f(x) - l| < \frac{|l|}{2}.$$

From the second of these, it follows that $f(x)$ and l have the same sign. For any $n \in \mathbb{N}$, let P_q stand for $\sqrt[n]{f(x)^q}$. It follows that for these values of x,

$$\begin{aligned} &|\sqrt[n]{f(x)} - \sqrt[n]{l}| \\ &= \left|\frac{(P_1 - \sqrt[n]{l})(P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}})}{(P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}})}\right| \\ &= \frac{|f(x) - l|}{|P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}}|} \\ &< \frac{\varepsilon|\sqrt[n]{l^{n-1}}|}{|\sqrt[n]{l^{n-1}}|} \\ &= \varepsilon. \end{aligned}$$

Hence by definition, $\lim_{x\to c^-} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

Case 3. $l < 0$. Then n is odd. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $0 < c - x < \delta$, we have

$$|f(x) - l| < \varepsilon|\sqrt[n]{l^{n-1}}|$$

$$|f(x) - l| < \frac{|l|}{2}.$$

From the second of these, it follows that $f(x)$ and l have the same sign, both negative and $|f(x)| = -f(x)$ and $|l| = -l$. It follows that if $0 < |x - c| < \delta$,

$$\begin{aligned} |\sqrt[n]{f(x)} - \sqrt[n]{l}| &= |\sqrt[n]{-|f(x)|} - \sqrt[n]{-|l|}| \\ &= |-\sqrt[n]{|f(x)|} + \sqrt[n]{|l|}| \\ &= |\sqrt[n]{|f(x)|} - \sqrt[n]{|l|}| \end{aligned}$$

and we can prove this less than ε by Case 2 above. Hence by definition, $\lim_{x\to c^-} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

3. There are three cases.

Case 1. $l = 0$. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $0 < x-c < \delta$, we have $|f(x)| < \varepsilon^n$. Hence, if $0 < x-c < \delta$, we have $|\sqrt[n]{f(x)}| = \sqrt[n]{|f(x)|} < \sqrt[n]{\varepsilon^n} = \varepsilon$. Hence by definition, $\lim_{x\to c^+} \sqrt[n]{f(x)} = l$.

Case2. $l > 0$. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $0 < x - c < \delta$, we have

$$|f(x) - l| < \varepsilon|\sqrt[n]{l^{n-1}}|$$

$$|f(x) - l| < \frac{|l|}{2}.$$

From the second of these, it follows that $f(x)$ and l have the same sign. For any $n \in \mathbb{N}$, let P_q stand for $\sqrt[n]{f(x)^q}$. It follows that for these values of x,

$$\begin{aligned}
&|\sqrt[n]{f(x)} - \sqrt[n]{l}| \\
&= \left| \frac{(P_1 - \sqrt[n]{l})(P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}})}{(P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}})} \right| \\
&= \frac{|f(x) - l|}{|P_{n-1} + P_{n-2}\sqrt[n]{l} + P_{n-3}\sqrt[n]{l^2} + \cdots + \sqrt[n]{l^{n-1}}|} \\
&< \frac{\varepsilon|\sqrt[n]{l^{n-1}}|}{|\sqrt[n]{l^{n-1}}|} \\
&= \varepsilon.
\end{aligned}$$

Hence by definition, $\lim_{x\to c^+} \sqrt[n]{f(x)} = \sqrt[n]{l}$.

Case 3. $l < 0$. Then n is odd. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $0 < x - c < \delta$, we have

$$|f(x) - l| < \varepsilon|\sqrt[n]{l^{n-1}}|$$

$$|f(x) - l| < \frac{|l|}{2}.$$

From the second of these, it follows that $f(x)$ and l have the same sign, both negative and $|f(x)| = -f(x)$ and $|l| = -l$. It follows that if $0 < |x - c| < \delta$,

$$\begin{aligned}
|\sqrt[n]{f(x)} - \sqrt[n]{l}| &= |\sqrt[n]{-|f(x)|} - \sqrt[n]{-|l|}| \\
&= |-\sqrt[n]{|f(x)|} + \sqrt[n]{|l|}| \\
&= |\sqrt[n]{|f(x)|} - \sqrt[n]{|l|}|
\end{aligned}$$

and we can prove this less than ε by Case 2 above. Hence by definition, $\lim_{x\to c^+} \sqrt[n]{f(x)} == \sqrt[n]{l}$.

LF7. We need to prove:

1. If $g(x) \leq f(x) \leq h(x)$ for all x in an interval whose right end-point is c, and if

$$\lim_{x\to c^-} g(x) = \lim_{x\to c^-} h(x) = l,$$

then

$$lim_{x\to c^-} f(x) = l.$$

2. If $g(x) \leq f(x) \leq h(x)$ for all x in an interval whose left end-point is c, and if

$$\lim_{x \to c^+} g(x) = \lim_{x \to c^+} h(x) = l,$$

then

$$lim_{x \to c^+} f(x) = l.$$

The proof is as follows:

1. Let $\varepsilon > 0$ be given. By hypothesis, we have
   (a) there is $\delta_1 > 0$ such that if $0 < c - x < \delta_1$, then $|g(x) - l| < \varepsilon$, and
   (b) there is $\delta_2 > 0$ such that if $0 < c - x < \delta_2$, then $|h(x) - l| < \varepsilon$.

   Let δ be the smaller of δ_1 and δ_2. Then if $0 < c - x < \delta$, we have $|g(x) - l| < \varepsilon$ and $|h(x) - l| < \varepsilon$. It follows that

$$l - \varepsilon < g(x) \leq f(x) \leq h(x) < l + \varepsilon,$$

   from which it follows that $|f(x) - l| < \varepsilon$. Hence by definition, $\lim_{x \to c^-} f(x) = l$.
2. Let $\varepsilon > 0$ be given. By hypothesis, we have
   (a) there is $\delta_1 > 0$ such that if $0 < x - c < \delta_1$, then $|g(x) - l| < \varepsilon$, and
   (b) there is $\delta_2 > 0$ such that if $0 < x - c < \delta_2$, then $|h(x) - l| < \varepsilon$.

   Let δ be the smaller of δ_1 and δ_2. Then if $0 < x - c < \delta$, we have $|g(x) - l| < \varepsilon$ and $|h(x) - l| < \varepsilon$. It follows that

$$l - \varepsilon < g(x) \leq f(x) \leq h(x) < l + \varepsilon,$$

   from which it follows that $|f(x) - l| < \varepsilon$. Hence by definition, $\lim_{x \to c^+} f(x) = l$.

LF8 We need to prove the following:

1. If $f(x) \leq g(x)$ for all x greater than some value, then

$$\lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} g(x),$$

   provided that both limits exist.
2. If $f(x) \leq g(x)$ for all x less than some negative value, then

$$\lim_{x \to -\infty} f(x) \leq \lim_{x \to -\infty} g(x),$$

   provided that both limits exist.

3. If $f(x) \leq g(x)$ for all x in an interval whose right end-point is c, then
$$\lim_{x \to c^-} f(x) \leq \lim_{x \to c^-} g(x),$$
provided that both limits exist.
4. If $f(x) \leq g(x)$ for all x in an interval whose left end-point is c, then
$$\lim_{x \to c^+} f(x) \leq \lim_{x \to c^+} g(x),$$
provided that both limits exist.

The proof is as follows:

1. Suppose that
$$\lim_{x \to \infty} f(x) = l \text{ and } \lim_{x \to \infty} g(x) = m,$$
and suppose $l > m$. Let $\varepsilon = (l - m)/2 > 0$. By hypothesis,
   (a) there is $M_1 > 0$ such that if $x > M_1$, then $|f(x) - l| < \varepsilon$, and
   (b) there is $M_2 > 0$ such that if $x > M_2$, then $|g(x) - m| < \varepsilon$.

   Let M be the larger of M_1 and M_2. Then for $x > M$, we have both $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$. Then
$$l - \varepsilon = l - \frac{l - m}{2} = \frac{2l - l + m}{2} = \frac{l + m}{2}$$
and
$$m + \varepsilon = m + \frac{l - m}{2} = \frac{2m + l - m}{2} = \frac{l + m}{2}$$
so $l - \varepsilon = m + \varepsilon$. Now the conditions $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$ imply that $g(x) < m + \varepsilon = l - \varepsilon < f(x)$, contradicting the hypothesis that $f(x) \leq g(x)$. Hence $\lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} g(x)$.
2. Suppose that
$$\lim_{x \to -\infty} f(x) = l \text{ and } \lim_{x \to -\infty} g(x) = m,$$
and suppose $l > m$. Let $\varepsilon = (l - m)/2 > 0$. By hypothesis,
   (a) there is $M_1 < 0$ such that if $x < M_1$, then $|f(x) - l| < \varepsilon$, and
   (b) there is $M_2 < 0$ such that if $x < M_2$, then $|g(x) - m| < \varepsilon$.

Let M be the smaller of M_1 and M_2. Then for $x < M$, we have both $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$. Then

$$l - \varepsilon = l - \frac{l-m}{2} = \frac{2l - l + m}{2} = \frac{l+m}{2}$$

and

$$m + \varepsilon = m + \frac{l-m}{2} = \frac{2m + l - m}{2} = \frac{l+m}{2}$$

so $l - \varepsilon = m + \varepsilon$. Now the conditions $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$ imply that $g(x) < m + \varepsilon = l - \varepsilon < f(x)$, contradicting the hypothesis that $f(x) \le g(x)$. Hence $\lim_{x\to-\infty} f(x) \le \lim_{x\to-\infty} g(x)$.

3. Suppose that

$$\lim_{x\to c^-} f(x) = l \text{ and } \lim_{x\to c^-} g(x) = m,$$

and suppose $l > m$. Let $\varepsilon = (l - m)/2 > 0$. By hypothesis,

(a) there is $\delta_1 > 0$ such that if $0 < c - x < \delta_1$, then $|f(x) - l| < \varepsilon$, and

(b) there is $\delta_2 > 0$ such that if $0 < c - x < \delta_2$, then $|g(x) - m| < \varepsilon$.

Let δ be the smaller of δ_1 and δ_2. Then for $0 < c - x < \delta$, we have both $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$. Then

$$l - \varepsilon = l - \frac{l-m}{2} = \frac{2l - l + m}{2} = \frac{l+m}{2}$$

and

$$m + \varepsilon = m + \frac{l-m}{2} = \frac{2m + l - m}{2} = \frac{l+m}{2}$$

so $l - \varepsilon = m + \varepsilon$. Now the conditions $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$ imply that $g(x) < m + \varepsilon = l - \varepsilon < f(x)$, contradicting the hypothesis that $f(x) \le g(x)$. Hence $\lim_{x\to c^-} f(x) \le \lim_{x\to c^-} g(x)$.

4. Suppose that

$$\lim_{x\to c^+} f(x) = l \text{ and } \lim_{x\to c^+} g(x) = m,$$

and suppose $l > m$. Let $\varepsilon = (l - m)/2 > 0$. By hypothesis,

(a) there is $\delta_1 > 0$ such that if $0 < x - c < \delta_1$, then $|f(x) - l| < \varepsilon$, and

(b) there is $\delta_2 > 0$ such that if $0 < x - c < \delta_2$, then $|g(x) - m| < \varepsilon$.

Let δ be the smaller of δ_1 and δ_2. Then for $0 < x - c < \delta$, we have both $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$. Then

$$l - \varepsilon = l - \frac{l-m}{2} = \frac{2l - l + m}{2} = \frac{l+m}{2}$$

and

$$m + \varepsilon = m + \frac{l-m}{2} = \frac{2m + l - m}{2} = \frac{l+m}{2}$$

so $l - \varepsilon = m + \varepsilon$. Now the conditions $|f(x) - l| < \varepsilon$ and $|g(x) - m| < \varepsilon$ imply that $g(x) < m + \varepsilon = l - \varepsilon < f(x)$, contradicting the hypothesis that $f(x) \leq g(x)$. Hence $\lim_{x\to c^+} f(x) \leq \lim_{x\to c^+} g(x)$.

LF9. We will prove that if l is a value and $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = l$ then $\lim_{x\to c} f(x) = l$.
Suppose that $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = l$. Let $\varepsilon > 0$. By hypothesis, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that if $0 < c - x < \delta_1$, then $|f(x) - l| < \varepsilon$ and if $0 < x - c < \delta_2$, then $|f(x) - l| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$ and let x such that $0 < |x - c| < \delta$.

- If $x > c$ then $0 < x - c < \delta < \delta_2$, and hence $|f(x) - l| < \varepsilon$.
- If $x < c$ then $0 < c - x < \delta < \delta_1$, and hence $|f(x) - l| < \varepsilon$.

Hence $|f(x) - l| < \varepsilon$ and by definition $\lim_{x\to c} f(x) = l$.

LF10. We need to prove the following:

1. If c is a value and if $\lim_{x\to c} f(x) = -\infty$, then $\lim_{x\to c} \dfrac{1}{f(x)} = 0$.
2. If c is a value and if $\lim_{x\to c^-} f(x) = -\infty$, then $\lim_{x\to c^-} \dfrac{1}{f(x)} = 0$.
3. If c is a value and if $\lim_{x\to c^-} f(x) = \infty$, then $\lim_{x\to c^-} \dfrac{1}{f(x)} = 0$.
4. If c is a value and if $\lim_{x\to c^+} f(x) = -\infty$, then $\lim_{x\to c^+} \dfrac{1}{f(x)} = 0$.
5. If c is a value and if $\lim_{x\to c^+} f(x) = \infty$, then $\lim_{x\to c^+} \dfrac{1}{f(x)} = 0$.
6. If $\lim_{x\to\infty} f(x) = -\infty$ then $\lim_{x\to\infty} \dfrac{1}{f(x)} = 0$.
7. If $\lim_{x\to\infty} f(x) = \infty$ then $\lim_{x\to\infty} \dfrac{1}{f(x)} = 0$.
8. If $\lim_{x\to-\infty} f(x) = -\infty$ then $\lim_{x\to-\infty} \dfrac{1}{f(x)} = 0$.
9. If $\lim_{x\to-\infty} f(x) = \infty$ then $\lim_{x\to-\infty} \dfrac{1}{f(x)} = 0$.

The proof is as follows:

1. We prove that if c is a value and if $\lim_{x\to c} f(x) = -\infty$, then $\lim_{x\to c} \dfrac{1}{f(x)} = 0$.
   Assume $\lim_{x\to c} f(x) = -\infty$. Let $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that for $0 < |x - c| < \delta$,
   $$f(x) < -\frac{1}{\varepsilon}.$$
   It follows that for these values of x, $f(x) < 0$, so $|f(x)| = -f(x)$, and
   $$\left|\frac{1}{f(x)} - 0\right| = -\frac{1}{f(x)} < \varepsilon.$$
   Hence $\lim_{x\to c} \dfrac{1}{f(x)} = 0$.
2. We prove that if c is a value and if $\lim_{x\to c^-} f(x) = -\infty$, then $\lim_{x\to c^-} \dfrac{1}{f(x)} = 0$.
   Assume $\lim_{x\to c^-} f(x) = -\infty$. Let $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that for $0 < c - x < \delta$,
   $$f(x) < -\frac{1}{\varepsilon}.$$

It follows that for these values of x, $f(x) < 0$, so $|f(x)| = -f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = -\frac{1}{f(x)} < \varepsilon.$$

Hence $\lim_{x\to c^-} \dfrac{1}{f(x)} = 0$.

3. We prove that if c is a value and if $\lim_{x\to c^-} f(x) = \infty$, then $\lim_{x\to c^-} \dfrac{1}{f(x)} = 0$.

   Assume $\lim_{x\to c^-} f(x) = \infty$. Let $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that for $0 < c - x < \delta$,

$$f(x) > \frac{1}{\varepsilon}.$$

   It follows that for these values of x, $f(x) > 0$, so $|f(x)| = f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = \frac{1}{f(x)} < \varepsilon.$$

   Hence $\lim_{x\to c^-} \dfrac{1}{f(x)} = 0$.

4. We prove that if c is a value and if $\lim_{x\to c^+} f(x) = -\infty$, then $\lim_{x\to c^+} \dfrac{1}{f(x)} = 0$.

   Assume $\lim_{x\to c^+} f(x) = -\infty$. Let $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that for $0 < x - c < \delta$,

$$f(x) < -\frac{1}{\varepsilon}.$$

   It follows that for these values of x, $f(x) < 0$, so $|f(x)| = -f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = -\frac{1}{f(x)} < \varepsilon.$$

   Hence $\lim_{x\to c^+} \dfrac{1}{f(x)} = 0$.

5. We prove that if c is a value and if $\lim_{x\to c^+} f(x) = \infty$, then $\lim_{x\to c^+} \dfrac{1}{f(x)} = 0$.

   Assume $\lim_{x\to c^+} f(x) = \infty$. Let $\varepsilon > 0$. By hypothesis, there is $\delta > 0$ such that for $0 < x - c < \delta$,

$$f(x) > \frac{1}{\varepsilon}.$$

It follows that for these values of x, $f(x) > 0$, so $|f(x)| = f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = \frac{1}{f(x)} < \varepsilon.$$

Hence $\lim_{x\to c^+} \frac{1}{f(x)} = 0$.

6. We prove that if $\lim_{x\to\infty} f(x) = -\infty$ then $\lim_{x\to\infty} \frac{1}{f(x)} = 0$. Assume $\lim_{x\to\infty} f(x) = -\infty$. Let $\varepsilon > 0$. By hypothesis, there is $M > 0$ such that for $x > M$,

$$f(x) < -\frac{1}{\varepsilon}.$$

It follows that for these values of x, $f(x) < 0$, so $|f(x)| = -f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = -\frac{1}{f(x)} < \varepsilon.$$

Hence $\lim_{x\to\infty} \frac{1}{f(x)} = 0$.

7. We prove that if $\lim_{x\to\infty} f(x) = \infty$ then $\lim_{x\to\infty} \frac{1}{f(x)} = 0$. Assume $\lim_{x\to\infty} f(x) = \infty$. Let $\varepsilon > 0$. By hypothesis, there is $M > 0$ such that for $x > M$,

$$f(x) > \frac{1}{\varepsilon}.$$

It follows that for these values of x, $f(x) > 0$, so $|f(x)| = f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = \frac{1}{f(x)} < \varepsilon.$$

Hence $\lim_{x\to\infty} \frac{1}{f(x)} = 0$.

8. We prove that if $\lim_{x\to-\infty} f(x) = -\infty$ then $\lim_{x\to-\infty} \frac{1}{f(x)} = 0$. Assume $\lim_{x\to-\infty} f(x) = -\infty$. Let $\varepsilon > 0$. By hypothesis, there is $M < 0$ such that for $x < M$,

$$f(x) < -\frac{1}{\varepsilon}.$$

It follows that for these values of x, $f(x) < 0$, so $|f(x)| = -f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = -\frac{1}{f(x)} < \varepsilon.$$

Hence $\lim_{x\to-\infty} \dfrac{1}{f(x)} = 0.$

9. We prove that if $\lim_{x\to-\infty} f(x) = \infty$ then $\lim_{x\to-\infty} \dfrac{1}{f(x)} = 0.$ Assume $\lim_{x\to-\infty} f(x) = \infty$. Let $\varepsilon > 0$. By hypothesis, there is $M < 0$ such that for $x < M$,

$$f(x) > \frac{1}{\varepsilon}.$$

It follows that for these values of x, $f(x) > 0$, so $|f(x)| = f(x)$, and

$$\left|\frac{1}{f(x)} - 0\right| = \frac{1}{f(x)} < \varepsilon.$$

Hence $\lim_{x\to-\infty} \dfrac{1}{f(x)} = 0.$

LF11. 1. We will prove that if c is a value, if

$$\lim_{x\to c} f(x) = 0,$$

and if for all x in an open interval containing c, $f(x) \le 0$ then

$$\lim_{x\to c} \frac{1}{f(x)} = -\infty.$$

Let $M < 0$ be given. By hypothesis, there is $\delta > 0$ such that if $0 < |x - c| < \delta$, $f(x) \le 0$ and

$$|f(x) - 0| = -f(x) < -\frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} < M.$$

Hence $\lim_{x\to c} \dfrac{1}{f(x)} = -\infty.$

2. We will prove that if

$$\lim_{x\to\infty} f(x) = 0,$$

and if $f(x) \ge 0$ for all x greater than a certain value, then

$$\lim_{x\to\infty} \frac{1}{f(x)} = \infty.$$

Let $M > 0$ be given. By hypothesis, there is $\delta > 0$ such that if $x > \delta$, $f(x) \geq 0$ and

$$|f(x) - 0| = f(x) < \frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} > M.$$

Hence $\lim_{x\to\infty} \dfrac{1}{f(x)} = \infty$.

3. We will prove that if

$$\lim_{x\to\infty} f(x) = 0,$$

and if $f(x) \leq 0$ for all x greater than a certain value, then

$$\lim_{x\to\infty} \frac{1}{f(x)} = -\infty.$$

Let $M < 0$ be given. By hypothesis, there is $\delta > 0$ such that if $x > \delta$, $f(x) \leq 0$ and

$$|f(x) - 0| = -f(x) < -\frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} < M.$$

Hence $\lim_{x\to\infty} \dfrac{1}{f(x)} = -\infty$.

4. We will prove that if

$$\lim_{x\to-\infty} f(x) = 0,$$

and if $f(x) \geq 0$ for all x less than a certain value, then

$$\lim_{x\to-\infty} \frac{1}{f(x)} = \infty.$$

Let $M > 0$ be given. By hypothesis, there is $\delta < 0$ such that if $x < \delta$, $f(x) \geq 0$ and

$$|f(x) - 0| = f(x) < \frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} > M.$$

Hence $\lim_{x\to-\infty} \dfrac{1}{f(x)} = \infty$.

5. We will prove that if

$$\lim_{x\to-\infty} f(x) = 0,$$

and if $f(x) \leq 0$ for all x less than a certain value, then

$$\lim_{x\to-\infty} \frac{1}{f(x)} = -\infty.$$

Let $M < 0$ be given. By hypothesis, there is $\delta < 0$ such that if $x < \delta$, $f(x) \leq 0$ and

$$|f(x) - 0| = -f(x) < -\frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} < M.$$

Hence $\lim_{x\to-\infty} \dfrac{1}{f(x)} = -\infty$.

6. We will prove that if c is a value, if

$$\lim_{x\to c^-} f(x) = 0,$$

and if for all x in an interval whose right end-point is c, $f(x) \geq 0$, then

$$\lim_{x\to c^-} \frac{1}{f(x)} = \infty.$$

Let $M > 0$ be given. By hypothesis, there is $\delta > 0$ such that if $0 < c - x < \delta$, $f(x) \geq 0$ and

$$|f(x) - 0| = f(x) < \frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} > M.$$

Hence $\lim_{x\to c^-} \dfrac{1}{f(x)} = \infty$.

7. If c is a value, if
$$\lim_{x\to c^-} f(x) = 0,$$
and if for all x in an interval whose right end-point is c, $f(x) \leq 0$, then
$$\lim_{x\to c^-} \frac{1}{f(x)} = -\infty.$$
Let $M < 0$ be given. By hypothesis, there is $\delta > 0$ such that if $0 < c - x < \delta$, $f(x) \leq 0$ and
$$|f(x) - 0| = -f(x) < -\frac{1}{M}.$$
Then for these same values of x,
$$\frac{1}{f(x)} < M.$$
Hence $\lim_{x\to c^-} \frac{1}{f(x)} = -\infty$.

8. If c is a value, if
$$\lim_{x\to c^+} f(x) = 0,$$
and if for all x in an interval whose right end-point is c, $f(x) \geq 0$, then
$$\lim_{x\to c^+} \frac{1}{f(x)} = \infty.$$
Let $M > 0$ be given. By hypothesis, there is $\delta > 0$ such that if $0 < x - c < \delta$, $f(x) \geq 0$ and
$$|f(x) - 0| = f(x) < \frac{1}{M}.$$
Then for these same values of x,
$$\frac{1}{f(x)} > M.$$
Hence $\lim_{x\to c^+} \frac{1}{f(x)} = \infty$.

9. If c is a value, if
$$\lim_{x\to c^+} f(x) = 0,$$
and if for all x in an interval whose right end-point is c, $f(x) \leq 0$, then
$$\lim_{x\to c^+} \frac{1}{f(x)} = -\infty.$$

Let $M < 0$ be given. By hypothesis, there is $\delta > 0$ such that if $0 < x - c < \delta$, $f(x) \leq 0$ and

$$|f(x) - 0| = -f(x) < -\frac{1}{M}.$$

Then for these same values of x,

$$\frac{1}{f(x)} < M.$$

Hence $\lim_{x \to c^+} \dfrac{1}{f(x)} = -\infty$.

Solution C.134. [Of Exercise 9.6.]

1. Prove directly from the definition that

$$\lim_{x \to 2} x^2 = 4.$$

Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon/5\}$. If $0 < |x - 2| < \delta$ then $2 - \delta < x < 2 + \delta$ and hence since $\delta \leq 1$ we can easily show that $|x + 2| < 5$. Now, let x be such that $0 < |x - 2| < \delta$. Now $|x^2 - 4| = |x - 2||x + 2| < (\varepsilon/5) \times 5 = \varepsilon$. Hence, $\lim_{x \to 2} x^2 = 4$.

2. Prove directly from the definition that for every value c,

$$\lim_{x \to c} |x| = |c|.$$

Note that $|x| = |x - c + c| \leq |x - c| + |c|$ hence $|x| - |c| \leq |x - c|$. Note also that $|c| \leq |c - x| + |x|$ and hence $|c| - |x - c| \leq |x|$. Hence $||x| - |c|| \leq |x - c|$.
Let $\varepsilon > 0$. Let $\delta = \varepsilon$. Let x be such that $0 < |x - c| < \delta$. Then $||x| - |c|| \leq |x - c| < \delta = \varepsilon$. Hence by definition $\lim_{x \to c} |x| = |c|$.

3. Prove directly from the definition that

$$\lim_{x \to 2} (5x - 11) = -1.$$

Let $\varepsilon > 0$. Let $\delta = \varepsilon/5$. Let x be such that $0 < |x - 2| < \delta = \varepsilon/5$. Then $|5x - 11 + 1| = 5|x - 2| < 5\delta = 5(\varepsilon/5) = \varepsilon$. Hence by definition $\lim_{x \to 2}(5x - 11) = -1$.

4. Prove directly from the definition that

$$\lim_{x \to 1} (x^2 + x - 1) = 1.$$

Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon/4\}$. If $0 < |x-1| < \delta$ then $1-\delta < x < 1+\delta$ and hence since $\delta \leq 1$ we can easily show that $|x+2| < 4$. Now, let x be such that $0 < |x-1| < \delta$. Then $|x^2+x-2| = |x-1||x+2| < (\varepsilon/4) \times 4 = \varepsilon$. Hence, $\lim_{x\to 1}(x^2+x-2) = -1$.

5. Prove directly from the definition that

$$\lim_{x\to 1}(x-3x^2) = -2.$$

Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon/8\}$. If $0 < |x-1| < \delta$ then $1-\delta < x < 1+\delta$ and hence since $\delta \leq 1$ we can easily show that $|3x+2| < 8$. Now, let x be such that $0 < |x-1| < \delta$. Then $|x-3x^2+2| = |1-x||3x+2| < (\varepsilon/8) \times 8 = \varepsilon$. Hence, $\lim_{x\to 1}(x-3x^2) = -2$.

6. Prove directly from the definition that

$$\lim_{x\to 4}(\sqrt{x}) = 2.$$

Let $\varepsilon > 0$ and let $\delta = \min\{1, 3\varepsilon\}$. If $0 < |x-4| < \delta$ then $4-\delta < x < 4+\delta$ and hence since $\delta \leq 1$ we can easily show that $\sqrt{x}+2 > 3$. Now, let x be such that $0 < |x-4| < \delta$. Then $|\sqrt{x}-2| = |\frac{x-4}{\sqrt{x}+2}| < 3\varepsilon/3 = \varepsilon$. Hence, $\lim_{x\to 4}(\sqrt{x}) = 2$.

7. Prove directly from the definition that

$$\lim_{x\to -2} x^3 = -8.$$

We will show the more general result that

$$\lim_{x\to a} x^3 = a^3 \text{ for all reals } a.$$

Let $\varepsilon > 0$. We want $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|x^3-a^3| < \varepsilon$. Let $\delta = \min\{1, \frac{\varepsilon}{3a^2+3a+1}\}$. Since $0 < |x-a| < \delta$ then $a-\delta < x < a+\delta$ and we can show that $x^2+xa+a^2 < (a+1)^2+a(a+1)+a^2 = 3a^2+3a+1$. Now, for such x, we have: $|x^3-a^3| = |x-a||x^2+xa+a^2| < \delta(3a^2+3a+1) < (\frac{\varepsilon}{3a^2+3a+1})(3a^2+3a+1) = \varepsilon$. Hence by definition $\lim_{x\to a} x^3 = a^3$.

8. Prove directly from the definition that

$$\lim_{x\to 1} \frac{4}{3x+2} = \frac{4}{5}.$$

Let $\varepsilon > 0$ and let $\delta = \min\{1, \frac{5}{6}\varepsilon\}$. If $0 < |x-1| < \delta$ then $1-\delta < x < 1+\delta$ and hence since $\delta \leq 1$ we can easily show that $x > 0$ and $\frac{1}{3x+2} < \frac{1}{2}$. Now, let x be such that $0 < |x-1| < \delta$. Then $\left|\frac{4}{3x+2} - \frac{4}{5}\right| = \left|\frac{12(1-x)}{5(3x+2)}\right| = \frac{12}{5}\frac{|1-x|}{3x+2} < \frac{12}{5}\frac{1}{2}|1-x| = \frac{6}{5}|1-x| < \frac{6}{5}\frac{5}{6}\varepsilon = \varepsilon$. Hence, $\lim_{x\to 1} \frac{4}{3x+2} = \frac{4}{5}$.

Solution C.135. [Exercise 9.7.]

1. Prove that the limit of a function is unique; i.e., that a function has at most one limit as $x \to c$.
   Assume a function $f(x)$ has two limits l_1 and l_2 as $x \to c$. By definition, this implies for any $\varepsilon > 0$, there are δ_1 and δ_2 such that for any x: if $0 < |x-c| < \delta_1$ then $|f(x) - l_1| < \varepsilon/2$ and if $0 < |x-c| < \delta_2$ then $|f(x) - l_2| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_1\}$. Then for any $0 < |x-c| < \delta$ we have $|f(x) - l_1| < \varepsilon/2$ and $|f(x) - l_2| < \varepsilon/2$. Hence for these x's $|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \leq |l_1 - f(x)| + |f(x) - l_2| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence no matter what ε we take, $|l_1 - l_2| < \varepsilon$. This means $l_1 = l_2$.

2. Suppose that f is a function defined on an open interval containing c except possibly at c itself.

   1. Suppose that $\lim_{x\to c} f(x)$ exists. Prove that $\lim_{x\to c} |f(x)|$ exists and $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|$.
      Assume $\lim_{x\to c} f(x) = l$. We will show that $\lim_{x\to c} |f(x)| = |l|$. Let $\varepsilon > 0$. By definition there is $\delta > 0$ such that for all x, if $0 < |x-c| < \delta$ then $|f(x) - l| < \varepsilon$. But, $||f(x)| - |l|| \leq |f(x) - l|$. Hence there is $\delta > 0$ such that for all x, if $0 < |x-c| < \delta$ then $||f(x)| - |l|| < \varepsilon$. Thus, $\lim_{x\to c} |f(x)| = |l|$.

   2. Suppose that $\lim_{x\to c} |f(x)|$ exists. Give an example to show that $\lim_{x\to c} f(x)$ may not exist.
      Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$
      Then, it is easy to prove that for any c, $\lim_{x\to c} |f(x)| = 1$ but $\lim_{x\to c} f(x)$ does not exist.

   3. Suppose that $\lim_{x\to c} |f(x)| = 0$. Prove that $\lim_{x\to c} f(x) = 0$.
      Let $\varepsilon > 0$. Since $\lim_{x\to c} |f(x)| = 0$, there is a δ such that if $0 < |x-c| < \delta$ then $||f(x)|| < \varepsilon$. But $||f(x)|| = |f(x)|$. Hence there is a δ such that if $0 < |x-c| < \delta$ then $|f(x)| < \varepsilon$. By definition, $\lim_{x\to c} f(x) = 0$.

3. Find
$$\lim_{x\to 2^+} \frac{1}{x^2-2x}$$
and prove using the definition that this limit is correct.
We will show that $\lim_{x\to 2^+} \frac{1}{x^2-2x} = \infty$. Let $M > 0$ and let $\delta = \min\{1, \frac{1}{3M}\}$. If $0 < x-2 < \delta$ then $0 < x < 3$ and $x(x-2) < 3\delta \le 3\frac{1}{3M} = \frac{1}{M}$. Hence for any x such that $0 < x-2 < \delta$ we have $\frac{1}{x^2-2x} = \frac{1}{x(x-2)} > M$. Hence by definition $\lim_{x\to 2^+} \frac{1}{x^2-2x} = \infty$.

4. Find
$$\lim_{x\to\infty} \frac{\sin x}{x}$$
and prove using the definition that this limit is correct.
We will show that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. Let $\varepsilon > 0$ and let $M = \frac{1}{\varepsilon}$. For any $x > M$ we have $|\frac{\sin x}{x}| \le \frac{1}{x} < \frac{1}{M} = \varepsilon$. Hence by definition $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

Solution C.136. [Exercise 9.8.] This solution is also taken from [25]. Assume $a < c_1 < c_2 < \cdots < c_p < b$. Let $x_0 = a$ and $x_p = b$ and for each $0 < i < p$, let x_i be such that $c_i < x_i < c_{i+1}$. Hence

$$x_0 = a < c_1 < x_1 < c_2 < x_2 < \cdots x_{p-1} < c_p < b = x_p.$$

Let $1 \le i \le p$. Since f is monotonically increasing, $\lim_{x\to c_i^+} f(x) \le f(x_i)$ and $f(x_{i-1}) \le \lim_{x\to c_i^-} f(x)$. Hence, $f(x_i) - f(x_{i-1}) \ge \lim_{x\to c_i^+} f(x) - \lim_{x\to c_i^-} f(x)$. Let $\sigma(c_i) = \lim_{x\to c_i^+} f(x) - \lim_{x\to c_i^-} f(x)$. Therefore:

$$\text{For all } 1 \le i \le p,\ f(x_i) - f(x_{i-1}) \ge \sigma(c_i).$$

Now,

$$f(b) - f(a) = f(x_p) - f(x_0) = \Sigma_{i=1}^p (f(x_i) - f(x_{i-1})) \ge \Sigma_{i=1}^p \sigma(c_i).$$

Since we are only interested in discontinuity points, assume that for each $1 \le i \le p$, f is discontinuous at c_i and hence $\sigma(c_i) > 0$. Hence, for some $n > 0$, we have $\sigma(c_i) > \frac{1}{n}$ for each $1 \le i \le p$. Therefore,

$$f(b) - f(a) \ge \Sigma_{i=1}^p \sigma(c_i) > \frac{p}{n}.$$

and

$$p < n(f(b) - f(a)).$$

So for each $n > 0$, we have a finite number p of points c in (a, b) such that $\sigma(c) > \frac{1}{n}$ and hence we have at most $p+2$ points c in $[a, b]$ such that $\sigma(c) > \frac{1}{n}$ (in case $\sigma(a) > \frac{1}{n}$ and/or $\sigma(b) > \frac{1}{n}$).

For each point c in A (the set of points at which f is discontinuous), $\sigma(c) > \frac{1}{n}$ for some $n > 0$. We can write the elements of A as a sequence as follows:

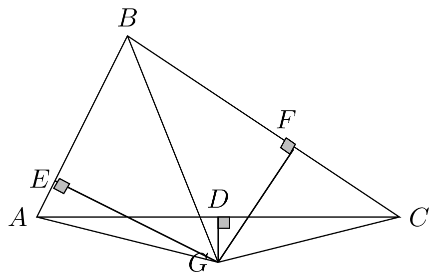
- First, list in sequence the finite set A_1 of points c of A such that $\sigma(c) > 1$.
- Then list in sequence the finite set A_2 of points c of $A \setminus A_1$ such that $\sigma(c) > \frac{1}{2}$.
- Then list in sequence the finite set A_3 of points c of $A \setminus (A_1 \cup A_2)$ such that $\sigma(c) > \frac{1}{3}$.
- Repeat this process for each $n > 0$.

This demonstrates that A is countable

C.10 Solutions for Chapter 10

Solution C.137. [Of Exercise 10.1.]

C2. Here is the figure for this case:



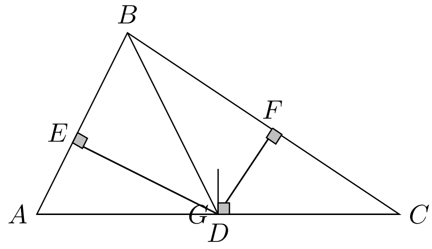
The proof for this case is exactly the same as the proof given at the start of Example 10.1.1. We repeat it here:
Since DG is the perpendicular bisector of AC, $AD = DC$. Also, angles $\angle ADG$ and $\angle GDC$ are both right angles. Then by Proposition 4 of Book I on triangles ADG and GDC, $AG = CG$.

Since BG is the angle bisector of $\angle ABC$, $\angle ABG = \angle GBC$. Also, $\angle BEG = \angle BFG$, since both are right angles by construction, and since BG is common, by Proposition 26 of Book I, it follows that $BE = BF$ and $GE = GF$.

Since $\angle GEA = \angle GFC$ are both right angles, by the Pythagorean triples, $AE^2 = CF^2 = AG^2 - EG^2 = CG^2 - FG^2$. Hence it follows that $AE = CF$.

Now $AB = AE + BE$ and $BC = BF + CF$, it follows that $AB = BC$, and the triangle ABC is isosceles.

C3. Here is the figure for this case:



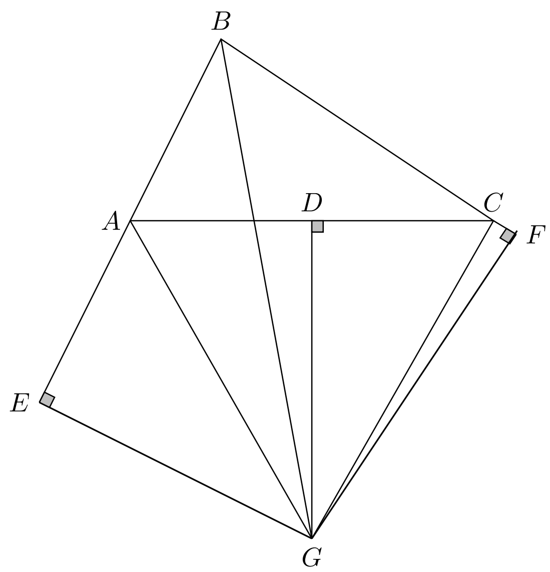
Since D is the middle point of AC, and D and G coincide, we have $AG = CG$.

Since BG is the angle bisector of $\angle ABC$, $\angle ABG = \angle GBC$. Also, $\angle BEG = \angle BFG$, since both are right angles by construction, and since BG is common, by Proposition 26 of Book I, it follows that $BE = BF$ and $GE = GF$.

Since $\angle GEA = \angle GFC$ are both right angles, by the Pythagorean triples, $AE^2 = CF^2 = AG^2 - EG^2 = CG^2 - FG^2$. Hence it follows that $AE = CF$.

Now $AB = AE + BE$ and $BC = BF + CF$, it follows that $AB = BC$, and the triangle ABC is isosceles.

C4. Here is the figure for this case:



Since DG is the perpendicular bisector of AC, $AD = DC$. Also, angles $\angle ADG$ and $\angle GDC$ are both right angles. Then by Proposition 4 of Book I on triangles ADG and GDC, $AG = CG$.

Since BG is the angle bisector of $\angle ABC$, $\angle ABG = \angle GBC$. Also, $\angle BEG = \angle BFG$, since both are right angles by construction, and since BG is common, by Proposition 26 of Book I, it follows that $BE = BF$ and $GE = GF$.

Since $\angle GEA = \angle GFC$ are both right angles, by the Pythagorean triples, $AE^2 = CF^2 = AG^2 - EG^2 = CG^2 - FG^2$. Hence it follows that $AE = CF$.

Now $AB = BE - AE$ and $BC = BF - CF$, it follows that $AB = BC$, and the triangle ABC is isosceles.

Solution C.138. [Of exercise 10.2] If in the model of Example 10.1.2, we consider only the xy-plane, then, for each line (i.e., the set of pairs (p, q) of rational numbers satisfying $ap + bq = c$ for rational numbers a, b, and c), we define its slope to be $-a/b$ if $b \neq 0$, otherwise, the line is $p = c/a$ and is parallel to the y axis (as in the red line in the figure below).[3]

Let us consider two lines which are the set of points $ap + bq = c$ respectively $a'p + b'q = c'$

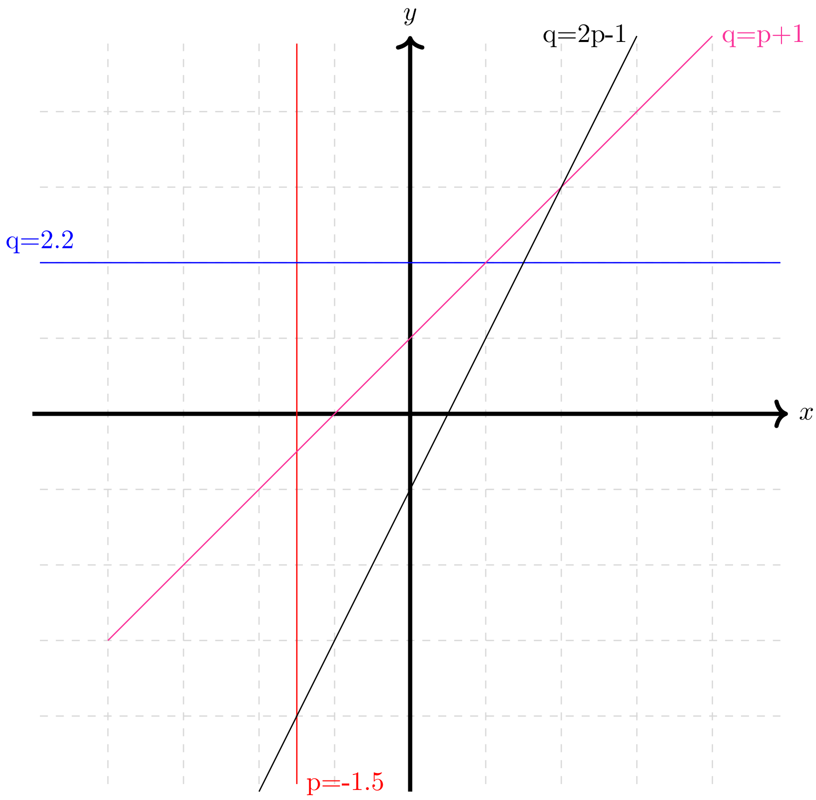
- If $b = b' = 0$ then each of the lines is parallel to the y axis and hence the two lines are parallel.
- If $b = 0$ and $b' \neq 0$, then $a \neq 0$ (otherwise the set of points $ap + bq = c$ is not a line). Let $A = (\frac{c}{a}, \frac{c'}{b'} - \frac{a'}{b'}\frac{c}{a})$. Note that the coefficients of A are rational. Furthermore, A is in both set of points $ap + bq = c$ respectively $a'p + b'q = c'$. That is: $a\frac{c}{a} + 0 = c$ and $a'\frac{c}{a} + b'\frac{c'}{b'} - b'\frac{a'}{b'}\frac{c}{a} = a'\frac{c}{a} + c' - a'\frac{c}{a} = c'$. Hence, A is the intersection point of both lines.
- If $b \neq 0$ and $b' \neq 0$ then the slopes are respectively a/b and a'/b'. Assume $a/b \neq a'/b'$. Hence $\frac{a}{b} - \frac{a'}{b'} \neq 0$. Let $A = (x, y)$ where $x = \dfrac{\frac{c}{b} - \frac{c'}{b'}}{\frac{a}{b} - \frac{a'}{b'}}$ and $y = \frac{c}{b} - x\frac{a}{b}$. Note that the coefficients of A are rational.

  Note also that since $x = \dfrac{\frac{c}{b} - \frac{c'}{b'}}{\frac{a}{b} - \frac{a'}{b'}}$ then $y = \frac{c}{b} - x\frac{a}{b} = \frac{c'}{b'} - x\frac{a'}{b'}$.

[3]Note that we cannot have both a and b be 0, since otherwise, we would be not be talking of a line.

Furthermore, A is in both set of points $ap + bq = c$ respectively $a'p + b'q = c'$ as can be seen from the following:

- $ax + by = ax + b\frac{c}{b} - xb\frac{a}{b} = ax + c - ax = c.$
- $a'x + b'y = a'x + b'\frac{c'}{b'} - xb'\frac{a'}{b'} = a'x + c' - a'x = c'.$



Solution C.139. [Of Exercise 10.3] By F9, the Archimedean ordered field includes 1, and by F1 it includes $1 + 1$, $1 + 1 + 1$, etc. Hence, it includes all positive integers. By F4 it contains 0 and by F5 it contains additive inverses. Hence, it contains all the integers. By F10 it contains multiplicative inverses and by F6 it contains the rationals.

Solution C.140. [Of Exercise 10.4] Assume an ordered field A. Let a and b be elements of A such that $a > 0$.

- Assume A satisfies AP and $b > a$. By AP, There is a positive integer n such that $b < an$. Hence, A satisfies AL.
- Assume A satisfies AL. We consider all the possible cases:
  - If $b < 0$ or $b = 0$ then let $n = 1$ and clearly there is a positive integer n such that $b < an$.
  - If $b > a$, then by AL, there is a positive integer n such that $b < an$.
  - If $a > b > 0$ then just take $n = 1$ and clearly there is a positive integer n such that $b < an$.

Solution C.141. [Of Exercise 10.5] Let $a = \sup S$. By hypothesis, $a \in S$. By Definition 10.3.1, $\forall x \in S$, $x \leq a$. Hence a is a maximum of S.

Solution C.142. [Of Exercise 10.6] If S has a least upper bound which is an element of S then by Exercise 10.5 above, this least upper bound is a maximum of S.
Assume S has a maximum a. Then, a is an upper bound of S and by the Axiom of Completeness AC, S has a least upper bound b. Now, since $a \in S$ then $a \leq b$. Since a is an upper bound of S then $b \leq a$. Hence $a = b$ and S has a least upper bound a which is an element of S.

Solution C.143. [Of Exercise 10.7] Let $S = \{a_1, \cdots, a_n\}$ where $n \geq 1$. By OF1 and OF2, we can order the finite set S to be $b_1 < b_2 < \cdots < b_n$. Hence b_n is a maximum element of S. If S has a least upper bound c then by Exercise 10.6 above, S has a least upper bound which belongs to S.
By Exercise 10.5 above, the least upper bound of S is a maximum element of S.

Solution C.144. [Of Exercise 10.8] By the Archimedean property (Theorem 10.4.4), since $1 > 0$, there is $m > 0$ such that $m > a$.

Solution C.145. [Of Exercise 10.9]
Let S be a nonempty bounded set of real numbers.

1. Since $S \neq \emptyset$, let $a \in S$. Let l and m be the greatest lower versus least upper bounds of S. By definition, $l \leq a \leq m$.
2. If $l = m$ then S is a singleton set. This is because, for any $a \in S$, $l \leq a \leq m$, but $l = m$, hence all elements of S are equal.

Solution C.146. [Of Exercise 10.10] Let $g \in \mathbb{R}$ such that $g > 0$ and $g < \frac{1}{a}$ for some $\frac{1}{a} \in A$. By the Archimedes Law AL, there is a positive integer n such that $\frac{1}{a} < ng$. Hence, for $\frac{1}{na} \in A$ we have $\frac{1}{na} < g$. Therefore is no $g \in \mathbb{R}$ such that $g > 0$ and $g < x$ for every $x \in A$. Hence 0 is a greatest lower bound (i.e., infimum) of A.

Solution C.147. [Of Exercise 10.11] Let S and T be nonempty bounded sets of real numbers.

1. By definition, $\forall a \in T$, inf$T \leq a$. But $S \subseteq T$. Hence $\forall a \in S$, inf$T \leq a$. Hence inf $T \leq$ inf S.
   By definition, $\forall a \in T$, $a \leq$ sup$T$. But $S \subseteq T$. Hence $\forall a \in S$, $a \leq$ sup T. Hence sup $S \leq$ sup T.
   By Exercise 10.9.1, inf $S \leq$ sup S. Hence inf $T \leq$ inf $S \leq$ sup $S \leq$ sup T.

2. I. If $a \in S$ then $a \in S \cup T$. Hence $a \leq \sup(S \cup T)$ and sup$S \leq \sup(S \cup T)$. Similarly, sup$T \leq \sup(S \cup T)$. Hence $\max\{\text{sup}S, \text{sup}T\} \leq \sup(S \cup T)$.
   II. If $a \in S \cup T$ then
   - Either $a \in S$ and hence $a \leq \text{sup}S \leq \max\{\text{sup}S, \text{sup}T\}$.
   - Or $a \in T$ and hence $a \leq \text{sup}T \leq \max\{\text{sup}S, \text{sup}T\}$.

   Hence $\sup(S \cup T) \leq \max\{\text{sup}S, \text{sup}T\}$.
   Hence by I and II, $\sup(S \cup T) = \max\{\text{sup}S, \text{sup}T\}$.

Solution C.148. [Of Exercise 10.12] Assume the contrary. I.e., assume there is $\varepsilon > 0$ such that for all $x \in S$, $x \leq \beta - \varepsilon$. Hence, sup$S = \beta \leq \beta - \varepsilon$. So, $\varepsilon \leq 0$. Contradiction. Hence, for every $\varepsilon > 0$, there exists an element x of S such that $x > \beta - \varepsilon$.

Solution C.149. [Of Exercise 10.13] Let $\varepsilon > 0$. Then $x - \varepsilon$ and $x + \varepsilon$ are two real numbers such that $x - \varepsilon < x + \varepsilon$. By the density theorem 10.4.5, there is a rational q such that $x - \varepsilon < q < x + \varepsilon$. Hence $x < q + \varepsilon$ and $q - \varepsilon < x$. I.e., $q - \varepsilon < x < q + \varepsilon$ or $-\varepsilon < x - q < \varepsilon$. In other words, $|x - q| < \varepsilon$.

Solution C.150. [Of Exercise 10.14] First we will show that if r and s are rationals such that $s \neq 0$ then $r + s\sqrt{2}$ is irrational. Let $r = m/n$ and $s = p/q$ where $n, p, q \neq 0$. If $r + s\sqrt{2}$ is a rational e/f, i.e., $m/n + (p/q)\sqrt{2} = e/f$ is rational, then $\frac{mq + np\sqrt{2}}{nq} = \frac{e}{f}$. Hence $\sqrt{2} = \frac{nq\frac{e}{f} - mq}{np}$ which is rational. Absurd. Hence $r + s\sqrt{2}$ is irrational.
Now, since $b > a$ then $b - a > 0$. Since $b - a$ and $\sqrt{2}$ are two positive reals, by the Archimedean property, there is a positive integer m such that $m(b - a) > \sqrt{2}$. Hence $ma + \sqrt{2} < mb$.
Let n be the greatest integer such that $ma \geq n$. Hence $ma < n+1 < n+\sqrt{2}$. Also, $ma + \sqrt{2} \geq n + \sqrt{2}$. Hence $ma < n + \sqrt{2} \leq ma + \sqrt{2} < mb$ and so, $a < \frac{n}{m} + \frac{1}{m}\sqrt{2} < b$. Since $a < \frac{n}{m} + \frac{1}{m}\sqrt{2} < b$ and $\frac{n}{m} + \frac{1}{m}\sqrt{2}$ is irrational, we are done.

Solution C.151. [Of Exercise 10.15] Since S and T are nonempty, let $t_o \in T$ and $s_o \in S$.

1. By hypothesis, for any $s \in S$, $s \leq t_0$. Hence S is bounded above by t_0. By hypothesis, for any $t \in T$, $s_0 \leq t$. Hence T is bounded below by s_0.

2. Since S is bounded above, by the Axiom of Completeness, sup S exists. Since T is bounded below, by Theorem 10.4.2, inf T exists. Since by hypothesis, any t in T is an upper bound of S, and since sup S is the least upper bound, then sup $S \leq t$ for any $t \in T$. Hence sup S is a lower bound of T. But inf T is the greatest lower bound. Hence sup $S \leq$ inf T.

3. Let $S = (2, 3]$ and $T = [3, 4)$. Then, for all $s \in S$, for all $t \in T$, $s \leq t$ and hence $S \cap T = \{3\}$.

4. Let $S = (2, 3)$ and $T = (3, 4)$. Then, for all $s \in S$, for all $t \in T$, $s \leq t$, sup $S =$ inf $T = 3$ and $S \cap T = \emptyset$.

Solution C.152. [Of Exercise 10.16] Since a and 1 are positive real numbers, then by the Archimedean property Theorem 10.4.4, there are p and m (strictly) positive integers such that $ap > 1$ and $m > a$. Let $n = \max\{p, m\}$. Hence $an > ap > 1$ and $n > a$. That is, $1/n < a < n$.

Solution C.153. [Of Exercise 10.17]

1. Since A and B are non empty and bounded above then by the Axiom of Completeness sup$A$ and sup$B$ exist. Let $a + b \in S$. Since $a \leq$ sup$A$ and $b \leq$ sup$B$ then $a + b \leq$ sup$A$ + sup$B$. Hence sup$A$ + sup$B$ is an upper bound of S. Since S is nonempty then by the Axiom of Completeness sup$S$ exists. Therefore, sup$S \leq$ sup$A$ + sup$B$.
   Now we show that for any upper bound α of S, we have $\alpha \geq$ sup$A$ + sup$B$. Assume on the contrary that $\alpha <$ sup$A$ + sup$B$. Then, $\alpha -$ sup$A <$ sup$B$ and since sup$B$ is the least upper bound of B, there must exist a $b \in B$ such that $\alpha -$ sup$A < b$. Hence, $\alpha - b <$ sup$A$. Again, there must exist an $a \in A$ such that $\alpha - b < a$. Hence, there is $a + b \in S$ such that $\alpha < a + b$ contradicting the fact that α is an upper bound of S. Hence $\alpha \geq$ sup$A$ + sup$B$ for any upper bound α of S. Hence sup$S =$ sup$A$ + sup$B$.

2. Since A and B are non empty and bounded below then by Theorem 10.4.2 inf A and inf B exist. Let $a + b \in S$. Since inf $A \leq a$ and inf $B \leq b$ then inf A + inf $B \leq a + b$. Hence inf A + inf B is a lower

bound of S. Since S is nonempty then by Theorem 10.4.2, $\inf S$ exists. Therefore, $\inf A + \inf B \leq \inf S$.
Now we show that for any lower bound α of S, we have $\alpha \leq \inf A + \inf B$. Assume on the contrary that $\alpha > \inf A + \inf B$. Then, $\alpha - \inf A > \inf B$ and since $\inf B$ is the greatest lower bound of B, there must exist a $b \in B$ such that $\alpha - \inf A > b$. Hence, $\alpha - b > \inf A$. Again, there must exist an $a \in A$ such that $\alpha - b > a$. Hence, there is $a + b \in S$ such that $\alpha > a + b$ contradicting the fact that α is a lower bound of S. Hence $\alpha \leq \inf A + \inf B$ for any lower bound α of S. Hence $\inf S = \inf A + \inf B$.

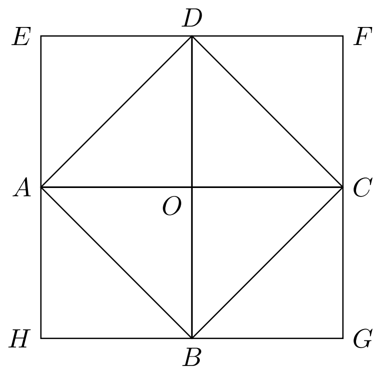
Solution C.154. [Of Exercise 10.18] Let $S = \{b + \frac{1}{n} : n \text{ is a positive integer}\}$. Note that a and b are both lower bounds of S and S is not empty. Hence by Theorem 10.4.2, $\inf S$ exists. We will show that $b = \inf S$. That is, we will show that if α is a lower bound of S then $\alpha \leq b$. Assume otherwise that $\alpha > b$. Then, $\alpha - b > 0$. By the Archimedean property, since $\alpha - b$ and 1 are real numbers, there is a positive integer n such that $n(\alpha - b) > 1$. I.e., $\alpha > b + \frac{1}{n}$. Contradicting the fact that α is a lower bound of S. Hence, if α is a lower bound of S then $\alpha \leq b$. This means that $b = \inf S$. Therefore $a \leq b$.

Solution C.155. [Of Exercise 10.19] First note that a is an upper bound of S_a and that S_a is not empty. Hence, by the Archimedean property, $\sup S_a$ exists and $\sup S_a \leq a$. If $\sup S_a < a$ then by the Density of rationals Theorem 10.4.5, there is a rational r such that $\sup S_a < r < a$. Since $r < a$ then $r \in S_a$. But $\sup S_a < r$ contradicts the fact that $\sup S_a$ is an upper bound of S_a. Hence, $\sup S_a = a$.

C.11 Solutions for Chapter 11

Solution C.156. [Of Exercise 11.1]] In the diagram below, we see that we doubled the area of the square ABCD into the square $EFGH$. We did this by drawing the two diagonals AC and BD and making a copy of each of the internal triangles (AOD, DOC, AOB and BOC) by taking their mirror image on the corresponding side of the square $ABCD$.

Another way of doubling the area of the square is by taking the square $AODE$, drawing the diagonal AD and then making 3 copies (DOC, COB and BOA) of the triangle AOD. The resulting square $ABCD$ is double the square $AODE$.



Solution C.157. [Of Exercise 11.2]]

1. The area of S is a^2. The area of T whose side is x is x^2. By our formula, $\frac{a}{x} = \frac{x}{2a}$ and so, $x^2 = 2a^2$.
2. The formula is:
$$a : x = x : 3a.$$
In this case, $x^2 = 3a^2$.
3. The formula is:
$$a : x = x : \frac{3}{4}a.$$
In this case, $x^2 = \frac{3}{4}a^2$.

Solution C.158. [Of Exercise 11.3] Using proportions as we did in Exercise 11.2, we need to find the side x of T so that $a^3 = 2x^3$. We use a temporary variable y such that $a : x = x : y = y : 2a$. Then,

$$\left(\frac{a}{x}\right)^3 = \frac{a}{x}\frac{x}{y}\frac{y}{2a} = \frac{1}{2} \text{ and } x^3 = 2a^3.$$

For the second case, for a temporary variable y we use the formula

$$a : x = x : y = y : \frac{3}{4}a.$$

Then,

$$\left(\frac{a}{x}\right)^3 = \frac{a}{x}\frac{x}{y}\frac{y}{\frac{3}{4}a} = \frac{4}{3} \text{ and } x^3 = \frac{3}{4}a^3.$$

Solution C.159. [Of Exercise 11.4]

1. $f(x) = 6x - 11$.
   Let $\varepsilon > 0$ and let $\delta = \varepsilon/6$. For any $c \in \mathbb{R}$, c is in the domain of f. Let $c \in \mathbb{R}$. If $|x - c| < \delta$ then $|f(x) - f(c)| = |6x - 6c| = 6|x - c| < 6(\varepsilon/6) = \varepsilon$. Hence by definition f is continuous at c for any c. Hence f is continuous everywhere.

2. $g(x) = |x|$.
   Let $\varepsilon > 0$ and let $\delta = \varepsilon$. For any $c \in \mathbb{R}$, c is in the domain of g. Let $c \in \mathbb{R}$. If $|x-c| < \delta$ then $|g(x)-g(c)| = ||x|-|c|| \le |x-c| < \varepsilon$. Hence by definition f is continuous at c for any c. Hence g is continuous everywhere.

3. $h(x) = x^2$.
   Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon/(2|c| + 1)\}$. For any $c \in \mathbb{R}$, c is in the domain of h. Let $c \in \mathbb{R}$. If $|x - c| < \delta$ then $|h(x) - h(c)| = |x^2 - c^2| = |x - c||x + c| < (2|c| + 1)(\varepsilon/(2|c| + 1)) = \varepsilon$. Hence by definition h is continuous at c for any c. Hence f is continuous everywhere.

Solution C.160. [Of Exercise 11.5] Let $\varepsilon > 0$ and let $\delta = \varepsilon/|a|$. For any $c \in \mathbb{R}$, c is in the domain of f. Let $c \in \mathbb{R}$. If $|x - c| < \delta$ then $|f(x) - f(c)| = |ax - ac| = |a||x - c| < |a|(\varepsilon/|a|) = \varepsilon$. Hence by definition f is continuous at c for any c. Hence f is continuous everywhere.

Solution C.161. [Of Exercise 11.6] Let $\varepsilon > 0$. For any $c \in \mathbb{R}$ such that $c \ge 0$, c is in the domain of f. Let $c \in \mathbb{R}$ where $c \ge 0$. Note that $|\sqrt{x}-\sqrt{c}| \le |\sqrt{x} + \sqrt{c}|$ and that $|\sqrt{x} - \sqrt{c}|^2 \le |\sqrt{x} - \sqrt{c}||\sqrt{x} + \sqrt{c}| = |x - c|$. Hence if $\delta = \varepsilon^2$ and $|x - c| < \varepsilon^2$ then $|\sqrt{x} - \sqrt{c}|^2 < \varepsilon^2$ and $|\sqrt{x} - \sqrt{c}| < \varepsilon$. Hence by definition f is continuous at c for all non negative values of c. Hence f is continuous everywhere.

Solution C.162. [Of Exercise 11.7] Let $m = f(c)/2 > 0$. Since f is continuous at c, there is $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < m$.

I.e., if $c - \delta < x < c + \delta$ then $m < f(x) < 3m$ (hence $f(x) \geq m$). Let $[u, v] = (c - \delta, c + \delta) \cap [a, b]$. Obviously $c \in [u, v] \subseteq [a, b]$. Now, if $x \in [u, v]$ then $c - \delta < x < c + \delta$ hence $|x - c| < \delta$ and $f(x) \geq m$.

Solution C.163. [Of Exercise 11.8] We need to show that

1. f is continuous at c iff
2. $\lim_{x \to c} f(x) = f(c)$ iff
3. for each $\{x_n\}$ in I, such that $\lim_{x \to \infty} x_n = c$, we have $\lim_{x \to \infty} f(x_n) = f(c)$.

We show $1 \iff 2$ as follows: Recall that

- f is continuous at c iff f is defined at c and for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.
- f has limit $f(c)$ at c (i.e., $\lim_{x \to c} f(x) = f(c)$) iff for all $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

$1 \Longrightarrow 2$. Obviously if f is continuous at c then $\lim_{x \to c} f(x) = f(c)$.

$2 \Longrightarrow 1$. On the other hand, if $\lim_{x \to c} f(x) = f(c)$ and $\varepsilon > 0$ then there is $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Hence

- If $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.
- If $|x - c| = 0$ then $x = c$ and hence $|f(x) - f(c)| = 0 < \varepsilon$.

Therefore, if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$ and f is continuous.

Now we will show $1 \iff 3$.

$3 \Longrightarrow 1$. Assume for each $\{x_n\}$ in I, such that $\lim_{n \to \infty} x_n = c$, we have $\lim_{x \to \infty} f(x_n) = f(c)$. We need to show f is continuous at c. Assume there is $\varepsilon > 0$ such that for every $\delta > 0$, if $|x - c| < \delta$ then $|f(x) - f(c)| \geq \varepsilon$. For all $n > 0$, let $\delta_n = 1/n$ and let x_n in I be such that $|x_n - c| < 1/n$. Hence $|f(x_n) - f(c)| \geq \varepsilon$. Now, $\{x_n\}$ is in I such that $\lim_{n \to \infty} x_n = c$ and $\lim_{n \to \infty} f(x_n) \neq f(c)$. Contradiction. Hence f is continuous at c.

$1 \Longrightarrow 3$. Let a sequence $\{x_n\}$ in I, such that $\lim_{n \to \infty} x_n = c$. We need to show that $\lim_{n \to \infty} f(x_n) = f(c)$. I.e., we need to show that for all $\varepsilon > 0$, there is $M > 0$ such that if $x_n > M$ then $|f(x_n) - f(c)| < \varepsilon$.

Let $\varepsilon > 0$. Since f is continuous, there $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Since $\lim_{n \to \infty} x_n = c$, for this δ, there is $M > 0$ such that if $x_n > M$ then $|x_n - c| < \delta$ and hence $|f(x_n) - f(c)| < \varepsilon$. Hence there is $M > 0$ such that if $x_n > M$ then $|f(x_n) - f(c)| < \varepsilon$.

Solution C.164. [Of Exercise 11.9]

$f+g$: Let $\varepsilon > 0$. There are δ_1 and δ_2 such that if $|x-c| < \delta_1$ then $|f(x) - f(c)| < \varepsilon/2$ and if $|x-c| < \delta_2$ then $g(x) - g(c)| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Now, if $|x-c| < \delta$ then $|f(x)+g(x)-f(c)-g(c)| \leq |f(x)-f(c)| + |g(x)-g(c)| < \varepsilon$. Hence $f+g$ is continuous at c.

$f-g$: Let $\varepsilon > 0$. There are δ_1 and δ_2 such that if $|x-c| < \delta_1$ then $|f(x) - f(c)| < \varepsilon/2$ and if $|x-c| < \delta_2$ then $g(x) - g(c)| < \varepsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Now, if $|x-c| < \delta$ then $|f(x)-g(x)-f(c)+g(c)| \leq |f(x)-f(c)| + |g(x)-g(c)| < \varepsilon$. Hence $f-g$ is continuous at c.

kf: Let $\varepsilon > 0$. There is δ such that if $|x-c| < \delta$ then $|f(x)-f(c)| < \varepsilon/k$. Now, if $|x-c| < \delta$ then $|kf(x) - kf(c)| = |k||f(x)-f(c)| < \varepsilon$.

fg: Note that $|f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)| \leq |g(x)||f(x)-f(c)| + |f(c)||g(x)-g(c)|$. Let $\varepsilon > 0$. We deal with the case $f(c) \neq 0$ and leave the case $f(c) = 0$ as an exercise. There are δ_1, δ_2 and δ_3 such that

- if $|x-c| < \delta_1$ then $|f(x)-f(c)| < \dfrac{\varepsilon}{2(|g(c)|+1)}$ and
- if $|f(c)| \neq 0$ and $|x-c| < \delta_2$ then $|g(x)-g(c)| < \dfrac{\varepsilon}{2|f(c)|}$ and
- if $|x-c| < \delta_3$ then $|g(x)-g(c)| < 1$ and hence $|g(x)| < |g(c)|+1$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Now, if $|x-c| < \delta$ then $|f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)| \leq |g(x)||f(x)-f(c)| + |f(c)||g(x)-g(c)| < (|g(c)|+1)\dfrac{\varepsilon}{2(|g(c)|+1)} + |f(c)|\dfrac{\varepsilon}{2|f(c)|} = \varepsilon$. Hence fg is continuous at c.

$\dfrac{f}{g}$: Let $g(c) \neq 0$. We prove that $\dfrac{1}{g}$ is continuous at c and use the above item to deduce that $\dfrac{f}{g}$ is continuous at c. Let $\varepsilon > 0$. There are δ_1, δ_2 such that

- If $|x-c| < \delta_1$ then $|g(x)-g(c)| < |g(c)|/2$ (hence $|g(x)| > |g(c)|/2$) and
- If $|x-c| < \delta_2$ then $|g(x)-g(c)| < (\varepsilon|g(c)|^2)/2$.

Let $\delta = \min\{\delta_1, \delta_1\}$ then $|\dfrac{1}{g(x)} - \dfrac{1}{g(c)}| = |\dfrac{g(c)-g(x)}{g(x)g(c)}| \leq \dfrac{|g(c)-g(x)|}{|g(x)g(c)|} < ((\varepsilon|g(c)|^2)/2)(2/|g(c)|^2) = \varepsilon$.

Solution C.165. [Of Exercise 11.10] Let I be an interval, let c be an element of I, let g be a function whose domain includes I, and let f be defined on an interval J that includes the image $g(I) = \{g(x) : x \in I\}$. Assume g is continuous at c and f is continuous at $g(c)$. Let $\varepsilon > 0$. Since f is continuous at c then there is a $\delta_1 > 0$ such that if $|x - c| < \delta_1$ then $|f(x) - f(c)| < \varepsilon$. But for δ_1, there is δ such that if $|x - c| < \delta$ then $|g(x) - g(c)| < \delta_1$. Hence, there is δ such that if $|x - c| < \delta$ then $|f(g(x)) - f(g(c))| < \varepsilon$ and $f \circ g$ is continuous at c.
Hence, if g is continuous on I and f is continuous on J, then $f \circ g$ is continuous on I.

Solution C.166. [Of Exercise 11.11]

1. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where for all $0 \leq i \leq n$, a_i is a constant (and of course for all $1 \leq i \leq n$, i is a positive integer). Let c be a quantity.

   - We first show that $f(x) = x$ is continuous at c. Note that f is defined for c and if $\varepsilon > 0$ then let $\delta = \varepsilon$. Now, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.
   - Then, we show by induction on $n \geq 0$ that for any $n \geq 0$, $g_n(x) = x^n$ is continuous at c. If $n = 0$ then it is easy to show $g_0(x) = 1$ is continuous. Assume the property holds for $n \geq 0$ then $g_{n+1}(x) = g_n(x)f(x)$ is continuous by IH and the above item and Theorem 11.2.4. Hence g_n is continuous for any n.
   - By Theorem 11.2.4, $p(x)$ is continuous.

   Hence the polynomial $p(x)$ is continuous at every quantity.

2. A rational function is continuous at every quantity for which it is defined.
   A rational function $f(x)$ is of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. $f(x)$ is defined on all quantities c such that $q(c) \neq 0$. Let c be such that $f(c)$ is defined. Then, by the above item, the polynomials $p(x)$ and $q(x)$ are continuous at c and by Theorem 11.2.4, $\frac{p(x)}{q(x)}$ is continuous at c.

Solution C.167. [Of Exercise 11.12]]

1. $f'(1) = \lim_{x\to 1} \frac{x^4 - 2x^2 - (1-2)}{x-1} = \lim_{x\to 1} \frac{x^4 - 2x^2 + 1}{x-1} = \lim_{x\to 1} \frac{(x^2-1)^2}{x-1} = \lim_{x\to 1} \frac{(x-1)^2(x+1)^2}{x-1} = \lim_{x\to 1} (x-1)(x+1)^2 = 0$.

2. $f'(x) = \lim_{x\to 1} \frac{\sqrt{x}-\sqrt{1}}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$.

3. $f'(x) = \lim_{x\to 1} \frac{x/\sqrt{x^2+1} - 1/\sqrt{1^2+1}}{x-1} = \lim_{x\to 1} \frac{\frac{x\sqrt{2}-\sqrt{x^2+1}}{\sqrt{2(x^2+1)}}}{x-1} = \lim_{x\to 1} \frac{x\sqrt{2}-\sqrt{x^2+1}}{(x-1)\sqrt{2(x^2+1)}} = \lim_{x\to 1} \frac{x\sqrt{2}-\sqrt{x^2+1}}{(x-1)\sqrt{2(x^2+1)}} \times \frac{x\sqrt{2}+\sqrt{x^2+1}}{x\sqrt{2}+\sqrt{x^2+1}} =$
$\lim_{x\to 1} \frac{2x^2-x^2-1}{(x-1)\sqrt{2(x^2+1)}\left(x\sqrt{2}+\sqrt{x^2+1}\right)} =$
$\lim_{x\to 1} \frac{x^2-1}{(x-1)\sqrt{2(x^2+1)}\left(x\sqrt{2}+\sqrt{x^2+1}\right)} =$
$\lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)\sqrt{2(x^2+1)}\left(x\sqrt{2}+\sqrt{x^2+1}\right)} =$
$\lim_{x\to 1} \frac{(x+1)}{\sqrt{2(x^2+1)}\left(x\sqrt{2}+\sqrt{x^2+1}\right)} = \frac{1}{2\sqrt{2}}$.

Solution C.168. [Of Exercise 11.13]]

1. By Corollary 11.3.13, $f'(x) = 4x^3 - 4x$.

2. $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$.

3. Let $g_1(x) = x^2+1$ and $g_2(x) = \sqrt{x}$. Note that $h(x) = g_2(g_1(x))$. By the item above, $g_2'(x) = \frac{1}{2\sqrt{x}}$. By Corollary 11.3.13, $g_1'(x) = 2x$. Hence by the chain rule (Theorem 11.3.15): $f'(x) = g_2'(g_1(x)).g_1'(x) = \frac{1}{2\sqrt{x^2+1}}.2x = \frac{x}{\sqrt{x^2+1}}$.

4. Let $h(x) = \sqrt{x^2+1}$. By the above item, $h'(x) = \frac{x}{\sqrt{x^2+1}}$. Since $f(x) = x/h(x)$ then by Corollary 11.3.13 and the quotient rule (Theorem 11.3.14), $f'(x) = \frac{h(x)-xh'(x)}{x^2+1} = \frac{\sqrt{x^2+1}-x^2/\sqrt{x^2+1}}{x^2+1} =$

$$\frac{x^2+1-x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+1)\sqrt{x^2+1}}.$$

Solution C.169. [Of Exercise 11.14] Use the definition of the derivative to find $f'(x)$.

1.

$$\begin{aligned}
& f'(x) \\
= \ & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
= \ & \lim_{h\to 0} \frac{(x+h)^3 - 5(x+h)^2 - x^3 + 5x^2}{h} \\
= \ & \lim_{h\to 0} \frac{(x+h)^3 - 5x^2 - 5h^2 - 10xh - x^3 + 5x^2}{h} \\
= \ & \lim_{h\to 0} \frac{(x+h)^3 - 5h^2 - 10xh - x^3}{h} \\
= \ & \lim_{h\to 0} \frac{x^3 + xh^2 + 2x^2h + hx^2 + h^3 + 2xh^2 - 5h^2 - 10xh - x^3}{h} \\
= \ & \lim_{h\to 0} \frac{3xh^2 + 3x^2h + h^3 - 5h^2 - 10xh}{h} \\
= \ & \lim_{h\to 0}(3xh + 3x^2 + h^2 - 5h - 10x) \\
= \ & 3x^2 - 10x.
\end{aligned}$$

2.

$$\begin{aligned}
f'(x) = \ & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
= \ & \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
= \ & \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\
= \ & \lim_{h\to 0} \frac{-h^2 - 2xh}{hx^2(x+h)^2} \\
= \ & \lim_{h\to 0} \frac{-h-2x}{x^2(x+h)^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}
\end{aligned}$$

3. $f(x) = \sqrt{x}$.

$$\begin{aligned}
f'(x) = \ & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
= \ & \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
= \ & \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
= \ & \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
= \ & \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Solution C.170. [Of Exercise 11.15]

- Let $\varepsilon > 0$. Let $\delta = \varepsilon$. If $|x - 0| = |x| < \delta$ then $||x| - 0| = |x| < \varepsilon$ and hence $|x|$ is continuous at 0.
- $\lim_{x\to 0^+} \dfrac{|x| - |0|}{x - 0} = \lim_{x\to 0^+} \dfrac{x}{x} = \lim_{x\to 0^+} 1 = 1$ whereas $\lim_{x\to 0^-} \dfrac{|x| - |0|}{x - 0} = \lim_{x\to 0^-} \dfrac{-x}{x} = \lim_{x\to 0^-} -1 = -1$.
  Hence by [LF9] of Theorem 9.3.9, $\lim_{x\to 0} \dfrac{|x| - |0|}{x - 0}$ does not exist and $|x|$ is not differentiable at 0.

Solution C.171. [Of Exercise 11.16] Let $c \in [0, 1]$. We will show that $\lim_{x\to c} \dfrac{|x| - |c|}{x - c}$ is defined. Since $x, c \in [0, 1]$, then $|x| = x$ and $|c| = c$. Hence $\lim_{x\to c} \dfrac{|x| - |c|}{x - c} = \lim_{x\to c} \dfrac{x - c}{x - c} = 1$. Hence $|x|$ is differentiable on the interval $[0, 1]$.

Solution C.172. [Of Exercise 11.17] $\lim_{x\to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \dfrac{2x - 1 - (2 - 1)}{x - 1} = \lim_{x\to 1^+} \dfrac{2x - 2}{x - 1} = \lim_{x\to 0^+} 2 = 2$.
$\lim_{x\to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \dfrac{x^2 - 1}{x - 1} = \lim_{x\to 1^-} \dfrac{(x - 1)(x + 1)}{x - 1} = \lim_{x\to 1^-} x + 1 = 2$.
Hence $\lim_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = 2$ and f is differentiable at 1.
Since it is differentiable at 1 then it is also continuous at 1.

Solution C.173. [Of Exercise 11.18] Let $n = \dfrac{1}{h}$. Note that $h \to 0$ iff $n \to \infty$. Now

$$f'(c) = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} = \lim_{n\to\infty} \frac{f(c + \frac{1}{n}) - f(c)}{\frac{1}{n}} =$$
$$\lim_{n\to\infty} \left(n \left(f(c + \tfrac{1}{n}) - f(c) \right) \right).$$

Solution C.174. [Of Exercise 11.19] Corollary 11.3.12 gives the result for r a positive integer: $f'(x) = rx^{r-1}$.

- First, we do the proof for r a negative integer. Let $n = -r$. Then, $x^r = x^{-n} = \frac{1}{x^n}$. By the quotient rule Theorem 11.3.14, $f'(x) = \left(\frac{1}{x^n}\right)' = \dfrac{0x^n - nx^{n-1}}{x^{2n}} = \dfrac{-n}{x^{n+1}} = -nx^{-n-1} = rx^{r-1}$.
- Next we do the proof for $r = \frac{1}{n}$ where n is a positive integer. For this, we leave it as an exercise to the reader to show the following:

1. $\lim_{x\to c^+} \dfrac{x^{\frac{1}{n}} - c^{\frac{1}{n}}}{x-c} = \frac{1}{n}c^{\frac{1}{n}-1}$.
2. $\lim_{x\to c^-} \dfrac{x^{\frac{1}{n}} - c^{\frac{1}{n}}}{x-c} = \frac{1}{n}c^{\frac{1}{n}-1}$.

Hence by [LF9] of Theorem 9.3.9, $\lim_{x\to c} \dfrac{x^{\frac{1}{n}} - c^{\frac{1}{n}}}{x-c} = \frac{1}{n}c^{\frac{1}{n}-1}$.

- Finally, if $f(x) = x^{\frac{m}{n}}$ where m, n are integers then let $g_2(x) = x^{\frac{1}{n}}$ and $g_1(x) = x^m$. Note that $f = g_2 \circ g_1$ and by the above, $g_2'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$ and $g_1'(x) = mx^{m-1}$. By the chain rule Theorem 11.3.15, $f'(x) = g_2'(g_1(x)).g_1'(x) = \frac{1}{n}(x^m)^{\frac{1}{n}-1}mx^{m-1} = \frac{m}{n}x^{\frac{m}{n}}x^{-m}x^{m-1} = \frac{m}{n}x^{\frac{m}{n}}x^{-1} = \frac{m}{n}x^{\frac{m}{n}-1}$.

Solution C.175. [Of Exercise 11.20]

- Case $r = 2$. $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \dfrac{x^2\cos(1/x) - 0}{x-0} = \lim_{x\to 0} \dfrac{x^2\cos(1/x)}{x} = \lim_{x\to 0}(x\cos(1/x)) = \lim_{x\to 0} x \lim_{x\to 0}\cos(1/x) = 0$ (since $\cos(1/x)$ is bounded).
- Case $r = 1$. $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \dfrac{x\cos(1/x) - 0}{x-0} = \lim_{x\to 0} \dfrac{x\cos(1/x)}{x} = \lim_{x\to 0}\cos(1/x)$.
  Now we show that $\lim_{x\to 0}\cos(1/x)$ does not exist. Intuitively, this is the case because in any interval around 0, no matter how small, we can find x's such that $\cos(1/x) = 1$ and x's such that $\cos(1/x) = -1$. Hence $\cos(1/x)$ has no limit at $x = 0$.

Solution C.176. [Of Exercise 11.21]

1. If f is neither strictly increasing nor strictly decreasing on I then let $a, b, c \in I$ such that $a < b < c$ and $f(a) < f(c) < f(b)$. By the hypothesis (IVT), there is a d such that $a < d < b$ and $f(d) = f(c)$. Since $d < b < c$, f cannot be one-to-one, absurd. Hence f is either strictly increasing or strictly decreasing on I.
2. Let $f(x)$ and $f(y)$ be elements in J such that $f(x) < f(y)$. Clearly $x \neq y$ and $x, y \in I$. Let $f(x) < M < f(y)$. By IVT, there is $z \in I$ such that $f(z) = M$. Hence, $f(z) \in J$ and J is an interval.

   We need to show f_{inv} is continuous on all points of J (whether interior points or end points). we only do the proof for interior points as the

proof for end points is similar.
Let a be an interior point of J. By the above, $f_{\text{inv}}(a)$ is an interior point of I. Let $\varepsilon > 0$. We need to find $\delta > 0$ such that $|y - a| < \delta$ implies $|f_{\text{inv}}(y) - f_{\text{inv}}(a)| < \varepsilon$. Since $f_{\text{inv}}(a)$ is interior in I, we can find $I' \subset I$ such that for all $x \in I'$, $|x - f_{\text{inv}}(a)| < \varepsilon$. By the above, $f(I')$ is interval and there is $\delta > 0$ such that $|y - a| < \delta$ implies $y \in f(I')$. Hence, $|y - a| < \delta$ implies $|f_{\text{inv}}(y) - f_{\text{inv}}(a)| < \varepsilon$.

3. Since f is continuous on I, by 2, f_{inv} is continuous on J. Let $c \in I$ such that $f'(c) \neq 0$.
Since f_{inv} is continuous and for any $y \in J$ there is $x \in I$ such that $f(x) = y$, we have $f(x) \to f(c)$ implies $x \to c$. Now,

$$\lim_{y \to f(c)} \frac{f_{\text{inv}}(y) - f_{\text{inv}}(f(c))}{y - f(c)} =$$

$$\frac{1}{\lim\limits_{y \to f(c)} \dfrac{y - f(c)}{f_{\text{inv}}(y) - f_{\text{inv}}(f(c))}} =$$

$$\frac{1}{\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{f_{\text{inv}}(f(x)) - f_{\text{inv}}(f(c))}} =$$

$$\frac{1}{\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}} =$$

$$\frac{1}{f'(c)}$$

Hence, $f'_{\text{inv}}(f(c)) = \dfrac{1}{f'(c)}$.

Solution C.177. [Of Exercise 11.22] By Theorem 11.3.5, f is continuous on I. Let x be an element of J for which $f'(f_{\text{inv}}(x)) \neq 0$. Since f is differentiable at $f_{\text{inv}}(x)$ and $f'(f_{\text{inv}}(x)) \neq 0$, then by Exercise11.21.3., f_{inv} is differentiable at x and $f'_{\text{inv}}(f(f_{\text{inv}}(x))) = \dfrac{1}{f'(f_{\text{inv}}(x))}$. Hence, $f'_{\text{inv}}(x) = \dfrac{1}{f'(f_{\text{inv}}(x))}$.

Solution C.178. [Of Exercise 11.23]

1. First recall LF17 that we proved in Exercise 8.4 where we showed that

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Recall also the trigonometric formula:

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

Now, $\lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \dfrac{2\cos\frac{2x+h}{2}\sin\frac{h}{2}}{h} = \lim_{h\to 0} \cos\dfrac{2x+h}{2}\dfrac{\sin\frac{h}{2}}{\frac{h}{2}} = \lim_{h\to 0}\cos\dfrac{2x+h}{2}\lim_{h\to 0}\dfrac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x$.

2. Again recall LF17 that we proved in Exercise 8.4 where we showed that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Recall also the trigonometric formula:

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Now, $\lim_{h\to 0} \dfrac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \dfrac{-2\sin\frac{2x+h}{2}\sin\frac{h}{2}}{h} = -\lim_{h\to 0} \sin\dfrac{2x+h}{2}\dfrac{\sin\frac{h}{2}}{\frac{h}{2}} = -\lim_{h\to 0}\sin\dfrac{2x+h}{2}\lim_{h\to 0}\dfrac{\sin\frac{h}{2}}{\frac{h}{2}} = -\sin x$.

3. First recall that $\tan x = \dfrac{\sin x}{\cos x}$ and that $\cos^2 x + \sin^2 x = 1$. By the quotient rule Theorem 11.3.14, $\tan' x = \dfrac{\cos x \sin' x - \sin x \cos' x}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$.

4. First recall that $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$ and that $\cos^2 x + \sin^2 x = 1$. By the quotient rule Theorem 11.3.14, $\cot' x = \dfrac{\sin x \cos' x - \cos x \sin' x}{\sin^2 x} = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$.

5. $\sec x = \dfrac{1}{\cos x}$. Hence, by the quotient rule Theorem 11.3.14, $\sec' x = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\dfrac{\sin x}{\cos x} = \sec x \tan x$.

6. $\csc x = \dfrac{1}{\sin x}$. Hence, by the quotient rule Theorem 11.3.14, $\csc' x = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\dfrac{\cos x}{\sin x} = -\csc x \cot x$.

7. Recall that $arcsin x = \sin_{\text{inv}} x$ and that $\cos^2 x + \sin^2 x = 1$. Hence $\cos^2(\sin_{\text{inv}} x) = 1 - \sin^2(\sin_{\text{inv}} x) = 1 - x^2$. We can easily show that this implies $\cos(\sin_{\text{inv}} x) = \sqrt{1 - \sin^2(\sin_{\text{inv}} x)} = \sqrt{1 - x^2}$ (the positive rather than negative sign).

   Now, we can easily establish the preconditions of Theorem 11.3.19 and hence, $arcsin' x = \sin'_{\text{inv}}(x) = \dfrac{1}{\sin'(\sin_{\text{inv}}(x))} = \dfrac{1}{\cos(\sin_{\text{inv}}(x))} = \dfrac{1}{\sqrt{1-x^2}}$.

8. Recall that $arctan x = \tan_{\text{inv}} x$. Recall also that $\cos^2 x + \sin^2 x = 1$ and hence $\dfrac{\cos^2 x}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$ and $1 + \tan^2 x = \sec^2 x$. Hence $1 + \tan^2(\tan_{\text{inv}} x) = \sec^2(\tan_{\text{inv}} x)$. I.e., $\sec^2(\tan_{\text{inv}} x) = 1 + x^2$.

   By Theorem 11.3.19, $arctan' x = \tan'_{\text{inv}}(x) = \dfrac{1}{\tan'(\tan_{\text{inv}}(x))} = \dfrac{1}{\sec^2(\tan_{\text{inv}}(x))} = \dfrac{1}{1+x^2}$.

9. Recall that $arcsec x = \sec_{\text{inv}} x$. Recall also that $\cos^2 x + \sin^2 x = 1$ and hence $\dfrac{\cos^2 x}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$ and $1 + \tan^2 x = \sec^2 x$. Hence $\tan^2 x = \sec^2 x - 1$. And so, $\tan^2(\sec_{\text{inv}}(x)) = \sec^2(\sec_{\text{inv}}(x)) - 1 = x^2 - 1$. I.e., $\tan(\sec_{\text{inv}}(x)) = \pm\sqrt{x^2 - 1}$.

   By Theorem 11.3.19, $arcsec' x = \sec'_{\text{inv}}(x) = \dfrac{1}{\sec'(\sec_{\text{inv}}(x))} = \dfrac{1}{\sec(\sec_{\text{inv}}(x))\tan(\sec_{\text{inv}}(x))} = \dfrac{1}{x\tan(\sec_{\text{inv}}(x))}$.

   Note that $\sec'_{\text{inv}}(x) = \dfrac{1}{\sec(\sec_{\text{inv}}(x))\tan(\sec_{\text{inv}}(x))}$ is always positive since $\sec(\sec_{\text{inv}}(x))$ and $\tan(\sec_{\text{inv}}(x))$ are either both positive or both negative. Hence to guarantee this, we replace x by $|x|$ in $\dfrac{1}{x\tan(\sec_{\text{inv}}(x))}$ and we replace $\tan(\sec_{\text{inv}}(x)) = \pm\sqrt{x^2-1}$ by $\tan(\sec_{\text{inv}}(x)) = \sqrt{x^2-1}$. That is, we have $arcsec' x = \sec'_{\text{inv}}(x) = \dfrac{1}{x\tan(\sec_{\text{inv}}(x))} = \dfrac{1}{|x|\sqrt{x^2-1}}$.

C.12 Solutions for Chapter 12

Solution C.179. [Of Exercise 12.1]

1. Let $\varepsilon > 0$. Then there is a positive N such that $\frac{1}{2^{N-1}} \leq \varepsilon$. Just take N any natural number such that $N \geq \log_2 \frac{2}{\varepsilon}$.
   Now, let $n, m > N$ and without loss of generality, assume $m > n$. Hence $m = n + k$ where $k > 0$. Note that $\frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} \leq \varepsilon$.

   $$|a_m - a_n| = |a_{n+k} - a_n| \leq$$
   $$|a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots |a_{n+1} - a_n| <$$
   $$2^{-(n+k-1)} + 2^{-(n+k-2)} + \cdots + 2^{-n} =$$
   $$2^{-(n-1)-k} + 2^{-(n-1-(k-1))} + \cdots + 2^{-(n-1)-1} =$$
   $$2^{-(n-1)}(2^{-k} + 2^{-(k-1)} + \cdots + 2^{-1}) =$$
   $$2^{-(n-1)}(1 - \frac{1}{2^k}) <$$
   $$2^{-(n-1)} = \frac{1}{2^{n-1}} < \varepsilon$$

   Hence $\{a_n\}$ is a Cauchy sequence and hence by Lemma 12.1.8, it is a sequence that converges to a limit.

2. No. Take the sequence $\{a_n\}$ where $a_n = \sum_{k=1}^{n} \frac{1}{k}$. For this sequence, we have that $|a_{n+1} - a_n| = \frac{1}{n+1} < \frac{1}{n}$ for all positive integers n. But this sequence is not convergent as we will see below and hence it is not a Cauchy sequence. The proof that it is not convergent is due to Jacob Bernouilli and goes as follows: Let $a_{p_n} = a_{2^n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots (\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n})$. Note that $\{a_{p_n}\}$ is a subsequence of $\{a_n\}$ and $a_{p_n} > 1 + \frac{1}{2} + (\frac{1}{2^2} + \frac{1}{2^2}) + \cdots (\frac{1}{2^n} + \cdots + \frac{1}{2^n}) = 1 + \frac{1}{2} + 2\frac{1}{2^2} + \cdots 2^{n-1}\frac{1}{2^n} = 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots \frac{1}{2}}_{n} = 1 + \frac{n}{2}$.

   Now, $\{a_{p_n}\}$ can be shown to diverge as follows: Let $M > 0$ and let N be a positive integer such that $N > 2(M-1)$. Hence $\frac{N}{2} + 1 > M$. For all $n > N$, $a_{p_n} > 1 + \frac{n}{2} > 1 + \frac{N}{2} > M$. Hence $\{a_{p_n}\}$ is not convergent.

Solution C.180. [Of Exercise 12.2] First we prove that the sequence $\{r^n\}$ converges. If $\varepsilon > 0$, let N be such that $r^N < \varepsilon$. Then, for any $n > N$ we have $|r^n| < |r^N| < \varepsilon$. Hence by Lemma 12.1.6, $\{r^n\}$ is a Cauchy sequence. Now we show that $\{a_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\{r^n\}$ is a

Cauchy, there is an $N > 0$ such that for all $n, m > N$ we have $|r^n - r^m| < (1-r)\varepsilon$. Let n, m such that $n > m > M$. Then $|a_n - a_m| \leq \sum_{k=m+1}^{n} |a_k - a_{k-1}| < \sum_{k=m+1}^{n} r^{k-1} = \frac{r^m - r^n}{1-r} < \varepsilon$. Hence $\{a_n\}$ is a Cauchy sequence and by Lemma 12.1.8 converges to a limit.

Solution C.181. [Of Exercise 12.3] Note that for $k > 1$ we have $|a_k - a_{k-1}| \leq r^{k-2}|a_2 - a_1|$. If $a_2 = a_1$ then for all k, $a_k = a_{k+1}$ and the sequence is the constant sequence and it is Cauchy. We assume that $a_2 \neq a_1$.
Now, for $n > m$ we have:
$|a_n - a_m| \leq \sum_{k=m+1}^{n} |a_k - a_{k-1}| \leq \sum_{k=m+1}^{n} r^{k-2}|a_2 - a_1| =$
$\frac{|a_2 - a_1|}{r} \sum_{k=m+1}^{n} r^{k-1} = \frac{|a_2 - a_1|}{r} \frac{r^m - r^n}{1-r} = \frac{|a_2 - a_1|}{r(1-r)}(r^m - r^n)$. By Exercise 12.2 above, we know that $\{r^n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. There is a number N such that for all $n, m > N$ we have $|r^m - r^n| < \frac{r(1-r)}{|a_2 - a_1|}\varepsilon$. Let $n > m > N$. Then $|a_n - a_m| \leq \frac{|a_2 - a_1|}{r(1-r)}(r^m - r^n) < \frac{|a_2 - a_1|}{r(1-r)} \frac{r(1-r)}{|a_2 - a_1|}\varepsilon = \varepsilon$.
Hence $\{a_n\}$ is a Cauchy sequence and by Lemma 12.1.8 converges to a limit.

Solution C.182. [Of Exercise 12.4] Let the Jacobsthal numbers be defined as follows:

$$J_n = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2 \\ 2J_{n-1} + (-1)^{n-1} & \text{if } n > 2 \end{cases}$$

Note that $J_n = J_{n-1} + 2J_{n-2}$.
We can easily show that $a_n = \frac{1}{2^{n-1}}(J_{n-1}a_0 + J_n a_1)$ for $n \geq 2$. Below we show some examples:

$$\begin{aligned} a_2 &= \frac{1}{2}(a_0 + a_1) \\ a_3 &= \frac{1}{2^2}(a_0 + 3a_1) \\ a_4 &= \frac{1}{2^3}(3a_0 + 5a_1) \\ a_5 &= \frac{1}{2^4}(5a_0 + 11a_1) \\ a_6 &= \frac{1}{2^5}(11a_0 + 21a_1) \end{aligned}$$

Note that

$$\begin{aligned} a_2 - a_1 &= -\frac{1}{2}(a_1 - a_0) \\ a_3 - a_2 &= \frac{1}{2^2}(a_1 - a_0) &&= -\frac{1}{2}(a_2 - a_1) \\ a_4 - a_3 &= -\frac{1}{2^3}(a_1 - a_0) &&= -\frac{1}{2}(a_3 - a_2) \end{aligned}$$

In general, for $n \geq 1$ we have

$$a_{n+1} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_0) = -\frac{1}{2}(a_n - a_{n-1}).$$

With these equations, we can show that

$$|a_{n+1} - a_n| = \frac{1}{2}|a_n - a_{n-1}| < \frac{3}{4}|a_n - a_{n-1}|.$$

Hence by Exercise 12.3 above we know that $\{a_n\}$ is a Cauchy sequence and by Lemma 12.1.8 it converges. Now,

$$\begin{aligned} a_n &= (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1) + a_1 \\ &= (a_1 - a_0)\left(\left(\frac{-1}{2}\right)^{n-1} + \left(\frac{-1}{2}\right)^{n-2} + \cdots + \frac{-1}{2}\right) + a_1 \\ &= a_1 + (a_1 - a_0)\Sigma_{k=1}^{n-1}\left(\frac{-1}{2}\right)^k. \end{aligned}$$

Since $|\frac{-1}{2}| < 1$, the geometric series $\Sigma_{k=1}^{n-1}\left(\frac{-1}{2}\right)^k$ converges to $\dfrac{1}{1+\dfrac{1}{2}} = \frac{2}{3}$.

Hence a_n converges to $\frac{2}{3}(a_1 - a_0) + a_1 = \dfrac{5a_1 - 2a_0}{3}$.

Solution C.183. [Of Exercise 12.5] Let $c = \sup S$ and and let $a \in S$. For each positive integer n, $c - \frac{1}{n}$ is not an upper bound of S and hence there is an $a_n \in S$ such that $c - \frac{1}{n} \leq a_n < c$. We know that $\lim_{n\to\infty}(c - \frac{1}{n}) = c$ and hence by LS9, $\lim_{n\to\infty} a_n = c$.

Solution C.184. [Of Exercise 12.6] Let $s_1 = 1$ and $s_{n+1} = (s_n + 1)/3$ for $n \geq 1$.

1. $s_2 = \frac{2}{3}$, $s_3 = \frac{5}{3^2}$, $s_4 = \frac{14}{3^3}$, $s_5 = \frac{41}{3^4}$.
2. $s_1 = 1 > \frac{1}{2}$ and $s_2 = \frac{2}{3} > \frac{1}{2}$. Assume $s_n > \frac{1}{2}$ for $n > 1$. Then, for $n \geq 1$, $s_{n+1} = (s_n + 1)/3 > (\frac{1}{2} + 1)/3 = \frac{1}{2}$. Hence, $s_n > 1/2$ for all n.
3. For $n \geq 1$, $s_{n+1} - s_n = \dfrac{s_n + 1}{3} - s_n = \dfrac{1 - 2s_n}{3} <^{by 2.} \frac{1}{3} - \frac{2}{3}\frac{1}{2} = 0$. Hence for $n \geq 1$, $s_{n+1} < s_n$. Hence $\{s_n\}$ is a nonincreasing sequence.
4. We have shown in 3. above that for all $n > 1$, $s_n < s_1 = 1$. We have also shown in 2. above that $s_n > 1/2$ for all n. Hence for all $n > 1$, $1/2 < s_n < 1$. Now, $\{s_n\}$ is a bounded and monotone sequence, hence by Theorem 12.1.2, $\{s_n\}$ has a limit l. Now, since $s_{n+1} = \dfrac{s_n + 1}{3}$ and both $\{s_n\}$ and $\{s_{n+1}\}$ have the same limit l, we have: $l = \dfrac{l+1}{3}$. That is, $l = \frac{1}{2}$.

Solution C.185. [Of Exercise 12.7] Let $t_1 = 1$ and $t_{n+1} = [1-1/(n+1)^2]t_n$ for $n \geq 1$.

1. Note that for all $n \geq 1$, $1/2 \leq 1 - 1/(n+1)^2 \leq 1$. Hence for all $n \geq 1$:

   - $t_n > 0$. We prove this by induction on $n \geq 1$. $t_1 = 1 > 0$. Assume $t_n > 0$ then $t_{n+1} = [1 - 1/(n+1)^2]t_n \geq t_n/2 > 0$.
   - Since $t_n > 0$ and $1 - 1/(n+1)^2 \leq 1$, then $t_{n+1} = [1 - 1/(n+1)^2]t_n \leq t_n$. Hence $\{t_n\}$ is a nonincreasing sequence.

   Hence for all $n \geq 1$, $0 < t_n \leq t_1 = 1$. Now, $\{t_n\}$ is a bounded and monotone sequence, hence by Theorem 12.1.2, $\{t_n\}$ has a limit l.

2. $t_1 = 1 = (1+1)/(2)$. Assume $t_n = (n+1)/(2n)$ for some $n \geq 1$. Then, $t_{n+1} = [1 - 1/(n+1)^2]t_n = [1 - 1/(n+1)^2](n+1)/(2n) = \dfrac{((n+1)^2-1)(n+1)}{2n(n+1)^2} = \dfrac{n^2+2n}{2n(n+1)} = \dfrac{n+2}{2(n+1)}$.
   Hence for all n, $t_n = (n+1)/(2n)$.

3. Since $\{t_n\}$ and $\{\frac{1}{2} + \frac{1}{2n}\}$ have limits, then $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\frac{1}{2} + \frac{1}{2n}) = \frac{1}{2} + \lim_{n\to\infty} \frac{1}{2n} = \frac{1}{2}$.

Solution C.186. [Of Exercise 12.8] We will prove that $5 \Rightarrow 4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 4 \Rightarrow 5$ and $1 \Leftrightarrow 3$ hence establishing that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$.

$5 \Rightarrow 4$. Let $I_n = \{[a_n, b_n]\}$ be a nested sequence of closed and bounded intervals. Hence for each n, I_{n+1} is contained in I_n and $a_1 \leq a_n \leq b_n \leq b_1$. Hence by the Bolzano-Weierstrass Theorem 12.1.15, there is a convergent subsequence $\{b_{p_n}\}$ of $\{b_n\}$. Let b be the limit of $\{b_{p_n}\}$. Note that $\{b_{p_n}\}$ is a decreasing sequence and that $p_n \geq n$ for all n. Hence $b_{p_n} \leq b_n$ for all n and $b_{p_n} \leq b_{p_m}$ for all $n > m$. It is the case that $b \leq b_n$ for all n, since otherwise, if there is an m_0 such that $b > b_{m_0}$, then $b_{p_n} \leq b_{p_{m_0}} \leq b_{m_0} < b$. Hence $b - b_{p_n} \geq b - b_{m_0}$ for all $n > m_0$ which contradicts the fact that b is a limit of $\{b_{p_n}\}$. Furthermore, since for all n, we have $a_n \leq b_{p_n}$, then $a_n \leq b$ for all n. Hence $a_n \leq b \leq b_n$ for all n, and b belongs to all intervals I_n.

$4 \Rightarrow 1$. Let A be a nonempty set of real numbers that has an upper bound b_1.

Since A is not empty, then there is $a_1 \in A$. If a_1 is an upper bound of A then a_1 is the least upper bound of A and we are done. Else, if a_1 is not an upper bound of A then $a_1 < b_1$ and let $[a_1, b_1]$ and $c_1 = \dfrac{a_1+b_1}{2}$.

Repeat the same process. If c_1 is an upper bound of A, let $[a_2, b_2] = [a_1, c_1]$ else, if c_1 is not an upper bound of A, let $[a_2, b_2] = [c_1, b_1]$.

Let $c_2 = \dfrac{a_2 + b_2}{2}$. Note that b_2 is an upper bound of A and $[a_2, b_2] \subseteq [a_1, b_1]$.

Again, if we have already constructed $[a_n, b_n]$ and $c_n = \dfrac{a_n + b_n}{2}$ such that b_n is an upper bound of A and $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$, then we repeat the same process. If c_n is an upper bound of A, let $[a_{n+1}, b_{n+1}] = [a_n, c_n]$ else, if c_n is not an upper bound of A, let $[a_{n+1}, b_{n+1}] = [c_n, b_n]$. Let $c_{n+1} = \dfrac{a_{n+1} + b_{n+1}}{2}$.

Note that b_{n+1} is an upper bound of A and $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$.

Clearly, we have $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_n, b_n], \cdots$ such that b_n is an upper bound of A. By the nested interval theorem, there is a $b \in \bigcap_{n\geq 1} I_n$. We show that $b = \sup A$.

- b **is an upper bound of** A**:** Assume otherwise there is a $c \in A$ such that $c > b$. Note that $c - b > 0$. But, for each n, we have $c \leq b_n$ and $a_n \leq b$. By the Archimedean law, there is a positive integer n such that $b_1 - a_1 < n(c-b)$ and hence here is an integer N such that $b_1 - a_1 < 2^{N-1}(c-b)$ and so, $\dfrac{b_1 - a_1}{2^{N-1}} < c - b$. Recall that $b_n - a_n = \dfrac{b_1 - a_1}{2^{n-1}}$ for each n. We have for each $n \geq N$:

$$c - b \leq b_n - b \leq b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \leq \frac{b_1 - a_1}{2^{N-1}} < c - b.$$

  Absurd. Hence b is an upper bound of A.

- b **is least upper bound of** A**:** Let d be an upper bound of A. We want to show that $b \leq d$. Assume otherwise that $d < b$. Then $b - d > 0$. Similarly as we did in the above item, let N such that $\dfrac{b_1 - a_1}{2^{N-1}} < b - d$. Recall that $b_n - a_n = \dfrac{b_1 - a_1}{2^{n-1}}$ for each n and that $a_n \leq d$ and $b \leq b_n$.

$$b - d \leq b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \leq \frac{b_1 - a_1}{2^{N-1}} < b - d.$$

  Absurd. Hence b is the least upper bound of A.

$1 \Rightarrow 2$. This was seen as part of the proof of Theorem 12.1.2 where it was shown for nondecreasing bounded sequences. Here, we show it for nonincreasing bounded sequences. Let $\{a_n\}$ be a nonincreasing sequence. Let A be the set of all real numbers a_n in the sequence, and since A is bounded and not empty, by the Completeness Axiom, it has a greatest lower bound, say l. Let $\varepsilon > 0$ be given. Then $l + \varepsilon$ cannot be a lower bound for A, so there is a positive integer N such that $a_N < l + \varepsilon$. Since a_n is nonincreasing, $a_N \geq a_n$ for all $n > N$. Of course, for all $n, a_n \geq l > l - \varepsilon$, and so if $n > N$, $l - \varepsilon < a_n < l + \varepsilon$. This latter implies that $|a_n - l| < \varepsilon$. This shows that $\lim_{n\to\infty} a_n = l$.

$2 \Rightarrow 4$. This was seen as part of the proof of Theorem 12.1.11 where $\{[a_n, b_n]\}$ is a nested sequence of closed and bounded intervals, and since the sequences $\{a_n\}$ and $\{b_n\}$ are monotone and bounded, they have limits a resp. b such that $a \leq b$ and any z satisfying $a \leq z \leq b$ is in all the intervals. Furthermore, if $\lim_{n\to\infty}(b_n - a_n) = 0$, then then we have

$$0 = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = b - a,$$

so $a = b$ and $z = b = a$ is unique.

$4 \Rightarrow 5$. Let $\{a_n\}$ be a bounded sequence. We will find a subsequence that converges. We do this by finding a sequence of nested closed bounded intervals $\{I_n\}$ such that for some a, $a \in \bigcap_{n\geq 1} I_n$ and we will find a subsequence of $\{a_n\}$ that converges to a. Since $\{a_n\}$ is bounded, let M such that $|a_n| \leq M$ for every M. Let $[l_1, r_1] = [-M, M]$ and let $c_1 = \dfrac{l_1 + r_1}{2}$. We construct $[l_2, r_2] = I_2$ such that

- $r_2 - l_2 = \dfrac{r_1 - l_1}{2}$ and
- if $[l_1, c_1]$ has infinitely many elements of $\{a_n\}$ then $I_2 = [l_1, c_1]$ else $[c_2, r_2]$ has infinitely many elements of $\{a_n\}$ and we let $I_2 = [c_2, r_2]$.

We iterate this process building $[l_{n+1}, r_{n+1}] = I_{n+1}$ such that for $c_n = \dfrac{l_n + r_n}{2}$, we have

- $r_{n+1} - l_{n+1} = \dfrac{r_n - l_n}{2}$ and
- if $[l_n, c_n]$ has infinitely many elements of $\{a_n\}$ then $I_{n+1} = [l_n, c_n]$ else $[c_n, r_n]$ has infinitely many elements of $\{a_n\}$ and we let $I_{n+1} = [c_n, r_n]$.

Obviously, $\{I_n\}$ is a sequence of nested closed bounded intervals and by the nested interval theorem 12.1.11, there is an a such that $a \in \bigcap_{n\geq 1} I_n$ and for each n, $l_n \leq l_{n+1} \leq a \leq r_{n+1} \leq r_n$. Since $\{l_n\}$ and $\{r_n\}$ are monotone bounded sequences, they converge to l resp. r such that $l_n \leq l_{n+1} \leq l \leq a \leq r \leq r_{n+1} \leq r_n$. We will show that $l = r$.
We know that $r_{n+1} - l_{n+1} = \dfrac{r_n - l_n}{2}$ and hence $2(\lim r_{n+1} - \lim l_{n+1}) =$

$\lim r_n - \lim l_n$. I.e., $2(r-l) = r-l$. I.e., $r = l$. This means $l = r = a$. Since for every n, I_n is not empty, we build our subsequence as follows: a_{n_1} is an arbitrary element of I_1. Since I_2 contains infinitely many elements of $\{a_n\}$, let a_{n_2} be one of these elements in I_2 such that $a_{n_1} < a_{n_2}$. We repeat the process building the subsequence $\{a_{n_k}\}$ of $\{a_n\}$. It is easy to show that $\{a_{n_k}\}$ converges to a. In fact, since for every k, $l_k \leq a_{n_k} \leq c_k$, by LS33, $\{a_{n_k}\}$ converges to a.

$1 \Rightarrow 3$. In the proof we gave for Lemma 12.1.8, we used both the Axiom of Completeness (our item 1 1), and Theorem 12.1.2 (our item 2). Here, reproduce the proof but where you replace the use of Theorem 12.1.2 (our item 2) by a proof of it as we did in the step $1 \Rightarrow 2$ above.

$3 \Rightarrow 1$. Let A be a non empty set that has an upper bound c_1. Since A is non empty, let $a_1 \in A$. If a_1 is an upper bound of A then a_1 is the least upper bound of A and we are done. Else, if a_1 is not an upper bound of A then $a_1 < c_1$ and let $m_1 = \frac{a_1 + c_1}{2}$. We will define a bounded Cauchy sequence whose limit is the least upper bound of A.

We start by building three sequences $\{a_n\}$ increasing, $\{c_n\}$ decreasing, and $\{m_n\}$ such that for every n, $a_n \leq m_n \leq c_n$, $m_n = \frac{a_n + c_n}{2}$ and $|c_{n+1} - a_{n+1}| \leq \frac{|c_n - a_n|}{2}$, $|c_{n+1} - c_n| \leq \frac{|c_n - a_n|}{2}$, $a_n \in A$ and c_n is an upper bound of A.

- If m_1 is an upper bound of A, let $a_2 = a_1$ and $c_2 = m_1$, else, if m_1 is not an upper bound of A, then there is $a_2 \in A$ such that $m_1 < a_2$. Let $c_2 = c_1$ and $m_2 = \frac{a_2 + c_2}{2}$. Note that c_2 is an upper bound of A, $|c_2 - a_2| \leq \frac{|c_1 - a_1|}{2}$, $|c_2 - c_1| \leq \frac{|c_1 - a_1|}{2}$, $a_1 \leq a_2$, $c_2 \leq c_1$ and $a_2 \leq m_2 \leq c_2$.
- Assume we have $a_n \in A$, c_n upper bound of A and $m_n = \frac{c_n + a_n}{2}$ as above. We repeat the process: If m_n is an upper bound of A, we let $a_{n+1} = a_n$ and $c_{n+1} = m_n$, else, there is $a_{n+1} \in A$ such that $m_n < a_{n+1}$ and $c_{n+1} = c_n$. Obviously $|c_{n+1} - a_{n+1}| \leq \frac{|c_n - a_n|}{2}$, $a_n \leq a_{n+1}$, $c_{n+1} \leq c_n$ and $a_{n+1} \leq m_{n+1} \leq c_{n+1}$ and $a_{n+1} \in A$ and c_{n+1} is an upper bound of A.

Hence, we can easily show that $|c_{n+1} - a_{n+1}| \leq \frac{|c_1 - a_1|}{2^n}$, $\lim_{n\to\infty} c_n - a_n = 0$ and $|c_{n+i} - c_n| \leq \frac{|c_1 - a_1|}{2^i}$ for any positive i and $|a_{n+i} - a_n| \leq \frac{|c_1 - a_1|}{2^i}$ for any positive i. Hence, we can show that $\{a_n\}$ and $\{c_n\}$ are both Cauchy sequences. Hence by Lemma 12.1.8, $\{a_n\}$ and $\{c_n\}$ converge to a and c

respectively. Since $\{c_n - a_n\}$ converges to 0, $a = c$. Note that $a_n \leq a = c \leq c_n$.
Now we show that c is a lub of A.

- For all $x \in A$, $x \leq c_n$ for all n. Hence, $x \leq c$ and c is an upper bound of A.
- Next, If b is upper bound of A, then $a_n \leq b$ for every n and hence $a \leq b$. Hence, $c \leq b$ and c is a least upper bound of A.

Solution C.187. [Of Exercise 12.9]

1. The tails of $\{a_n\}$ where $a_n = (-1)^{n+1} = 1, -1, 1, -1, \cdots$, were already given in Section 12.2. If N is odd, T_N is the same as the entire sequence, while if N is even, $T_N = \{-1, 1, -1, 1, \cdots\}$. Clearly, for each N,
$$\inf T_N = -1 \text{ and } \sup T_N = 1.$$
Hence, $\liminf a_n = -1$ and $\limsup a_n = 1$.

   The floor terms of $\{a_n\}$ are all the terms a_N such that N is even and all these floor terms are equal to -1. The sequential limits are $\{-1, 1\}$ because we have found a subsequence that converges to 1 and another subsequence that converges to -1.

2. The tails of $\{a_n\}$ where $a_n = (-1)^n = -1, 1, -1, 1, \cdots$, are as follows: If N is even, T_N is the same as the entire sequence, while if N is odd, $T_N = \{1, -1, 1, -11, \cdots\}$. Clearly, for each N,
$$\inf T_N = -1 \text{ and } \sup T_N = 1.$$
Hence, $\liminf a_n = -1$ and $\limsup a_n = 1$.

   The floor terms of $\{a_n\}$ are all the terms a_N such that N is odd and all these floor terms are equal to -1. The sequential limits are $\{-1, 1\}$ because we can found a subsequence that converges to 1 and another subsequence that converges to -1.

3. Left as an exercise.

Solution C.188. [Of Exercise 12.10] We do the proof first for $f(a) < v < f(b)$. The proof for $f(b) < v < f(a)$ follows. Let $S = \{x \in [a, b] : f(x) < v\}$. Since S is nonempty (it contains a) and bounded above (b is an upper bound), S has a least upper bound $c = \sup S$ by the Completeness Axiom. Note that for $x \in [a, b]$:

1. If $f(x) < v$ then $a \leq x \leq c \leq b$.

2. Hence if $x > c$ then $f(x) \geq v$.

3. Furthermore, since c is the least upper bound such that $f(x) < v$ then there is no $y < c$ such that $f(x) \geq v$ for $x > y$.

4. Finally, since f is continuous, $\lim_{x \to c} f(x) = f(c)$.

We show now that the value c satisfies $f(c) = v$.

- We show first that $f(c) \leq v$. For any positive integer n, let $c_n = c - \frac{c-a}{2^n}$. Note that $c_1 = c - \frac{c-a}{2} = \frac{c+a}{2} > a$ and that for all n, $c_n < c$ and by 3. above, $f(c_n) < v$. Also, since $2^n < 2^{n+1}$, we have $c_n < c_{n+1}$. So, $\{c_n\}$ is an increasing sequence of elements of S such that for all n, $a < c_n < c$. Hence by Bounded monotone sequences Theorem 12.1.2, $\{c_n\}$ has a limit. In fact, we have that $\lim_{n\to\infty} c_n = c$. By Theorem 11.2.3, $\lim_{n\to\infty} f(c_n) = f(c)$. Since $f(c_n) < v$ for all n then $f(c) \leq v$.

- Now we show that $f(c) \geq v$. Let $d_n = c + \frac{b-c}{n}$ for each positive integer n and note that by 2. above, $\{d_n\}$ is a sequence in the complement of S that converges to c (we prove this similarly to the above item). Since $f(d_n) \geq v$ for all n and the sequence $\{f(d_n)\}$ converges to $f(c)$, the inequality $f(c) \geq v$ must hold.

It follows that $f(c) = v$.
As for the proof when $f(b) < v < f(a)$, let $g(x) = -f(x)$. Then g is defined on the same interval as f and g is continuous on $[a, b]$ and $g(a) < -v < g(b)$. By what we proved above, there is a $c \in (a, b)$ such that $g(c) = -v$. Hence $f(c) = v$.

Solution C.189. [Of Exercise 12.11] Assume f is not bounded on closed interval I. Then, for each positive integer n, there is $x_n \in I$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass Theorem 12.1.15, $\{x_n\}$ has a subsequence $\{x_{p_n}\}$ that converges to a limit a. Since $\inf I \leq x_{p_n} \leq \sup I$, then $\inf I \leq a \leq \sup I$ and $a \in I$. Since f is continuous, by Theorem 11.2.3, $\lim_{n\to\infty} f(x_{p_n}) = f(a)$ and by Exercise 9.7.2, $\lim_{n\to\infty} |f(x_{p_n})| = |f(a)|$. But for all n, $|f(x_n)| > n$, hence $|f(x_{p_n})| > p_n \geq n$. So, $|f(a)| = \lim_{n\to\infty} |f(x_{p_n})| \geq \lim_{n\to\infty} n$. Absurd since f is defined at a.
Since $f(I)$ is bounded non empty, by the Completeness Axiom, let g and l be the greatest lower versus least upper bounds of $f(I)$.

- For each positive n, since l is a least upper bound of $f(I)$, there is $y_n \in I$ such that $|f(y_n) - l| < \frac{1}{n}$. Let $\varepsilon > 0$ and let N a positive integer such that $N \geq \frac{1}{\varepsilon}$. Then for each $n > N$ we have: $|f(y_n) - l| < \frac{1}{n} < \frac{1}{N} \leq \varepsilon > 0$. Hence $\lim_{n\to\infty} f(y_n) = l$. Since I is closed and $y_n \in I$ for all n, then $\lim_{n\to\infty} y_n = y \in I$. Since f is continuous then $l = \lim_{n\to\infty} f(y_n) = f(\lim_{n\to\infty} y_n) = f(y)$.

- The proof that there is $x \in I$ such that $f(x) = g$ is similar to the above.

Hence there are x and y in I such that $f(x) = g$ and $f(y) = l$.
Let $h(x) = \frac{1}{x^2}$ and $J = [-1, 0)$. Then, h is continuous on $[-1, 0)$ with greatest lower bound -1 and least upper bound 0. But there is no $x \in [-1, 0)$ such that $h(x) = 0$.
Let $h'(x) = x$ and $J = [-1, 0)$. Then, h' is continuous on $J = [-1, 0)$ with greatest lower bound of J being -1 and least upper bound of J being 0. But there is no $x \in [-1, 0)$ such that $h'(x) = 0$.

Solution C.190. [Of Exercise 12.12]

1. Let $f(x) = x - \tan x$. Since f is continuous on $[2\pi, 2\pi + \frac{\pi}{2})$, $f(2\pi) = 2\pi$ and $\lim_{\varepsilon\to 0} f(2\pi + \frac{\pi}{2} - \varepsilon) = -\infty$, by Intermediate Value Theorem 12.3.2, for ε very small, there is $x \in [2\pi, 2\pi + \frac{\pi}{2} - \varepsilon]$ such that $f(x) = 0$. Hence there is $x > 2\pi$ such that $\tan x = x$.

2. Let $f(x) = x - \cos x$. Since f is continuous on $[0, \frac{\pi}{2}]$, $f(0) = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi}{2} > 0$, by Intermediate Value Theorem 12.3.2, there is $x \in [0, \frac{\pi}{2}]$, such that $f(x) = 0$. Since $f(0) \neq 0$ and $f(\frac{\pi}{2}) \neq 0$, there is $x \in (0, \frac{\pi}{2})$, such that $\cos x = x$.

3. If $f(a) = a$ or $f(b) = b$ we are done. Assume $f(a) \neq a$ and $f(b) \neq b$ and let $g(x) = f(x) - x$. Then, since $f([a, b]) \subseteq [a, b]$, $a < f(a), f(b) < b$. Hence $g(b) < 0 < g(a)$ and since g is continuous on $[a, b]$, by Intermediate Value Theorem 12.3.2, there is $x \in [a, b]$ such that $g(x) = 0$. Hence there is $x \in [a, b]$ such that $f(x) = x$.

4. $f(x) = x2^x x - 1$ is continuous on $[0, 1]$ and $f(0) = -1$ and $f(1) = 1$. Since $f(0) < 0 < f(1)$, by Intermediate Value Theorem 12.3.2, there is $x \in [0, 1]$ such that $f(x) = 0$. But $f(0) \neq 0$ and $f(1) \neq 0$. Hence there is $x \in (0, 1)$ such that $x2^x x = 1$.

5. Assume $a < b$. Note that f is continuous on $[a, b]$. Since $f(a)f(b) < 0$, we know that $f(a) \neq 0$, $f(b) \neq 0$ and 0 is strictly between $f(a)$ and $f(b)$. Hence by Intermediate Value Theorem 12.3.2, there is a number x between a and b such that $f(x) = 0$.

Solution C.191. [Of Exercise 12.13] Since for any x real number, $\lim_{n\to\infty}\frac{x}{n}=0$, given $\varepsilon>0$ and a real number x, there is a positive integer $N(\varepsilon,x)$ such that $N(\varepsilon,x)>\frac{|x|}{\varepsilon}$. Hence there is a positive integer $N(\varepsilon,x)$ such that for all $n>N(\varepsilon,x)$, $|\frac{x}{n}|<\frac{|x|}{N(\varepsilon,x)}<\varepsilon$.

Solution C.192. [Of Exercise 12.14] Since for all $x\in(0,1)$, $\lim_{n\to\infty}x^n=0$, given $\varepsilon>0$ and a real number x, there is a positive integer $N(\varepsilon,x)$ such that $N(\varepsilon,x)>\log_x\varepsilon$. Hence there is a positive integer $N(\varepsilon,x)$ such that for all $n>N(\varepsilon,x)$, $|x^n|<|x^{N(\varepsilon,x)}|<x^{\log_x\varepsilon}=\varepsilon$.

Solution C.193. [Of Exercise 12.15]

1. Let $\varepsilon=1$. Since $\{f_n\}$ uniformly converges to f, there is a positive integer N such that for each $n>N$, for each $x\in I$, $|f_n(x)-f(x)|<1$. Hence, for each $n>N$, for each $x\in I$, $|f_n(x)|\le|f_n(x)-f(x)|+|f(x)-f_{N+1}(x)|+|f_{N+1}(x)|<2+M_{N+1}$. Let $M=\max\{M_1,M_2,...M_N,2+M_{N+1}\}$. Obviously, for all n, for all $x\in I$, $|f_n(x)|\le M$.

2. Let $\varepsilon=1$. Since $\{f_n\}$ uniformly converges to f, there is a positive integer N such that:
for each $n>N$, for each $x\in I$, $|f_n(x)-f(x)|<1$. Hence, for each $x\in I$, $|f(x)|\le|f_{N+1}(x)-f(x)|+|f_{N+1}(x)|<1+M_f$. Hence f is bounded.

Solution C.194. [Of Exercise 12.16] Let I be an interval. Suppose that $\{f_n\}$ converges uniformly to f on I and that $\{g_n\}$ converges uniformly to g on I.

1. Let $\varepsilon>0$. Since $\{f_n\}$ converges uniformly to f on I and $\{g_n\}$ converges uniformly to g on I, there are positive integers N_1 and N_2 such that:
for all $x\in I$ and all $n>N_1$, $|f(x)-f_n(x)|<\varepsilon/2$ and
for all $x\in I$ and all $n>N_2$, $|g(x)-g_n(x)|<\varepsilon/2$.

   Let $N=\max\{N_1,N_2\}$. Then for all $x\in I$ and all $n>N$

$$
\begin{array}{ll}
|(f(x)-g(x))-(f_n(x)+g_n(x))| & \le \\
|f(x)-f_n(x)|+|g(x)-g_n(x)| & < \\
\varepsilon/2+\varepsilon/2 & = \\
\varepsilon. &
\end{array}
$$

   Hence $\{f_n+g_n\}$ converges uniformly to $f+g$ on I.

2. For any $x\in\mathbb{R}$ and n positive integer, let $f_n(x)=\frac{1}{n}$, $g_n(x)=x$, $f(x)=0$ and $g(x)=x$. Obviously $\{f_n\}$ uniformly converges to f and $\{g_n\}$

uniformly converges to g. Furthermore, $\{f_n g_n\}$ converges pointwise to fg ($\lim_{n\to\infty} f_n(x)g_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0 = f(x)g(x)$). However, $\{f_n g_n\}$ does not uniformly converge to fg. To see this, for each n, let $M_n = \sup\{|f_n(x)g_n(x) - f(x)g(x)| : x \in \mathbb{R}\} = \sup\{|\frac{x}{n}| : x \in \mathbb{R}\} = \infty$. Hence $\{M_n\}$ does not converges to 0 and by Theorem 12.4.7, $\{f_n g_n\}$ does not uniformly converge to fg.

We could give another example where the domain of the function is not the whole $\mathbb{R}$. Here is such an example:
For any $x \in (0,1)$ and positive integer n, let $f_n(x) = \frac{1}{n}$, $g_n(x) = \frac{1}{x}$, $f(x) = 0$ and $g(x) = \frac{1}{x}$. Obviously $\{f_n\}$ uniformly converges to f on $(0,1)$ and $\{g_n\}$ uniformly converges to g on $(0,1)$. Furthermore, $\{f_n g_n\}$ converges pointwise to fg ($\lim_{n\to\infty} f_n(x)g_n(x) = \lim_{n\to\infty} \frac{1}{xn} = 0 = f(x)g(x)$). However, $\{f_n g_n\}$ does not uniformly converge to fg. To see this, for each n, let $M_n = \sup\{|f_n(x)g_n(x) - f(x)g(x)| : x \in (0,1)\} = \sup\{|\frac{1}{nx}| : x \in (0,1)\} = \infty$. Hence $\{M_n\}$ does not converges to 0 and by Theorem 12.4.7, $\{f_n g_n\}$ does not uniformly converge to fg.

3. Since f and g are bounded on I, then there are M_f and M_g such that for all $x \in I$, $|f(x)| \leq M_f$ and $|g(x)| \leq M_g$.

   Since $\{f_n\}$ converges uniformly to f on I there is a positive integer N_1 such that:
   for all $x \in I$ and all $n > N_1$, $|f(x) - f_n(x)| < 1$. Hence for all $x \in I$ and all $n > N_1$, $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + M_f$

   Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly to f on I and $\{g_n\}$ converges uniformly to g on I, there are positive integers N_f and N_g such that:
   for all $x \in I$ and all $n > N_f$, $|f(x) - f_n(x)| < \dfrac{\varepsilon}{2(M_g+1)}$ and
   for all $x \in I$ and all $n > N_g$, $|g(x) - g_n(x)| < \dfrac{\varepsilon}{2(M_f+1)}$.

   Let $N = \max\{N_1, N_f, N_g\}$. Then for all $x \in I$ and all $n > N$

$$\begin{array}{ll}
|(f(x)g(x)) - (f_n(x)g_n(x))| & \leq \\
|f(x)g(x) - f_n(x)g(x)| + |f_n(x)g(x) - f_n(x)g_n(x)| & = \\
|f(x) - f_n(x)||g(x)| + |f_n(x)||g(x) - g_n(x)| & < \\
M_g \dfrac{\varepsilon}{2(M_g+1)} + (1+M_f)\dfrac{\varepsilon}{2(M_f+1)} & = \\
M_g \dfrac{\varepsilon}{2(M_g+1)} + \frac{\varepsilon}{2} & < \\
(M_g+1)\dfrac{\varepsilon}{2(M_g+1)} + \frac{\varepsilon}{2} & = \\
\frac{\varepsilon}{2} + \frac{\varepsilon}{2} & = \\
\varepsilon. &
\end{array}$$

Hence $\{f_n g_n\}$ converges uniformly to fg on I.

4. If for all n, f_n and g_n are bounded on I, then by Exercise 12.15.(1. and 2.) above, there are M_f, M_g such that for each n, for each $x \in I$, $|f_n(x)| \leq M_f$ and $|g_n(x)| \leq M_g$ and f and g are bounded. Then use 3. above to conclude that $\{f_n g_n\}$ converges uniformly to fg on I.

Solution C.195. [Of Exercise 12.17] Define f on $(0,1)$ such that $f(x) = \frac{1}{x}$. Let $x \in (0,1)$. By applying the definition of limit, we can show that $\lim_{n\to\infty} f_n(x) = f(x)$. Hence the sequence $\{f_n\}$ converges pointwise to $f(x)$.
Note that for any n, for any $x \in (0,1)$, $\frac{n}{n+1} < \frac{n}{nx+1} < n$. Hence each f_n is bounded. However, $f(x)$ is not bounded on $(0,1)$ since $\lim_{x\to 0^+} f(x) = \infty$. Hence the sequence $\{f_n\}$ where $f_n(x) = \frac{n}{x^n+1}$ cannot converge uniformly to the pointwise $f(x)$ on $(0,1)$. Otherwise, by Exercise 12.15.2 above, f would be bounded.

Solution C.196. [Of Exercise 12.18] For each of the sequences of functions given below, determine its pointwise limit on $[0,3]$ and give a proof whether convergence to this limit is uniform.

1. Let $x \in [0,3]$. $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{n+1} = 0$.
   Hence $f_n(x) = \frac{x^2}{n+1}$ converges pointwise to $f(x) = 0$ on $[0,3]$.
   Let $\varepsilon > 0$ and take N be a positive integer such that $N > \frac{9-\varepsilon}{\varepsilon}$. Then, $\frac{9}{N+1} < \varepsilon$.
   Now, for all $n > N$, for all $x \in I$, $|f_n(x) - f(x)| = |\frac{x^2}{n+1} - 0| < \frac{9}{N+1} < \varepsilon$. Hence the sequence $\{f_n\}$ where $f_n(x) = \frac{x^2}{n+1}$ converges uniformly to $f(x) = 0$ on $[0,3]$.

2. Let $x \in [0,3]$. $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x-n} = 0$.
   Hence $f_n(x) = \frac{x}{x-n}$ converges pointwise to $f(x) = 0$ on $[0,3]$.
   Let $\varepsilon > 0$ and take N be a positive integer such that $N > \frac{3+3\varepsilon}{\varepsilon}$. Then, $\frac{3}{N-3} < \varepsilon$.
   Now, for all $n > N$, for all $x \in I$, $|f_n(x) - f(x)| = |\frac{x}{x-n} - 0| < \frac{3}{N-3} < \varepsilon$. Hence the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{x-n}$ converges uniformly to $f(x) = 0$ on $[0,3]$.

3. Let $f(x) = 0$ for all $x \in [0,3]$. It is easy to show that for any $x \in [0,3]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$.

Now, for all n, for all $x \in [0,3]$, $f_n(x) = \frac{x}{nx+1} = \frac{1}{n}\frac{nx}{nx+1} = \frac{1}{n}\left(\frac{nx+1}{nx+1} - \frac{1}{nx+1}\right) = \frac{1}{n}\left(1 - \frac{1}{nx+1}\right) < \frac{1}{n}$.
Hence for all n, for all $x \in [0,3]$, $|f_n(x)| = \frac{x}{nx+1} < \frac{1}{n}$. Let $M_n = \sup\{|f_n(x) - f(x)| : x \in [0,3]\}$. Then, for all n, $0 \le M_n \le \frac{1}{n}$. Hence, $\lim_{n\to\infty} M_n = 0$ and by Theorem 12.4.7, the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{nx+1}$ uniformly converges to $f(x)$ on $[0,3]$.

4. Let $f(x) = 0$ for all $x \in [0,3]$. It is easy to show that for any $x \in [0,3]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$.
Since $(\sqrt{n}|x| - 1)^2 \ge 0$, then $nx^2 + 1 \ge 2\sqrt{n}|x|$ and hence $\frac{1}{2\sqrt{n}} \ge \frac{|x|}{nx^2+1}$.

   Hence for all n, $|f_n(x)| = \frac{|x|}{nx^2+1} \le \frac{1}{2\sqrt{n}}$. Let $M_n = \sup\{|f_n(x) - f(x)| : x \in [0,3]\}$. Then, for all n, $0 \le M_n \le \frac{1}{2\sqrt{n}}$. Hence, $\lim_{n\to\infty} M_n = 0$ and by Theorem 12.4.7, the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{nx^2+1}$ uniformly converges to $f(x)$.

5. Let
$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \le 3 \end{cases}$$
It is easy to show that for any $x \in [0,3]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$.
Note that f is not continuous on $[0,3]$ whereas for each n, f_n is continuous on $[0,3]$. Hence by Corollary 12.4.5, the sequence $\{f_n\}$, where $f_n(x) = \frac{nx}{nx+1}$ cannot uniformly converge to $f(x)$.

6. Let
$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 3 \end{cases}$$
It is easy to show that for any $x \in [0,3]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$.
Note that f is not continuous on $[0,3]$ whereas for each n, f_n is continuous on $[0,3]$. Hence by Corollary 12.4.5, the sequence $\{f_n\}$, where $f_n(x) = \frac{x^n}{x^n+1}$ cannot uniformly converge to $f(x)$.

Solution C.197. [Of Exercise 12.19] For each of the sequences given below, determine whether there is a function to which the sequence converges pointwise on $[0,1]$ and if such a function exist:

- Formally show the pointwise convergence.
- Determine whether this convergence is uniform and give a proof for your claim.

1. Define f on $[0,1]$ such that $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$
   If $x = 0$ then $\lim_{n\to\infty} f_n(0) = 0 = f(0)$.
   Let x such that $0 < x \leq 1$. We show $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(1 - x^2)^n = 0$ as follows:
   Let $\varepsilon > 0$. There is $N(\varepsilon, x)$ such that $N(\varepsilon, x) > \log_{1-x^2} \varepsilon$. Note that since $0 \leq 1 - x^2 < 1$ then if $n > m$ then $(1 - x^2)^n < (1 - x^2)^m$.

   Hence there is a positive integer $N(\varepsilon, x)$ such that for all $n > N(\varepsilon, x)$, $|(1 - x^2)^n| < |(1 - x^2)^{N(\varepsilon,x)}| < (1 - x^2)^{\log_{1-x^2} \varepsilon} = \varepsilon$.

   Hence for any $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$.
   Note that f is not continuous on $[0,1]$ whereas for each n, f_n is continuous on $[0,1]$. Hence by Corollary 12.4.5, the sequence $\{f_n\}$, where $f_n(x) = (1 - x^2)^n$ cannot uniformly converge to $f(x)$.

2. Let $f(x) = 0$ for all $x \in [0,1]$. If $x = 0$ then $\lim_{n\to\infty} f_n(0) = 0 = f(0)$. Similarly, if $x = 1$ then $\lim_{n\to\infty} f_n(1) = 0 = f(1)$.
   Let x such that $0 < x < 1$. We show $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x(1 - x)^n = 0$ as follows:
   Let $\varepsilon > 0$. There is $N(\varepsilon, x)$ such that $N(\varepsilon, x) > \log_{1-x} \frac{\varepsilon}{x}$. Note that since $0 < 1 - x < 1$ then if $n > m$ then $(1 - x)^n < (1 - x)^m$.

   Hence there is a positive integer $N(\varepsilon, x)$ such that for all $n > N(\varepsilon, x)$, $|x(1 - x)^n| < |x(1 - x)^{N(\varepsilon,x)}| < x(1 - x)^{\log_{1-x} \frac{\varepsilon}{x}} = x\frac{\varepsilon}{x} = \varepsilon$.

   Hence for any $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$ on $[0,1]$.

   Furthermore, the functions f and f_n for each n are continuous and $[0,1]$ is closed.
   Since $0 \leq 1-x \leq 1$ then for each n, $0 \leq (1-x)^n \leq 1$ and hence for each n, $0 \leq (1-x)^{n+1} \leq (1-x)^n$. Since $0 \leq x$ then $x(1-x)^{n+1} \leq x(1-x)^n$ and for each x, $\{f_n\}$ is decreasing.

Hence, by Dini's theorem 12.4.9, the sequence $\{f_n\}$, where $f_n(x) = x(1-x)^n$ uniformly converges to f.

3. Let $f(x) = 0$ for all $x \in [0,1]$. If $x = 0$ then $\lim_{n\to\infty} f_n(0) = 0 = f(0)$. Similarly, if $x = 1$ then $\lim_{n\to\infty} f_n(1) = 0 = f(1)$.
Let x be such that $0 < x < 1$. We show $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n = 0$ as follows:
First note that for any $a > 0$ and $n \geq 2$ we have $(1+a)^n = 1 + na + \dfrac{n(n-1)a^2}{2} + \cdots > \dfrac{n(n-1)a^2}{2}$ and hence $\dfrac{1}{(1+a)^n} < \dfrac{2}{n(n-1)a^2}$.
So, we look for replacing $(1-x)^n$ by $\dfrac{1}{(1+a)^n}$. But, it is possible to find $a > 0$ such that $1 - x = \dfrac{1}{1+a}$ (take $a = \dfrac{x}{1-x}$). Hence, $nx(1-x)^n = nx\dfrac{1}{(1+a)^n} < nx\dfrac{2}{n(n-1)a^2} = \dfrac{2x}{(n-1)a^2}$.
Let $\varepsilon > 0$. There is $N(\varepsilon, x)$ such that $N(\varepsilon, x) > \dfrac{2x}{a^2\varepsilon} + 1$ and hence $\dfrac{2x}{(N(\varepsilon,x)-1)a^2} < \varepsilon$.
Hence there is a positive integer $N(\varepsilon, x)$ such that for all $n > N(\varepsilon, x)$, $|nx(1-x)^n| < \dfrac{2x}{(n-1)a^2}\dfrac{2x}{(N(\varepsilon,x)-1)a^2} < \varepsilon$.
Hence for any $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $\{f_n\}$ converges pointwise to $f(x)$ on $[0,1]$.

As for uniform convergence, recall Theorem 12.4.7. So, let us look at $M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \sup\{|nx(1-x)^n| : x \in [0,1]\}$. Since $\frac{1}{n} \in [0,1]$, $M_n \geq n\frac{1}{n}(1-\frac{1}{n})^n = (1-\frac{1}{n})^n$. Hence $\lim_{n\to\infty} M_n \geq \lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$ and $\lim_{n\to\infty} M_n > 0$. Hence by Theorem 12.4.7, the sequence $\{f_n\}$, where $f_n(x) = nx(1-x)^n$ does not converge uniformly to $f(x)$.

4. Let $f(x) = 0$ for all $x \in [0,1]$. If $x = 0$ then $\lim_{n\to\infty} f_n(0) = 0 = f(0)$. Similarly, if $x = 1$ then $\lim_{n\to\infty} f_n(1) = 0 = f(1)$.
Let x such that $0 < x < 1$. We show $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n = 0$ as follows:

   – First note that for any $a > 0$ and $n \geq 2$ we have $(1+a)^n = 1+na+\dfrac{n(n-1)a^2}{2}+\cdots > \dfrac{n(n-1)a^2}{2}$ and hence $\dfrac{1}{(1+a)^n} < \dfrac{2}{n(n-1)a^2}$.
   – We will prove that if $0 < a < 1$ then $\lim_{n\to\infty} na^n = 0$.
   Let a such that $0 < a < 1$ and let b such that $a = \dfrac{1}{1+b}$. Then

obviously, $b > 0$ and if $n \geq 2$ then by above, $na^n = \dfrac{1}{(1+b)^n} < \dfrac{2}{n(n-1)b^2} \leq \dfrac{1}{(n-1)b^2}$.
Let $\varepsilon > 0$ and let N be a positive integer such that $N > 1 + \dfrac{1}{\varepsilon b^2}$.
Then, $N \geq 2$ and $\dfrac{1}{(N-1)b^2} < \varepsilon$ and for all $n > N$, $|na^n| \leq \dfrac{1}{(n-1)b^2} < \dfrac{1}{(N-1)b^2} < \varepsilon$. Hence $\lim_{n\to\infty} na^n = 0$.

Now, let $a = 1 - x^2$. Since $0 < x < 1$ then $0 < a < 1$ and $0 < nx(1-x^2)^n < n(1-x^2)^n = na^n$. Hence $0 \leq \lim_{n\to\infty} nx(1-x^2)^n \leq \lim_{n\to\infty} na^n = 0$. So, for any $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$ and $\{f_n\}$ converges pointwise to $f(x)$ on $[0,1]$.

As for uniform convergence, recall Theorem 12.4.7. So, let us look at $M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \sup\{|nx(1-x^2)^n| : x \in [0,1]\}$.
Since $\dfrac{1}{\sqrt{n}} \in [0,1]$, $M_n \geq n\dfrac{1}{\sqrt{n}}(1-\frac{1}{n})^n = \sqrt{n}(1-\frac{1}{n})^n$.
Now, since $\lim_{n\to\infty} \sqrt{n} = \infty$ and $\lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e} > 0$ then by LS13, $\lim_{n\to\infty} \sqrt{n}(1-\frac{1}{n})^n = \infty$. Hence $\lim_{n\to\infty} M_n = \infty$. By Theorem 12.4.7, the sequence $\{f_n\}$, where $f_n(x) = nx(1-x^2)^n$ does not converge uniformly to $f(x)$.

C.13 Solutions for Chapter 13

Solution C.198. [Of Exercise 13.1] Assume $P_1 = \{x_i : 0 \leq i \leq n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $P_2 = \{y_j : 0 \leq j \leq m\}$ and $a = y_0 < y_1 < y_2 < \cdots < y_m = b$ where $P_2 \subseteq P_1$. Clearly, for every $1 \leq j \leq m$, there is $1 \leq i \leq n$ such that $[y_{j-1}, y_j] \subseteq [x_{i-1}, x_i]$. Hence, for every $1 \leq j \leq m$, there is $1 \leq i \leq n$ such that $y_j - y_{j-1}] \leq x_i - x_{i-1}]$. Hence, $\|P_1\| = max\{x_i - x_{i-1} : 1 \leq i \leq n\} \leq max\{y_j - y_{j-1} : 1 \leq j \leq m\} = \|P_2\|$.

Solution C.199. [Of Exercise 13.2]

1. This statement says that the area between a and b under the curve of f is unique.
   Assume there are two distinct values L and L' which are the Riemann integral of f on $[a, b]$. Then, let $\varepsilon = |L - L'| > 0$. By definition, there are δ and δ' such that for any tagged partition ${}^tP$ of $[a, b]$,

   - $\|{}^tP\| < \delta$, we get $|S(f,{}^t P) - L| < \frac{\varepsilon}{2}$, and
   - $\|{}^tP\| < \delta'$, we get $|S(f,{}^t P) - L'| < \frac{\varepsilon}{2}$.

   Let $\delta'' = \min\{\delta, \delta'\}$ and a tagged partition ${}^tP$ of $[a, b]$ such that $\|{}^tP\| < \delta''$. Then, $|S(f,{}^t P) - L| < \frac{\varepsilon}{2}$ and $|S(f,{}^t P) - L'| < \frac{\varepsilon}{2}$. Now, $|L - L'| = |L - S(f,{}^t P) + S(f,{}^t P) - L'| \leq |L - S(f,{}^t P)| + |S(f,{}^t P) - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = |L - L'|$. Absurd.

2. Obviously, since the function h is constant, the area between a and b under the graph of h is the area of the rectangle whose sides are k and $(b - a)$. Hence, the area is $k(b - a)$. The proof is as follows:
   For any tagged partition ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ of $[a, b]$ we have: $S(h,{}^t P) = \Sigma_{i=1}^n h(t_i)(x_i - x_{i-1}) = \Sigma_{i=1}^n k(x_i - x_{i-1}) = k(b - a)$. Let $\varepsilon > 0$ and let δ be any positive number. Then, for any tagged partition ${}^tP$ of $[a, b]$ such that $\|{}^tP\| < \delta$ we have $|S(h,{}^t P) - k(b-a)| = 0 < \varepsilon$. Hence, by definition, h is Riemann integrable on $[a, b]$ and has $k(b - a)$ as its Riemann integral.

3. Obviously, since the function kf always multiplies the value of f by k, the area between a and b under the graph of kf is k-times the area between a and b under the graph of f. The proof is as follows:
   Let $\varepsilon > 0$. By definition, there is a $\delta > 0$ such that for all tagged partitions ${}^tP$ of $[a, b]$ where $\|{}^tP\| < \delta$ we have $|S(f,{}^t P) - \int_a^b f| < \frac{\varepsilon}{|k|}$.
   Note that if ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ then $S(kf,{}^t P) = \Sigma_{i=1}^n (kf)(t_i)(x_i - x_{i-1}) = k\Sigma_{i=1}^n f(t_i)(x_i - x_{i-1})$. Hence, for all

tagged partitions ${}^tP$ where $\|{}^tP\| < \delta$ we have $|S(kf, {}^tP) - k\int_a^b f| = |k||S(f, {}^tP) - \int_a^b f| < |k|\frac{\varepsilon}{|k|} = \varepsilon$. Hence, kf is Riemann integrable and $\int_a^b kf = k\int_a^b f$.

4. Obviously, the area between a and b under the graph of $f + g$ is the sum of the area between a and b under the graph of f and the area between a and b under the graph of g. The proof is as follows:
   Let $\varepsilon > 0$. By definition, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for all tagged partitions ${}^tP$ where $\|{}^tP\| < \delta$ we have $|S(f, {}^tP) - \int_a^b f| < \frac{\varepsilon}{2}$ and $|S(g, {}^tP) - \int_a^b g| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Note that if ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ then $S(f+g, {}^tP) = \Sigma_{i=1}^n (f+g)(t_i)(x_i - x_{i-1}) = \Sigma_{i=1}^n f(t_i)(x_i - x_{i-1}) + \Sigma_{i=1}^n g(t_i)(x_i - x_{i-1}) = S(f, {}^tP) + S(g, {}^tP)$. Hence, for all tagged partitions ${}^tP$ where $\|{}^tP\| < \delta$ we have $|S(f, {}^tP) - \int_a^b f| < \frac{\varepsilon}{2}$ and $|S(g, {}^tP) - \int_a^b g| < \frac{\varepsilon}{2}$. Hence $|S(f+g, {}^tP) - (\int_a^b f + \int_a^b g)| \leq |S(f, {}^tP) - \int_a^b f| + |S(g, {}^tP) - \int_a^b g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$. Hence $f + g$ is Riemann integrable and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

Solution C.200. [Of Exercise 13.3]

1. Recall that $\omega(f, I) = \sup\{f(z) : z \in I\} - \inf\{f(z) : z \in I\} \geq 0$. Since for any $x, y \in I$, $\inf\{f(z) : z \in I\} \leq f(x) \leq \sup\{f(z) : z \in I\}$ and $-\sup\{f(z) : z \in I\} \leq -f(y) \leq -\inf\{f(z) : z \in I\}$ then $-(\sup\{f(z) : z \in I\} - \inf\{f(z) : z \in I\}) \leq f(x) - f(y) \leq \sup\{f(z) : z \in I\} - \inf\{f(z) : z \in I\}$ and so, $0 \leq |f(x) - f(y)| \leq \sup\{f(z) : z \in I\} - \inf\{f(z) : z \in I\}$.

2. Since by 1, $\omega(f, I) \geq |f(x) - f(y)|$ then $\omega(f, I) \geq \sup\{|f(x) - f(y)| : x, y \in I\}$.
   Furthermore, since for all $x, y \in I$, $f(x) - f(y) \leq |f(x) - f(y)| \leq \sup\{|f(x) - f(y)| : x, y \in I\}$ then for all $x, y \in I$, $f(x) \leq \sup\{|f(x) - f(y)| : x, y \in I\} + f(y)$ and hence, $\sup\{f(x) : x \in I\} \leq \sup\{|f(x) - f(y)| : x, y \in I\} + f(y)$ for all $y \in I$. Therefore, $\sup\{f(x) : x \in I\} - \sup\{|f(x) - f(y)| : x, y \in I\} \leq f(y)$ for all $y \in I$, and finally $\sup\{f(x) : x \in I\} - \sup\{|f(x) - f(y)| : x, y \in I\} \leq \inf\{f(x) : x \in I\}$. Hence, $\omega(f, I) = \sup\{|f(x) - f(y)| : x, y \in I\}$.

   Now, since for any $x, y \in I$, $f(x) - f(y) \leq |f(x) - f(y)|$, then $\sup\{f(x) - f(y) : x, y \in I\} \leq \sup\{|f(x) - f(y)| : x, y \in I\}$. Moreover, since for any $x, y \in I$, $f(x) - f(y) \leq \sup\{f(x) - f(y) : x, y \in I\}$ then for any $x, y \in I$, $|f(x) - f(y)| \leq \sup\{f(x) - f(y) : x, y \in I\}$ and hence, $\sup\{|f(x) - f(y)| : x, y \in I\} \leq \sup\{f(x) - f(y) : x, y \in I\}$.
   Therefore, $\sup\{|f(x) - f(y)| : x, y \in I\} = \sup\{f(x) - f(y) : x, y \in I\}$.

3. Since $\sup\{f(x) : x \in [c, d]\} \leq \sup\{f(x) : x \in [a, b]\}$ and $-\inf\{f(x) : x \in [c, d]\} \leq -\inf\{f(x) : x \in [a, b]\}$ then $\omega(f, [c, d]) \leq \omega(f, [a, b])$.

4. By the above case, $\omega(f, [c, d]) \leq \omega(f, [c, e])$ and $\omega(f, [d, e]) \leq \omega(f, [c, e])$. Hence $\max\{\omega(f, [c, d]), \omega(f, [d, e])\} \leq \omega(f, [c, e])$.
   As for the second inequality, let $m = \inf\{f(x) : x \in [c, e]\}$, $M = \sup\{f(x) : x \in [c, e]\}$, $m_1 = \inf\{f(x) : x \in [c, d]\}$, $M_1 = \sup\{f(x) : x \in [c, d]\}$, $m_2 = \inf\{f(x) : x \in [d, e]\}$ and $M_2 = \sup\{f(x) : x \in [d, e]\}$. Note that $m = min\{m_1, m_2\} \leq= max\{M_1, M_2\} = M$. Now do the proof by the cases on the order between the m_i and M_i. For example, if $m = m_1 \leq m_2 \leq M_1 \leq M_2 = M$ then $M - m = M_2 - m_1 \leq M_2 + M_1 - m_2 - m_1$.

5. Since P_2 is a refinement of P_1, the set $P_2 \cap [z_{i-1}, z_i]$ is a partition of $[z_{i-1}, z_i]$. Consequently, since we can work with each interval $[z_{i-1}, z_i]$ separately, it is sufficient to consider the case in which $P_1 = \{a, b\}$; this simplifies the notation in the proof considerably. Let ${}^tP_1 = \{(v, [a, b])\}$, let $P_2 = \{x_i : 0 \leq i \leq p\}$, and let

$$^tP_2 = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq p\}.$$

   It follows that

$$\begin{array}{ll}
|S(f,^t P_2) - S(f,^t P_1)| & = \\
|\Sigma_{i=1}^p f(t_i)(x_i - x_{i-1}) - f(v)(b - a)| & = \\
|\Sigma_{i=1}^p f(t_i)(x_i - x_{i-1}) - f(v)\Sigma_{i=1}^p (x_i - x_{i-1})| & = \\
|\Sigma_{i=1}^p f(t_i)(x_i - x_{i-1}) - \Sigma_{i=1}^p f(v)(x_i - x_{i-1})| & = \\
|\Sigma_{i=1}^p (f(t_i) - f(v))(x_i - x_{i-1})| & \leq \\
\Sigma_{i=1}^p |f(t_i) - f(v)|(x_i - x_{i-1}) & \leq \\
\Sigma_{i=1}^p \omega(f, [a, b])(x_i - x_{i-1}) & = \\
\omega(f, [a, b])(b - a). &
\end{array}$$

   Now we prove the general result. Assume $P_1 = \{z_0, z_1, \cdots, z_n\}$ and that for all $1 \leq i \leq n$, $P_2 \cap [z_{i-1}, z_i] = \{x_{0(i-1)}, x_{1(i-1)}, \cdots x_{p_{(i-1)}(i-1)}\}$ where $z_{i-1} = x_{0(i-1)} < x_{1(i-1)} < \cdots < x_{p_{(i-1)}(i-1)} = z_i$.
   Assume also that ${}^tP_1 = \{(t_i, [z_{i-1}, z_i]) : 1 \leq i \leq n\}$ and for all $1 \leq i \leq n$, ${}^tP_2|[z_{i-1}, z_i] = \{(t_{j(i-1)}, [x_{(j-1)(i-1)}, x_{j(i-1)}]) : 1 \leq j \leq p_{i-1}\}$.
   By above, for all $1 \leq i \leq n$, $|S(f,^t P_2|[z_{i-1}, z_i]) -$

$S(f, \{(t_i, [z_i, z_{i-1}])\})| \leq \omega(f, [z_{i-1}, z_i])(z_i - z_{i-1})$. Hence

$$\begin{array}{ll} |S(f,^t P_2) - S(f,^t P_1)| & = \\ |\Sigma_{i=1}^n S(f,^t P_2|[z_{i-1}, z_i]) - \Sigma_{i=1}^n S(f, \{(t_i, [z_{i-1}, z_i])\})| & = \\ |\Sigma_{i=1}^n (S(f,^t P_2|[z_{i-1}, z_i]) - S(f, \{(t_i, [z_{i-1}, z_i])\}))| & \leq \\ \Sigma_{i=1}^n |S(f,^t P_2|[z_{i-1}, z_i]) - S(f, \{(t_i, [z_{i-1}, z_i])\})| & \leq \\ \Sigma_{i=1}^n \omega(f, [z_{i-1}, z_i])(z_i - z_{i-1}) & = \\ O(f, P_1) & \end{array}$$

6. Assume $P_2 = P_1 \cup \{y\}$ where for some $1 \leq j \leq n$, $z_{j-1} < y < z_j$. Since $\omega(f, [z_{j-1}, y]) \leq \omega(f, [z_{j-1}, z_j])$ and $\omega(f, [y, z_j]) \leq \omega(f, [z_{j-1}, z_j])$, we have

$$\begin{array}{rcl} O(f, P_2) & = & \Sigma_{i=1}^{j-1} \omega(f, [z_{i-1}, z_i])(z_i - z_{i-1}) + \\ & & \omega(f, [z_{j-1}, y])(y - z_{j-1}) + \omega(f, [y, z_j])(z_j - y) + \\ & & \Sigma_{i=j+1}^n \omega(f, [z_{i-1}, z_i])(z_i - z_{i-1}) \\ & \leq & \Sigma_{i=1}^{j-1} \omega(f, [z_{i-1}, z_i])(z_i - z_{i-1}) + \\ & & \omega(f, [z_{j-1}, z_j])(y - z_{j-1}) + \omega(f, [z_{j-1}, z_j])(z_j - z_{j-1}) + \\ & & \Sigma_{i=j+1}^n \omega(f, [z_{i-1}, z_i])(z_i - z_{i-1}) \\ & = & O(f, P_1) \end{array}$$

If $P_2 = P_1 \cup \{y_1, \ldots, y_m\}$, use induction on m.

7. $O(f, P_1) = \Sigma_{i=1}^n \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) = \Sigma_{i=1}^n (\sup\{f(x) : x \in [x_{i-1}, x_i]\} - \inf\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1}) = \Sigma_{i=1}^n \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) - \Sigma_{i=1}^n \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) = S^+(f, P_1) - S^-(f, P_1)$.

8. Assume ${}^t P_1 = \{(t_i, [z_{i-1}, z_i]) : 1 \leq i \leq n\}$. Since for all $1 \leq i \leq n$, $\inf\{f(x) : x \in [x_{i-1}, x_i]\} \leq f(t_i) \leq \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ then $\Sigma_{i=1}^n \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) \leq \Sigma_{i=1}^n f(t_i)(x_i - x_{i-1}) \leq \Sigma_{i=1}^n \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1})$.
Hence $S^-(f, P_1) \leq S(f,^t P_1) \leq S^+(f, P_1)$.

9. Obviously $S^-(f, P_1) \leq \sup\{S^-(f, P) : P \text{ is a partition of } [a, b]\}$ and $\inf\{S^+(f, P) : P \text{ is a partition of } [a, b]\} \leq S^+(f, P_1)$. Also, since by 8. above, $S^-(f, P_1) \leq S^+(f, P_1)$ for any P_1, we have
$\sup\{S^-(f, P) : P \text{ is a partition of } [a, b]\} \leq$
$\inf\{S^+(f, P) : P \text{ is a partition of } [a, b]\}$.

   Hence $S^-(f, P_1) \leq S^-(f) \leq S^+(f) \leq S^+(f, P_1)$.

10. By 9. and 7. above, $S^+(f) - S^-(f) \leq S^+(f, P_1) - S^-(f, P_1) = O(f, P_1)$.

Solution C.201. [Of Exercise 13.4] **Proof by Contradiction:** Since f is Riemann integrable on $[a,b]$, there exists a positive number δ such that $|S(f,{}^tP) - \int_a^b f| < 0.5$ (and hence $|S(f,{}^tP)| < |\int_a^b f| + 0.5$) for all tagged partitions ${}^tP$ on $[a,b]$ that satisfy $\|{}^tP\| < \delta$. Assume f is unbounded on $[a,b]$. Let $Q = \{x_i : 0 \leq i \leq n\}$ be a partition of $[a,b]$ such that $\|Q\| < \delta$. Since f is unbounded on $[a,b]$, there is a $1 \leq j \leq n$ such that f is unbounded on $[x_{j-1}, x_j]$. Let $M = \frac{1}{x_j - x_{j-1}}(|\int_a^b f| + 0.5 + |\Sigma_{i=1}^n f(x_i)(x_i - x_{i-1}) - f(x_j)(x_j - x_{j-1})|)$. Since f is unbounded on $[x_{j-1}, x_j]$, let $v \in [x_{j-1}, x_j]$ such that $|f(v)| > M$ and hence $|f(v)(x_j - x_{j-1})| - |\Sigma_{i=1}^n f(x_i)(x_i - x_{i-1}) - f(x_j)(x_j - x_{j-1})| > |\int_a^b f| + 0.5$.

Let ${}^tQ = (\{(x_i, [x_{i-1}, x_i]); 1 \leq i \leq n\} \cup \{(v, [x_{j-1}, x_j])\}) \setminus \{(x_j, [x_{j-1}, x_j])\}$. Obviously, $\|{}^tQ\| < \delta$, but $|S(f,{}^tQ)| = |\Sigma_{i=1}^n f(x_i)(x_i - x_{i-1})) - f(x_j)(x_j - x_{j-1}) + f(v)(x_j - x_{j-1})| \geq |f(v)(x_j - x_{j-1})| - |\Sigma_{i=1}^n f(x_i)(x_i - x_{i-1})) - f(x_j)(x_j - x_{j-1})| > |\int_a^b f| + 0.5$ contradiction.

Solution C.202. [Of Exercise 13.5] Let f be Riemann integrable on $[a,b]$. We first show that f is Riemann integrable on $[a,c]$ and on $[c,b]$. Note that by Theorem 13.2.1, f is bounded on $[a,b]$. Let $\varepsilon > 0$. By Theorem 13.2.3, there exists a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a,b]$ such that

$$O(f,P) = \Sigma_{i=1}^n \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \varepsilon.$$

- If for some $0 \leq j \leq n$, $c = x_j$ then $P_1 = \{x_i : 0 \leq i \leq j\}$ is a partition of of $[a,c]$ and $P_2 = \{x_i : j \leq i \leq n\}$ is a partition of of $[c,b]$ such that $O(f,P) = O(f,P_1) + O(f,P_2) < \varepsilon$. Since each of $O(f,P_1)$ and $O(f,P_2)$ are positive, then $O(f,P_1) < \varepsilon$ and $O(f,P_2) < \varepsilon$. Since f is bounded on each of $[a,c]$ and $[c,b]$, by Theorem 13.2.3, f is Riemann integrable on each of $[a,c]$ and $[c,b]$.

- If for some $0 \leq j \leq n$, $x_{j-1} < c < x_j$ then $P' = P \cup \{c\}$ is a refinement of P and $O(f,P') \leq O(f,P)$ can be seen as follows:

$$\begin{aligned}
O(f,P') = \;& \Sigma_{i=1}^{j-1} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) + \\
& \omega(f, [x_{j-1}, c])(c - x_{j-1}) + \omega(f, [c, x_j])(x_j - c) + \\
& \Sigma_{i=j+1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) \\
\leq \;& \Sigma_{i=1}^{j-1} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) + \\
& \omega(f, [x_{j-1}, x_j])(c - x_{j-1}) + \omega(f, [x_{j-1}, x_j])(x_j - c) + \\
& \Sigma_{i=j+1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) \\
= \;& \Sigma_{i=1}^{j-1} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) + \\
& \omega(f, [x_{j-1}, x_j])(x_j - x_{j-1}) + \\
& \Sigma_{i=j+1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) \\
= \;& O(f,P).
\end{aligned}$$

But, $P_1 = \{x_i : 0 \leq i \leq j\} \cup \{c\}$ is a partition of of $[a, c]$ and $P_2 = \{x_i : j \leq i \leq n\} \cup \{c\}$ is a partition of of $[c, b]$ such that $O(f, P') = O(f, P_1) + O(f, P_2) \leq O(f, P) < \varepsilon$. Since each of $O(f, P_1)$ and $O(f, P_2)$ are positive, then $O(f, P_1) < \varepsilon$ and $O(f, P_2) < \varepsilon$. Since f is bounded on each of $[a, c]$ and $[c, b]$, by Theorem 13.2.3, f is Riemann integrable on each of $[a, c]$ and $[c, b]$.

Now, since f is Riemann integrable of each of $[a, c]$ and $[c, b]$, by what we just proved above, f is also integrable on each of $[c, d]$ and $[d, b]$. Hence, f is Riemann integrable on $[c, d]$.

Solution C.203. [Of Exercise 13.6] By Exercise 13.5, if f is Riemann integrable on $[a, b]$ then f is Riemann integrable on $[a, c]$ and on $[c, b]$. Conversely, assume f is Riemann integrable on $[a, c]$ and on $[c, b]$. By Theorem 13.2.1, f is bounded on $[a, c]$ and on $[c, b]$. Hence f is bounded on $[a, b]$. Let $\varepsilon > 0$. By Theorem 13.2.3, there are P_1 partition of $[a, c]$ and P_2 partition of $[c, b]$ such that $O(f, P_1) < \frac{\varepsilon}{2}$ and $O(f, P_2) < \frac{\varepsilon}{2}$. Let $P = P_1 \cup P_2$. P is a partition of $[a, b]$ and $O(f, P) = O(f, P_1) + O(f, P_2) < \varepsilon$. By Theorem 13.2.3, f is bounded on $[a, b]$.

Solution C.204. [Of Exercise 13.7] We first show that f^2 is Riemann integrable on $[a, b]$.

By Theorem 13.2.1, f is bounded on $[a, b]$. Let M be a bound for f on $[a, b]$. Hence, f^2 is also bounded (by M^2) on $[a, b]$. Let $\varepsilon > 0$. By Theorem 13.2.3, there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that $O(f, P) = \Sigma_{i=1}^n \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \frac{\varepsilon}{2M}$.

Now, let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Then $\omega(f^2, [x_{i-1}, x_i]) = M_i^2 - m_i^2 = (M_i + m_i)(M_i - m_i) < 2M(M_i - m_i)$. Hence, $O(f^2, P) = \Sigma_{i=1}^n \omega(f^2, [x_{i-1}, x_i])(x_i - x_{i-1}) = \Sigma_{i=1}^n (M_i + m_i)(M_i - m_i)(x_i - x_{i-1}) < \Sigma_{i=1}^n 2M(M_i - m_i)(x_i - x_{i-1}) = 2M\Sigma_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) = 2MO(f, P) < 2M\frac{\varepsilon}{2M} = \varepsilon$. Hence by Theorem 13.2.3, f^2 is Riemann integrable on $[a, b]$.

Since $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$, we use Theorem 13.1.8 and what we just proved above to deduce that fg is Riemann integrable on $[a, b]$.

Solution C.205. [Of Exercise 13.8]

1. Note that for any partition P of $[a, b]$, $O(f_+, P) \leq O(f, P)$ and $O(f_-, P) \leq O(f, P)$. Since f is Riemann, by Theorem 13.2.1, f is bounded on $[a, b]$ and hence f_+ and f_- are also bounded on $[a, b]$. By Theorem 13.2.3 for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $O(f, P) < \varepsilon$. Hence for each $\varepsilon > 0$, there exists a partition P of

$[a,b]$ such that $O(f_+,P) \leq O(f,P) < \varepsilon$ and $O(f_-,P) \leq O(f,P) < \varepsilon$. Hence, by Theorem 13.2.3, f_+ and f_- are Riemann integrable on $[a,b]$.

2. Note that $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Since by 1. above f_+ and f_- are Riemann integrable on $[a,b]$ then by Theorem 13.1.8.1, $|f|$ is Riemann integrable on $[a,b]$. Furthermore, we have $-|f| \leq f \leq |f|$ and by Theorem 13.1.8.(2+3), $-\int_a^b |f| = \int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|$. Hence $|\int_a^b f| \leq \int_a^b |f|$.

Solution C.206. [Of Exercise 13.9] f is not necessarily Riemann integrable on $[a,b]$. Take for example the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even;} \\ -1 & \text{otherwise} \end{cases}$$

Since $|f|$ is the constant function, it is Riemann integrable on $[a,b]$ by Theorem 13.1.8.2. However, f is not Riemann integrable on $[a,b]$. To see this, let $0 < \varepsilon < b-a$ and let $P = \{x_i : 0 \leq i \leq n\}$ be any partition of $[a,b]$. Let ${}^tP_1 = \{(t_i,[x_{i-1},x_i]) : 1 \leq i \leq p\}$ and ${}^tP_2 = \{(t_i',[x_{i-1},x_i]) : 1 \leq i \leq p\}$ such that all t_is are even and all t_i's are odd. Then, $|S(f,{}^tP_2) - S(f,{}^tP_1)| = |\Sigma_{i=1}^n (f(t_i) - f(t_i'))(x_i - x_{i-1})| = 2|\Sigma_{i=1}^n (x_i - x_{i-1})| = 2(b-a) > \varepsilon$. Since f is bounded, by Theorem 13.2.2, f is not Riemann integrable on $[a,b]$.

Solution C.207. [Of Exercise 13.10] First note that f is bounded on $[a+c,b+c]$ iff g is bounded on $[a,b]$.
Assume f is Riemann integrable on $[a+c,b+c]$. Let $\varepsilon > 0$. By Theorem 13.2.3 there exists a partition $P = \{x_i + c : 0 \leq i \leq n\}$ of $[a+c,b+c]$ such that $O(f,P) < \varepsilon$. Now, $P' = \{x_i : 0 \leq i \leq n\}$ is a partition of $[a,b]$. Note that $\omega(f,[x_{i-1}+c,x_i+c]) = \omega(g,[x_{i-1},x_i])$. Hence $\Sigma_{i=1}^n \omega(f,[x_{i-1}+c,x_i+c])(x_i+c-x_{i-1}-c) = \Sigma_{i=1}^n \omega(f,[x_{i-1},x_i])(x_i - x_{i-1})$. I.e., $O(f,P) = O(g,P')$. Hence g is Riemann integrable on $[a,b]$.
Conversely, assume g is Riemann integrable on $[a,b]$. Let $a = a' - c$ and $b = b' - c$ and $g'(x) = g(x-c) = f(x)$ (i.e., $g' = f$). Then, since g is Riemann integrable on $[a'-c,b'-c]$, by the previous case, f is Riemann integrable on $[a',b']$. I.e., f is Riemann integrable on $[a+c,b+c]$.

We will show that if f is Riemann integrable on $[a+c,b+c]$ then for all $\varepsilon > 0$, $|\int_a^b g - \int_{a+c}^{b+c} f| < \varepsilon$. Let $\varepsilon > 0$. Since f is Riemann integrable on $[a+c,b+c]$ then by above, g is Riemann integrable on $[a,b]$. By definition, there are δ_1 and δ_2 such that for any ${}^tP_1$ and ${}^tP_2$ partitions of $[a+c,b+c]$ resp. $[a,b]$ we have:

$$\text{if } \|{}^tP_1\| < \delta_1 \text{ then } |S(f,{}^tP_1) - \int_{a+c}^{b+c} f| < \frac{\varepsilon}{2}.$$

$$\text{if } \|^tP_2\| < \delta_2 \text{ then } |S(g,^t P_2) - \int_a^b g| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. And let $^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$ such that $\|^tP\| < \delta$. Then $^tP' = \{(t_i + c, [x_{i-1} + c, x_i + c]) : 1 \leq i \leq n\}$ is tagged a partition of $[a + c, b + c]$ such that $\|^tP'\| < \delta$ and $S(g,^t P) = S(f,^t P')$. Hence:
$|\int_a^b g - \int_{a+c}^{b+c} f| \leq |\int_a^b g - S(g,^t P)| + |S(f,^t P') - \int_{a+c}^{b+c} f| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} =< \varepsilon.$

Solution C.208. [Of Exercise 13.11]

1. Note that f is bounded by 1. We will use two methods to prove that this function is not Riemann integrable.

   - **Cauchy Criterion for Riemann Inegrability.** By Theorem 13.2.2, f is Riemann integrable on $[a, b]$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S(f,^t P_1) - S(f,^t P_2)| < \varepsilon$ for all tagged partitions $^tP_1$ and $^tP_2$ of $[a, b]$ with norms less than δ. Let $\varepsilon < 1$. Let $^tP_1 = \{(t_i, [x_{i-1}, x_i] : 1 \leq i \leq n\}$ be an arbitrary tagged partition of $[a, b]$ whose tags t_i are all rational. Let $^tP_2 = \{(t'_i, [x'_{i-1}, x'_i] : 1 \leq i \leq m\}$ be an arbitrary tagged partition of $[a, b]$ whose tags t'_i are all irrational. Then, $S(f,^t P_1) = \Sigma_{i=1}^n (x_i - x_{i-1}) = b - a$ and $S(f,^t P_2) = \Sigma_{i=1}^n (x_i - x_{i-1})0 = 0$. Hence, $|S(f,^t P_1) - S(f,^t P_2)| = 1 > \varepsilon$. Hence, by Theorem 13.2.2, f is not Riemann integrable on $[a, b]$.
   - **Partition with small oscillation.** By Theorem 13.2.3, f is Riemann integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exists a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that $\Sigma_{i=1}^n \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \varepsilon$. By the density of the rationals Theorem 10.4.5, for any $[a, b]$ and any partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$, $\omega(f, [x_{i-1}, x_i]) = 1$ and $\Sigma_{i=1}^n \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) = \Sigma_{i=1}^n (x_i - x_{i-1}) = b - a$. Hence for any $\varepsilon < b - a$, the property fails and by Theorem 13.2.3, f is not Riemann integrable on any interval $[a, b]$.

2. We will show that $\int_0^1 g = 0$. Let $\varepsilon > 0$. There is $n > 0$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Hence, $A_n = \{x \in [0, 1] : g(x) \geq \frac{\varepsilon}{2}\}$ is finite because if $x \in A_n$ then $x = \frac{k}{l}$ where $k, l < n$. Let $|A_n|$ be the number of elements of A_n and let $\delta = \begin{cases} 1 & \text{if } |A_n| = 0 \\ \frac{\varepsilon}{4|A_n|} & \text{otherwise.} \end{cases}$

Let ${}^tP$ be a tagged partition of $[0,1]$ such that $\|P\| < \delta$. Let ${}^tP_1$ be the subset of ${}^tP$ where all the tags belong to A_n and let ${}^tP_2$ be the subset of ${}^tP$ where all the tags do not belong to A_n. Obviously, $S(g,{}^tP) = S(g,{}^tP_1) + S(g,{}^tP_2)$. Since $g(x) \geq 0$ for all $x \in [0,1]$ then $S(g,{}^tQ) \geq 0$ for any tagged partition of $[0,1]$.

Now, if $P = \{x_i : 0 \leq i \leq l\}$ and $A_n = \{a_1, \cdots, a_{|A_n|}\}$ then ${}^tP_1$ can contain at most $2|A_n|$ (where for example, $\{(x_i,[x_{i-1},x_i]),(x_i,[x_i,x_{i+1}])\} \subseteq {}^tP_1$). Hence, $S(g,{}^tP_1) = \Sigma_{t_i \in A_n} g(t_i)(x_i - x_{i-1}) \leq \Sigma_{t_i \in A_n}(x_i - x_{i-1}) \leq 2|A_n|\delta \leq \frac{\varepsilon}{2}$.

Furthermore, $S(g,{}^tP_2) = \Sigma_{t_i \notin A_n} g(t_i)(x_i - x_{i-1}) \leq \Sigma_{t_i \notin A_n} \frac{\varepsilon}{2}(x_i - x_{i-1}) \leq \frac{\varepsilon}{2}\Sigma_{t_i \notin A_n}(x_i - x_{i-1}) \leq \frac{\varepsilon}{2}$.

Hence $S(g,{}^tP) = S(g,{}^tP_1) + S(g,{}^tP_2) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

So we have shown that for every $\varepsilon > 0$, there is $\delta > 0$ such that for all tagged partition P of $[0.1]$, $S(g,{}^tP) \leq \varepsilon$. Hence by definition, $\int_0^1 g = 0$.

Note that this function is discontinuous at every rational and a function need not be continuous to be Riemann integrable.

3. Assume that $\int_0^2 h$ exists. Then by definition, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_0^2 h - \varepsilon < S(h,{}^tP) < \int_0^2 h + \varepsilon$ for all tagged partitions ${}^tP$ of $[0,2]$ that satisfy $\|{}^tP\| < \delta$.

   Since $S(h,{}^tP) \geq 0$, for any ${}^tP$, we have $\int_0^2 h \geq 0$. There are two possibilities:

   - If $\int_0^2 h = 0$ then let $\varepsilon = 2$. For this ε, there is a $\delta > 0$ such that $S(h,{}^tP) < 2$ for all tagged partitions ${}^tP$ of $[0,2]$ that satisfy $\|{}^tP\| < \delta$.
     Let ${}^tP = \{(t_i,[x_{i-1},x_i]) : 1 \leq i \leq n\}$ be a tagged partition whose tags are all irrationals such that $\|{}^tP\| < \delta$. For this ${}^tP$, we have $2 > S(h,{}^tP) = \Sigma_{i=1}^n h(t_i)(x_i - x_{i-1}) > \Sigma_{i=1}^n (x_i - x_{i-1}) = 2$, absurd.
   - If $\int_0^2 h > 0$ then let $\varepsilon = \int_0^2 h > 0$. For this ε, there is a $\delta > 0$ such that $0 = \int_0^2 h - \varepsilon < S(h,{}^tP)$ for all tagged partitions ${}^tP$ of $[0,2]$ that satisfy $\|{}^tP\| < \delta$. Let ${}^tP$ be a tagged partition whose tags are all rationals such that $\|{}^tP\| < \delta$. For this ${}^tP$, we have $0 < S(h,{}^tP) = 0$ absurd.

   Note here that we could have given an easier proof as follows: $\lim_{x\to 0} h(x) = \infty$, $h(x)$ is not bounded on $[0,2]$ and hence by Theorem 13.2.1, h is not Riemann integrable.

4. We show that r is Riemann integrable on $[a, b]$ by applying the definition of Riemann integrability. Let $\varepsilon > 0$ and let $\delta > 0$ such that $\delta < \frac{\varepsilon}{2}$ and $c + \delta < d - \delta$. Let ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$ such that $\|{}^tP\| < \delta$. Since $[c, c + \delta]$ is of length δ and $\|{}^tP\| < \delta$, there is an $1 \leq l \leq n$ such that $c < t_l < c+\delta$. Similarly, there is an $1 \leq k \leq n$ such that $d-\delta < t_k < d$. Hence $[c + \delta, d - \delta] \subseteq \bigcup_{t_i \in [c,d]}[x_{i-1}, x_i] \subseteq [c - \delta, d + \delta]$. Hence $d - c - 2\delta \leq \Sigma_{t_i \in [c,d]}(x_i - x_{i-1}) \leq d - c + 2\delta$.
But $S(r,{}^t P) = \Sigma_{i=1}^n r(t_i)(x_i - x_{i-1}) = \Sigma_{t_i \in [c,d]}(x_i - x_{i-1})$. That is, $d - c - 2\delta \leq S(r,{}^t P) \leq d - c + 2\delta$ and $|S(r,{}^t P) - (d - c)| \leq 2\delta < \varepsilon$. So by definition r is Riemann integrable on $[a, b]$ and $\int_a^b r = d - c$.

Solution C.209. [Of Exercise 13.12]

- If f is Riemann integrable on $[a, b]$ and $\varepsilon > 0$, let $g_\varepsilon = f = h_\varepsilon$. Then, obviously g_ε and h_ε are Riemann integrable on $[a, b]$ and $g_\varepsilon(x) \leq f(x) \leq h_\varepsilon(x)$ for all $x \in [a, b]$. By Theorem 13.1.8, $\int_a^b h_\varepsilon - g_\varepsilon = \int_a^b h_\varepsilon - \int_a^b g_\varepsilon = \int_a^b f - \int_a^b f = 0 < \varepsilon$.
- Assume that for every $\varepsilon > 0$, there are two Riemann integrable functions g_ε and h_ε on $[a, b]$ such that $g_\varepsilon(x) \leq f(x) \leq h_\varepsilon(x)$ for all $x \in [a, b]$ and $\int_a^b h_\varepsilon - g_\varepsilon < \varepsilon$.
  Let $\varepsilon > 0$ and let g_ε and h_ε be two Riemann integrable functions on $[a, b]$ such that $g_\varepsilon(x) \leq f(x) \leq h_\varepsilon(x)$ for all $x \in [a, b]$ and $\int_a^b h_\varepsilon - g_\varepsilon < \frac{\varepsilon}{3}$.
  By Theorem 13.2.1, both h_ε and g_ε are bounded on $[a, b]$ and hence f is bounded on $[a, b]$. By definition, there are $\delta_1, \delta_2 > 0$ such that for all tagged partition ${}^tP$ of $[a, b]$,
  - if $\|{}^tP\| < \delta_1$ then $|S(g_\varepsilon,{}^t P) - \int_a^b g_\varepsilon| < \frac{\varepsilon}{3}$;
  - if $\|{}^tP\| < \delta_1$ then $|S(h_\varepsilon,{}^t P) - \int_a^b h_\varepsilon| < \frac{\varepsilon}{3}$.

  Let $\delta = \min\{\delta_1, \delta_2\}$ and let ${}^tP$ and ${}^tQ$ be tagged partitions of $[a, b]$ such that $\|{}^tP\| < \delta$. We have $S(g_\varepsilon,{}^t P) \leq S(f,{}^t P) \leq S(h_\varepsilon,{}^t P)$ and:

  $$\int_a^b g_\varepsilon - \frac{\varepsilon}{3} < S(g_\varepsilon,{}^t P) \text{ and } S(h_\varepsilon,{}^t P) < \int_a^b h_\varepsilon + \frac{\varepsilon}{3}$$

  Hence
  $$\int_a^b g_\varepsilon - \frac{\varepsilon}{3} < S(f,{}^t P) < \int_a^b h_\varepsilon + \frac{\varepsilon}{3}$$

Similarly we have

$$\int_a^b g_\varepsilon - \frac{\varepsilon}{3} < S(f,^t Q) < \int_a^b h_\varepsilon + \frac{\varepsilon}{3}$$

Hence

$$-(\int_a^b h_\varepsilon - \int_a^b g_\varepsilon + \frac{2\varepsilon}{3}) < S(f,^t P) - S(f,^t Q) < \int_a^b h_\varepsilon - \int_a^b g_\varepsilon + \frac{2\varepsilon}{3}$$

That is,

$$|S(f,^t P) - S(f,^t Q)| < \int_a^b h_\varepsilon - \int_a^b g_\varepsilon + \frac{2\varepsilon}{3} < +\frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

By Cauchy Criterion for Riemann Inegrability Theorem 13.2.2, f is Riemann integrable.

Solution C.210. [Of Exercise 13.13] Assume f is Riemann integrable and let $\varepsilon > 0$. By Theorem 13.2.2, there exists $\delta > 0$ such that $|S(f,^t P_1) - S(f,^t P_2)| < \varepsilon$ for all tagged partitions ${}^tP_1$ and ${}^tP_2$ of $[a, b]$ with norms less than δ. Let $P = \{x_i : 0 \leq i \leq n\}$ a partition of $[a, b]$ such that $\|P\| < \delta$.

Note that $\sup\{f(x) - f(y) : x, y \in [x_{i-1}, x_i]\} = \omega(f, [x_{i-1}, x_i])$ for all $1 \leq i \leq n$. Hence for all $1 \leq i \leq n$, choose $t_i, t'_i \in [x_{i-1}, x_i]$ such that $f(t_i) - f(t'_i) = \sup\{f(x) - f(y) : x, y \in [x_{i-1}, x_i]\}$.

Let ${}^tP_1 = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ and ${}^tP_2 = \{(t'_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$. Now, $|S(f,^t P_1) - S(f,^t P_2)| = |\Sigma_{i=1}^n (f(t_i) - f(t'_i)(x_i - x_{i-1})| = |\Sigma_{i=1}^n \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1})| < \varepsilon$ and we are done.

Solution C.211. [Of Exercise 13.14]

1. If f and g are continuous, then by Theorem 11.2.7 $f \circ g$ is continuous on $[c, d]$ and hence by Theorem 13.2.7, $f \circ g$ is Riemann integrable on $[c, d]$.

2. [4] Let $\varepsilon > 0$. We will show that there is a partition P of $[c, d]$ such that $O(f \circ g, P) < \varepsilon$ (by Theorem 13.2.3).

   Since f is continuous on $[a, b]$ then

   – By Lemma 13.2.6, there exists $\delta > 0$ such that for all $x, y \in [a, b]$ that satisfy $|y - x| < \delta$, we have $|f(y) - f(x)| < \dfrac{\varepsilon}{2(d-c)}$

[4] This solution has been taken from [?] and [?]. In [?] you will also find an example of two functions f and g where f is continuous on $[a, b]$ and g is Riemann integrable on $[c, d]$, but $f \circ g$ is not Riemann integrable on $[c, d]$.

- f is bounded on $[a,b]$ (this comes from either the Extreme Value Theorem 12.3.12 or from both Theorems 13.2.7 and 13.2.1 which state that a continuous function is Riemann integrable and a Riemann integrable function is bounded). Hence, let U be such that $|f(x)| < U$ for all $x \in [a,b]$.

Since g is Riemann integrable on $[c,d]$ then by Theorem 13.2.3, there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[c,d]$ such that $O(g,P) = \Sigma_{i=1}^{n}\omega_i(x_i - x_{i-1}) < \frac{\varepsilon\delta}{4U}$ where $I_i = [x_{i-1}, x_i]$ for $1 \leq i \leq n$ and $\omega_i = \omega(g, I_i)$. Recall that $\omega(g,[l,u]) = \sup\{g(x) : x \in [l,u]\} - \inf\{g(x) : x \in [l,u]\} = \sup\{|g(x) - g(y)| : x, y \in I\} = \sup\{g(x) - g(y) : x, y \in I\}$ by Lemma 13.1.11.

Let $\omega'_i = \omega(f \circ g, I_i)$. Note that $\omega'_i \leq \sup\{|f(x)| + |f(x)| : x, y \in [a,b]\} < 2U$. There are two cases:

- Either $\omega_i < \delta$ and hence $\omega'_i < \frac{\varepsilon}{2(d-c)}$ and $\Sigma_{i/\omega_i<\delta}\omega'_i(x_i - x_{i-1}) < \Sigma_{i/\omega_i<\delta}\frac{\varepsilon}{2(d-c)}(x_i - x_{i-1}) = \frac{\varepsilon}{2(d-c)}\Sigma_{i/\omega_i<\delta}(x_i - x_{i-1}) < \frac{\varepsilon}{2(d-c)}\Sigma_{i=1}^{n}(x_i - x_{i-1}) = \frac{\varepsilon}{2(d-c)}(d-c) = \frac{\varepsilon}{2}$
- Or $\omega_i \geq \delta$ and hence $\Sigma_{i/\omega_i\geq\delta}\omega'_i(x_i - x_{i-1}) < 2U\Sigma_{i/\omega_i\geq\delta}(x_i - x_{i-1})$. But $\Sigma_{i=1}^{n}\omega_i(x_i - x_{i-1}) \geq \Sigma_{i/\omega_i\geq\delta}\omega_i(x_i - x_{i-1}) \geq \Sigma_{i/\omega_i\geq\delta}\delta(x_i - x_{i-1}) = \delta\Sigma_{i/\omega_i\geq\delta}(x_i - x_{i-1})$ and hence $\Sigma_{i/\omega_i\geq\delta}(x_i - x_{i-1}) \leq \frac{\Sigma_{i=1}^{n}\omega_i(x_i - x_{i-1})}{\delta} < \frac{\varepsilon\delta}{4U\delta} = \frac{\varepsilon}{4U}$. Hence, $\Sigma_{i/\omega_i\geq\delta}\omega'_i(x_i - x_{i-1}) < 2U\Sigma_{i/\omega_i\geq\delta}(x_i - x_{i-1}) < 2U\frac{\varepsilon}{4U} = \frac{\varepsilon}{2}$.

Since $O(f \circ g, P) = \Sigma_{i/\omega_i\geq\delta}\omega'_i(x_i - x_{i-1}) + \Sigma_{i/\omega_i<\delta}\omega'_i(x_i - x_{i-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, we are done.

Solution C.212. [Of Exercise 13.15] We will first do the proof for the case that f and g only differ on one value. That is, suppose there is $c \in [a,b]$ such that $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$. We will show that f is Riemann integrable implies g is Riemann integrable. Assume f is Riemann integrable on $[a,b]$. Hence by Theorem 13.2.1, f is bounded. Hence g is bounded. Let $M > 0$ such that $|f|, |g| < M$.

Let $\varepsilon > 0$. By definition, there is a $\delta > 0$ such that for any tagged partition ${}^tP$ of $[a,b]$, if $\|{}^tP\| < \delta$ then $|S(f, {}^tP) - \int_a^b f| < \frac{\varepsilon}{2}$. Let $\delta' = \min\{\delta, \frac{\varepsilon}{8M}\}$.

Let ${}^tP$ be a tagged partition of $[a,b]$ such that $\|{}^tP\| < \delta'$ and $P = \{x_i : 0 \leq i \leq n\}$. Let ${}^tP_1$ be the subset of ${}^tP$ whose tags are all c. Let ${}^tP_2$ be the subset of ${}^tP$ whose tags exclude c. Obviously,

$S(f,{}^t P) = S(f,{}^t P_1) + S(f,{}^t P_2)$ and $S(g,{}^t P) = S(g,{}^t P_1) + S(g,{}^t P_2)$. Note that $S(f,{}^t P_2) = S(g,{}^t P_2)$ and $|{}^t P_1| \leq 2$ (since either $c \neq x_i$ for any i and hence $|{}^t P_1| \leq 1$; or $c \in \{a, b\}$ and hence $|{}^t P_1| \leq 1$; or $c = x_j$ for some $1 \leq j \leq n-1$ and $|{}^t P_1| \leq 2$).

Hence $|S(f,{}^t P) - S(g,{}^t P)| = |S(f,{}^t P_1) - S(g,{}^t P_1)| =$

$$\begin{cases} 0 & \text{if } |{}^t P_1| = 0 \\ |g(c) - f(c)|(x_j - x_{j-1}) & \text{if } {}^t P_1 = \{(c, [x_{j-1}, x_j])\} \\ |g(c) - f(c)|(c - x_{j-1}) + |g(c) - f(c)|(x_{j+1} - c) & \\ \qquad \text{if } {}^t P_1 = \{(c, [x_{j-1}, c]), (c, [c, x_{j+1}])\} \end{cases}$$

That is, $|S(f,{}^t P) - S(g,{}^t P)| < 2|g(c) - f(c)|\delta' \leq 4M\delta' \leq 4M\frac{\varepsilon}{8M} = \frac{\varepsilon}{2}$.

Now, $|S(g,{}^t P) - \int_a^b f| = |S(g,{}^t P) - S(f,{}^t P) + S(f,{}^t P) - \int_a^b f| \leq |S(g,{}^t P) - S(f,{}^t P)| + |S(f,{}^t P) - \int_a^b f| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Hence, g is Riemann integrable on $[a, b]$ and $\int_a^b f = \int_a^b g$.

If f and g differ on more than one point, we do the proof by induction on the number of points on which they differ, using the above result.

Solution C.213. [Of Exercise 13.16]

1. Since F is an indefinite integral of f on interval I then $F' = f$. By Theorems 11.3.6 and 11.3.8, $(F + c)' = F' + c' = F' = f$. Hence $F + c$ is an indefinite integral of f.

2. By Theorem 11.3.8, $(F + G)' = F' + G'$.

3. By Corollary 11.3.11, $(cF)'(x) = cF'(x) = cf(x)$. Hence cF is an indefinite integral of cf.