

# Simplified Reducibility Proofs of Church-Rosser for $\beta$ - and $\beta\eta$ -reduction

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## Abstract

The simplest proofs of the Church Rosser Property are usually done by the syntactic method of parallel reduction. On the other hand, reducibility is a semantic method which has been used to prove a number of properties in the  $\lambda$ -calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we concentrate on simplifying a semantic method based on reducibility for proving Church-Rosser for both  $\beta$ - and  $\beta\eta$ -reduction. Interestingly, this simplification results in a syntactic method (so the semantic aspect disappears) which is nonetheless projectable into a semantic method. Our contributions are as follows:

- We give a simplification of a semantic proof of CR for  $\beta$ -reduction which unlike some common proofs in the literature, avoids any type machinery and is solely carried out in a completely untyped setting.
- We give a new proof of CR for  $\beta\eta$ -reduction which is a generalisation of our simple proof for  $\beta$ -reduction.
- Our simplification of the semantic proof results into a syntactic proof which is projectable into a semantic method and can hence be used as a bridge between syntactic and semantic methods.

*Keywords:* Church-Rosser, Reducibility, Parallel reductions

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## 1 Introduction

Reducibility is a method based on realisability semantics [9], developed by Tait [14] in order to prove normalisation of some functional theories. The idea is to interpret types by sets of  $\lambda$ -terms closed under some properties. Since its introduction, this method has gone through a number of improvements and generalisations. In particular, Krivine [12] uses reducibility to prove the strong normalisation (SN) of his intersection type system called system  $\mathcal{D}$ . Koletsos [10] uses reducibility to prove that the set of simply typed  $\lambda$ -terms holds the Church-Rosser property (CR, also known as confluence property) w.r.t.  $\beta$ -reduction. Although it is well known that  $\beta$ -reduction satisfies CR, reducibility proofs of CR are in line with proofs of SN and hence, one can establish both SN and CR for some calculus using the same method. Moreover, CR proofs can be quite involved. So, reducibility proofs can help within

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the same machinery to prove the most important properties of a  $\lambda$ -calculus (such as SN, CR or standardisation).

In this paper we simplify a reducibility method for proving CR for  $\beta$ -reduction and we generalise our proof to proving CR for  $\beta\eta$ -reduction. Our proof of CR for  $\beta$ -reduction is simpler than the ones given by Ghilezan and Kunčák [6], by Koletsos and Stavrinou [11] and by Barendregt, Bergstra, Klop and Volken [1,2]. Furthermore, our proof is generalisable into a new proof of CR for  $\beta\eta$ -reduction. The CR theorem is a strong form of a theorem stated by Church and Rosser [5] proving the consistency of the  $\lambda$ -calculus. The Church-Rosser property of the untyped  $\lambda$ -calculus w.r.t.  $\beta$ -reduction can be stated as follows: for any  $\lambda$ -terms  $M, M_1, M_2$  such that  $M \rightarrow_{\beta}^* M_1$  and  $M \rightarrow_{\beta}^* M_2$  there exists  $M_3$  such that  $M_1 \rightarrow_{\beta}^* M_3$  and  $M_2 \rightarrow_{\beta}^* M_3$ , where  $\rightarrow_{\beta}^*$  is the  $\beta$ -reduction relation.

As in a number of other works [11,8], our method to prove CR for a given set of terms w.r.t. a reduction relation (we consider both  $\beta$ -reduction and  $\beta\eta$ -reduction) consists of two main steps:

- Our first step is based on a simplification of a reducibility method used to prove the confluence of developments. We started the construction of our method with a “common” reducibility method. We observed that not all the types were actually needed in the method and that the interpretations of the few needed types corresponded to simple sets of terms satisfying simple closure properties (such as saturation). This led us to the removal of the whole type machinery and we obtained as a result a syntactic method. Interestingly, although our method has turned into a syntactic one, it can still be embedded into a semantic method based on a type system such as Krivine’s system  $\mathcal{D}$  [12].<sup>2</sup> As it is crucial to a reducibility method to use a soundness result, we call soundness the corresponding result in our method.
- The second step of our method consists in reducing the problem of the confluence of the  $\lambda$ -calculus w.r.t. the considered reduction relation to the problem of the confluence of the defined set of terms w.r.t. the defined reduction. This second step is achieved through a common simulation method used in most proofs of CR. The simulation consists in defining a new simple confluent reduction whose transitive closure is equal to the considered reduction.

To achieve their goals, all of [1,6,11,8] use the notion of developments. Both Koletsos and Stavrinou [11] as well as Kamareddine and Rahli [8] use a complicated handling of developments. On the other hand, Barendregt et al. [1], Ghilezan and Kunčák [6] as well as this article are based on some simpler and sufficient notions of developments. These notions of developments are simpler because, as in the so called method of parallel reductions [13,15], they do not deal with residuals. Because this article does not make use of a type system and does not deal with residuals, it is a simplification of the work done by [11] and by [8]. However, this article can also be regarded as a simplification and a generalisation of the work done by:

- Barendregt et al. [1], because we do not introduce a new calculus and we do not use the finiteness of developments,

<sup>2</sup> The methods of [6] and [11] use the Simply Typed Lambda Calculus  $\lambda_{\rightarrow}$  and the type system  $\mathcal{D}$  [4,3].

- Ghilezan and Kunčák [6], because we do not make use of a type system.

In Section 2 we introduce the needed machinery about the  $\lambda$ -calculus and our developments. In Section 3 we give our proof of the Church-Rosser property of the untyped  $\lambda$ -calculus w.r.t.  $\beta$ -reduction. In Section 4 we give our proof of the Church-Rosser property of the untyped  $\lambda$ -calculus w.r.t.  $\beta\eta$ -reduction. Finally, in Section 5, we compare our solution to the related work in the literature, especially to the work of Ghilezan and Kunčák [6], of Koletsos and Stavrinos [11] and to the developments of Tait and Martin-Löf [13,2]. Due to space limitations, we omit the details of the proofs. However, full proofs can be found in the extended version of this article which will always be available at the authors' web pages.

## 2 The Formal Machinery

In this section we provide some known formal machinery and introduce new definitions and lemmas that are necessary for the paper. Let  $n, m$  be metavariables which range over the set of natural numbers  $\mathbb{N} = \{0, 1, 2, \dots\}$ . We take as convention that if a metavariable  $v$  ranges over a set  $s$  then the metavariables  $v_i$  such that  $i \geq 0$  and the metavariables  $v', v'', \text{etc.}$  also range over  $s$ .

### 2.1 Background on the $\lambda$ -calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the  $\lambda$ -calculus and one basic lemma.

**Definition 2.1** (i) Let  $x, y, z$  range over  $\text{Var}$ , a countable infinite set of term variables (or just variables). The set of terms of the  $\lambda$ -calculus is defined by:

$$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$$

We let  $M, N, P, Q, R$  range over  $\Lambda$ . We assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write  $MN_0 \cdots N_n$  instead of  $(\cdots((MN_0)N_1) \cdots N_{n-1})N_n$ . We also assume the usual definition of subterms and write  $N \subseteq M$  if  $N$  is a subterm of  $M$  (note that  $M \subseteq M$ ). We call a term of the form  $\lambda x.M$ , a  $\lambda$ -abstraction (or just abstraction) and a term of the form  $M_1 M_2$  an application.

We take terms modulo  $\alpha$ -conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms  $M$  and  $N$  are equal (modulo  $\alpha$ ), we write  $M = N$ . We write  $\text{fv}(M)$  for the set of the free variables of term  $M$ .

- (ii) We define as usual the substitution  $M[x := N]$  of  $N$  for all free occurrences of  $x$  in  $M$ . We let  $M[x_1 := N_1, \dots, x_n := N_n]$  be the simultaneous substitution of  $N_i$  for all free occurrences of  $x_i$  in  $M$  for  $1 \leq i \leq n$ .
- (iii) We assume the usual definition of compatibility (see the last line of Figure 1). For  $r \in \{\beta, \beta\eta\}$ , we define the reduction relation  $\rightarrow_r$  on  $\Lambda$  as the least compatible relation closed under rule  $(r) : L \rightarrow_r R$  below, and we call  $L$  an  $r$ -redex and  $R$  the  $r$ -contractum of  $L$  (or the  $L$   $r$ -contractum).
  - $(\beta)$ :  $(\lambda x.M)N \rightarrow_\beta M[x := N]$ .
  - $(\eta)$ :  $\lambda x.Mx \rightarrow_\eta M$  where  $x \notin \text{fv}(M)$ .

let $\mathcal{R}$ be a binary relation on $\Lambda$ .		
$\frac{}{M \mathcal{R} M} \text{ (refl)}$	$\frac{M_1 \mathcal{R} M_2 \quad M_2 \mathcal{R} M_3}{M_1 \mathcal{R} M_3} \text{ (tr)}$	
$\frac{P \mathcal{R} Q}{\lambda x. P \mathcal{R} \lambda x. Q} \text{ (abs)}$	$\frac{Q \mathcal{R} Q'}{PQ \mathcal{R} PQ'} \text{ (app}_1\text{)}$	$\frac{P \mathcal{R} P'}{PQ \mathcal{R} P'Q} \text{ (app}_2\text{)}$

Fig. 1. Closure rules

We define  $\rightarrow_{\beta\eta} = \rightarrow_{\beta} \cup \rightarrow_{\eta}$ .

- (iv) Let  $r \in \{\beta, \eta, \beta\eta\}$ . We use  $\rightarrow_r^*$  to denote the reflexive transitive closure (see the rules (refl) and (tr) of Figure 1) of  $\rightarrow_r$ . We let  $\simeq_r$  denote the equivalence relation induced by  $\rightarrow_r$ . If the  $r$ -reduction from  $M$  to  $N$  is in  $k$  steps, we write  $M \rightarrow_r^k N$ .
- (v) Let  $r \in \{\beta, \beta\eta\}$ . A term  $(\lambda x. M')N'$  is a *direct  $r$ -reduct* of  $(\lambda x. M)N$  iff  $M \rightarrow_r^* M'$  and  $N \rightarrow_r^* N'$ .
- (vi) Let  $r \in \{\beta, \beta\eta\}$ . We say that  $M$  has the Church-Rosser property for  $r$  (has  $r$ -CR) if whenever  $M \rightarrow_r^* M_1$  and  $M \rightarrow_r^* M_2$  then there exists  $M_3$  such that  $M_1 \rightarrow_r^* M_3$  and  $M_2 \rightarrow_r^* M_3$ . We define  $\text{CR}^r = \{M \mid M \text{ has } r\text{-CR}\}$ . We use CR to denote  $\text{CR}^\beta$ .
- (vii) We define the set SAT of the sets satisfying the saturation property as follows:  
 $\text{SAT} = \{s \subseteq \Lambda \mid M[x := N] \in s \Rightarrow (\lambda x. M)N \in s\}$ .
- (viii) We define the set VAR of the sets satisfying the variable property as follows:  
 $\text{VAR} = \{s \subseteq \Lambda \mid n \geq 0 \wedge (\forall i \in \{1, \dots, n\}. M_i \in s) \Rightarrow xM_1 \cdots M_n \in s\}$ .
- (ix) We define the set ABS of the sets satisfying the abstraction property as follows:  
 $\text{ABS} = \{s \subseteq \Lambda \mid M \in s \Rightarrow \lambda x. M \in s\}$ .

**Lemma 2.2** *Let  $r \in \{\beta, \beta\eta\}$ . The following hold:*

- (i) *If  $M \rightarrow_r^* N$  and  $P \rightarrow_r^* Q$  then  $M[x := P] \rightarrow_r^* N[x := Q]$ .*
- (ii)  *$\text{fv}(M[x := N]) \subseteq \text{fv}((\lambda x. M)N)$ .*
- (iii) *If  $M \rightarrow_r^* N$  then  $\text{fv}(N) \subseteq \text{fv}(M)$ .*
- (iv) *If  $\lambda x. M \rightarrow_{\beta\eta}^* N$  then either  $N = \lambda x. M'$  such that  $M \rightarrow_{\beta\eta}^* M'$  or  $M \rightarrow_{\beta\eta}^* Nx$  such that  $x \notin \text{fv}(N)$ .*
- (v) *If  $x \notin \text{fv}(M)$  and  $Mx \rightarrow_{\beta\eta}^* N$  then  $M \rightarrow_{\beta\eta}^* P$  and either  $N = Px$  or  $P = \lambda x. N$ .*
- (vi) *If  $n \geq 0$ ,  $Q = (\lambda x. M)N \rightarrow_r^k P$  and  $P$  is not a direct  $r$ -reduct of  $Q$  then (a)  $k \geq 1$ , (b) if  $k = 1$  then  $P = M[x := N]$  and (c) there exists a direct  $r$ -reduct  $(\lambda x. M')N'$  of  $Q$  such that  $M'[x := N'] \rightarrow_r^* P$ .*
- (vii) *Let  $n \geq 0$  and  $(\lambda x. M)N \rightarrow_r^* P$ . There exists  $P'$  such that  $P \rightarrow_r^* P'$  and  $M[x := N] \rightarrow_r^* P'$ .*
- (viii) *a)  $\text{CR}^r \in \text{SAT}$       b)  $\text{CR}^r \in \text{VAR}$       c)  $\text{CR}^r \in \text{ABS}$*

## 2.2 Pseudo Development Definitions

Throughout, we take  $c$  to be a distinct metavariables ranging over  $\text{Var}$ .

**Remark 2.3** Such a distinct variable is usually given not as metavariable, but as a new variable or constant [6,11,12]. We noted that this usual way leads to problems.

For example, Ghilezan and Kunčák [6] use two of these distinct variables (called  $f$  and  $g$  and introduced as “predefined constants” not belonging to the untyped  $\lambda$ -calculus). But the “freezing” function  $\Psi$  defined by Ghilezan and Kunčák (similar to our function  $\Psi_c$ ) is proved to be a function from  $\Lambda$  to  $\Lambda_0$  where  $\Lambda_0$  is defined as follows:  $\Lambda_0 = \{M \in \Lambda \mid (\exists x_1, \dots, x_n)\Gamma_0, x_1 : 0, \dots, x_n : 0 \vdash M : 0\}$ , which is the set of terms in  $\Lambda$  which are typable in  $\lambda_{\rightarrow}$  (0 is a ground type and  $\Gamma_0$  is a predefined type environment). Hence, by their definition,  $\Lambda_0 \subset \Lambda$ . So, it is obvious that their function  $\Psi$  does not associate a term in  $\Lambda_0$  to each term in  $\Lambda$  since  $\Psi$  adds some  $f$  and  $g$  to the terms (for example  $\Psi(xx) = fxx$ , but  $fxx \notin \Lambda$ , so  $fxx \notin \Lambda_0$ ).

Moreover, typing environments (contexts) are defined as sets of type assignments of the form  $x : \varphi$  where  $x$  is a term variable and  $\varphi$  is a simple type. Later, some contexts are built with type assignments of the form  $f : \varphi$ , but  $f$  is not defined as a term variable. More generally, the introduction of a new variable or a new constant implies that the considered type system has to be defined on the new calculus.

Koletsos and Stavrinos [11], define two sets  $\text{CR}, \text{CR}_0 \subseteq \Lambda$  which turn out to be equal to ours. Koletsos and Stavrinos prove that each term typable in the type system  $\mathcal{D}$  has the Church-Rosser property. The proof of this statement fails, for example, for terms with free variables not belonging to the set of variables of the initial  $\lambda$ -calculus ( $c$  is defined as a variable not belonging to this set), since the proof uses the fact that the free variables of the term belong to the set  $\text{CR}_0$ . But, further, this statement is used for a term which may contain some  $c$ .

We call *current redex* the occurrence of a redex in a given term  $M$ . We call *potential redex* an application which is not an occurrence of a redex in a given term  $M$  but which is the occurrence of a redex in the term obtained after at least one reduction step from  $M$ . As done by Krivine [12] and followed by many others [6,11,8], we use  $c$  to “freeze” some current or potential redexes in a term. The parametrised calculi (with parameter  $c$ ) presented in Definition 2.4 are the “frozen” calculi based on the  $\lambda$ -calculus where some reductions are blocked by the use of  $c$ . For example  $(\lambda x.xy)(\lambda z.z) \rightarrow_{\beta} (\lambda z.z)y \rightarrow_{\beta} y$ , but  $(\lambda x.cxy)(\lambda z.z) \rightarrow_{\beta} c(\lambda z.z)y$  which does not reduce further. It is easy to see that  $\Lambda_c^{\beta} \subset \Lambda_c^{\beta\eta} \subset \Lambda$ .

**Definition 2.4** [ $\Lambda_c^{\beta}, \Lambda_c^{\beta\eta}$ ] Let  $\bar{x}, \bar{y} \in \text{Var}_c = \text{Var} \setminus \{c\}$ .

$$\bar{M} \in \Lambda_c^{\beta} ::= \bar{x} \mid \lambda \bar{x}.\bar{M} \mid (\lambda \bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2$$

$$\bar{M} \in \Lambda_c^{\beta\eta} ::= \bar{x} \mid \lambda \bar{x}.\bar{M} \mid (\lambda \bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2 \mid c\bar{M}$$

We let  $\bar{M}, \bar{N}, \bar{P}, \bar{Q}, \bar{R}$  range over both  $\Lambda_c^{\beta}$  and  $\Lambda_c^{\beta\eta}$ .

Definition 2.5, introduces the “freezing” function which allows to “freeze” the potential redexes of a term. Unlike definitions in the literature [6,11,12,8], with our definition (the third clause below), the current  $\beta$ -redexes of a term are all “unfrozen”. Furthermore, our definition does not freeze any of the current or potential

$\eta$ -redexes. For example,  $M = \lambda x.(\lambda y.czx)z$  does not contain any  $\eta$ -redex but contains a potential  $\eta$ -redex, since  $M \rightarrow_\beta \lambda x.czx = N$  and  $N$  contains a  $\eta$ -redex. As we will see in this paper, it is not necessary to “freeze” the potential  $\eta$ -redexes.

**Definition 2.5** [ $\Psi_c(-)$ ]  $\Psi_c(-)$  is defined as follows:

- (i)  $\Psi_c(x) = x$
- (ii)  $\Psi_c(\lambda x.N) = \lambda x.\Psi_c(N)$ , where  $x \notin \{c\}$
- (iii) If  $P$  is a  $\lambda$ -abstraction then  $\Psi_c(PQ) = \Psi_c(P)\Psi_c(Q)$
- (iv) If  $P$  is not a  $\lambda$ -abstraction then  $\Psi_c(PQ) = c\Psi_c(P)\Psi_c(Q)$ .

Definition 2.6 introduces the reduction  $\rightarrow_c$  which allows to remove the  $c$ 's from a term. This reduction can be regarded as a simplification of the reduction  $\rightarrow_o$  defined by Ghilezan and Kunčák [6].

**Definition 2.6** [ $\rightarrow_c$ ] Let the  $c$ -reduction relation  $\rightarrow_c$  be the least compatible relation on  $\Lambda$  closed under the rule:  $(c) : cM \rightarrow_c M$   
As usual  $\rightarrow_c^*$  is the reflexive and transitive closure of  $\rightarrow_c$ .

In Definition 2.7, we introduce our  $\beta$ -developments ( $\rightarrow_1$ ) as well as our  $\beta\eta$ -developments ( $\rightarrow_2$ ).

**Definition 2.7** [Developments:  $\rightarrow_1, \rightarrow_2$ ] Let  $\langle r, s \rangle \in \{\langle 1, \beta \rangle, \langle 2, \beta\eta \rangle\}$ .

$$M \rightarrow_r N \iff \exists P. \Psi_c(M) \rightarrow_s^* P \wedge P \rightarrow_c^* N \wedge c \notin \text{fv}(M) \cup \text{fv}(N)$$

As usual,  $\rightarrow_r^*$  is the reflexive and transitive closure of  $\rightarrow_r$ . (Note that  $\rightarrow_r$  is reflexive, but in order not to have to introduce a new symbol for its transitive closure, we consider  $\rightarrow_r^*$ .)

Definition 2.8 defines the parametric set of terms  $\mathbf{A}_c$  built over the parameter  $c$  using application. Such terms contain only  $c$ 's and no abstraction. This set of terms is especially needed to state Lemma 2.9.vii. The particularity of such terms being that they can be completely erased by the  $c$ -reduction (see Lemma 2.9.v).

**Definition 2.8**  $d \in \mathbf{A}_c ::= c \mid dd$

Lemma 2.9 is informative about the reduction relation  $\rightarrow_c$ .

**Lemma 2.9** (i)  $\Psi_c(M) \rightarrow_c^* M$ .

- (ii) If  $M \rightarrow_c^* N$  then  $\text{fv}(M) \setminus \{c\} = \text{fv}(N) \setminus \{c\}$ .
- (iii)  $\text{fv}(M) \setminus \{c\} = \text{fv}(\Psi_c(M)) \setminus \{c\}$ .
- (iv)  $\Lambda_c^\beta \cap \mathbf{A}_c = \emptyset = \Lambda_c^{\beta\eta} \cap \mathbf{A}_c$ .
- (v)  $dM \rightarrow_c^* M$ .
- (vi) If  $M \rightarrow_c^* N$  then  $M \in \mathbf{A}_c$  iff  $N \in \mathbf{A}_c$ .
- (vii) Let  $M \rightarrow_c^* N$ .
  - If  $M = x$  then  $N = x$ .
  - If  $M = \lambda x.M_1$  then  $N = \lambda x.N_1$  such that  $M_1 \rightarrow_c^* N_1$ .
  - If  $M = M_1M_2$  then either  $M_1 \in \mathbf{A}_c$  and  $M_2 \rightarrow_c^* N$  or  $N = N_1N_2$  and  $M_1 \rightarrow_c^* N_1$  and  $M_2 \rightarrow_c^* N_2$ .

- (viii) If  $M \rightarrow_c^* M'$ ,  $N \rightarrow_c^* N'$  and  $x \neq c$  then  $M[x := N] \rightarrow_c^* M'[x := N']$ .
- (ix) If  $c \notin \text{fv}(M)$  and  $M \rightarrow_c^* N$  then  $M = N$ .
- (x) If  $M \rightarrow_c^* N$ ,  $M \rightarrow_c^* P$  and  $c \notin \text{fv}(N)$  then  $P \rightarrow_c^* N$ .

**Proof.** i,viii,x. By induction on the structure of  $M$ .

iii. Corollary of Lemma 2.9.i and Lemma 2.9.ii.

iv. Let  $\bar{M} \in \Lambda_c^{\beta\eta}$ . By induction on the structure of  $\bar{M}$  that  $\bar{M} \notin A_c$ .

v. By induction on the structure of  $d$ .

vi.  $\Rightarrow$ ) By induction on the length of the reduction  $M \rightarrow_c^* d$ .  $\Leftarrow$ ) By induction on the reduction  $d \rightarrow_c^* N$ .

vii,ix. By induction on the length of the reduction  $M \rightarrow_c^* N$ . □

Lemma 2.9.i stresses the relation between  $\rightarrow_c$  and the freezing function.

Lemma 2.9.vii presents the shape of a term w.r.t. a  $c$ -reduction.

Lemma 2.9.x is a sort of weak confluence property w.r.t.  $\rightarrow_c^*$ .

### 3 A simple Church-Rosser proof for $\beta$ -reduction

Koletsos and Stavrinou [11] gave a proof of the Church-Rosser property for the set of terms typable in an intersection type system called system  $\mathcal{D}$  [12] w.r.t.  $\beta$ -reduction and showed that this can be used to establish the confluence of  $\beta$ -developments without using strong normalisation. Ghilezan and Kunčak [6] gave a proof of the Church-Rosser property for the set of terms typable in  $\lambda_{\rightarrow}$  w.r.t.  $\beta$ -reduction and showed that this can be used to establish the confluence of a weak form of  $\beta$ -developments without using strong normalisation.

The first aim of this section, was to simplify the proof of Koletsos and Stavrinou [11]. During this simplification, we obtained a proof that bore some resemblance to the proof of Ghilezan and Kunčak [6]. A second simplification of our proof started with the observation that in both proofs of [6,11] only few types were really needed and that one can actually completely get rid of the type system. We considered two type interpretations based on the sets  $\text{CR}^{\beta}$  and  $\text{CR}^{\beta\eta}$  and interpreted the few needed types by sets of terms satisfying simple properties: saturation, variable and abstraction (see Definition 2.1). Since the calculus used by Koletsos and Stavrinou to prove the confluence of developments is simpler than the one used by Ghilezan and Kunčak, a third simplification which led to our actual simple proof has been to come back to the use of a calculus similar to the one used by Koletsos and Stavrinou as well as Krivine [12] before them (see Definition 2.4). As mentioned above, our proof is carried out in an untyped setting but one can relate the first part of the method to the reducibility proof in a semantic method using, for example, the type system  $\mathcal{D}$ . Interestingly, our proof can also be related to the proof given in [1,2].

The second aim of this section is to provide a framework for our main result: the extension of our proof to  $\beta\eta$ -reduction where we give a non semantic proof of Church-Rosser for  $\beta\eta$ -reduction which can be projectable into a semantic proof (Section 4).

As part of our simplification of a reducibility method, Lemma 3.1 states the “soundness” of our simple calculus (based on the set of terms  $\Lambda_c^{\beta}$  w.r.t. the re-

mains of our type interpretation using a set  $s$  satisfying the saturation, variable and abstraction properties given in Definition 2.1).

**Lemma 3.1 (Soundness)** *If  $\bar{M} \in \Lambda_c^\beta$ ,  $\text{fv}(\bar{M}) \setminus \{c\} = \{x_1, \dots, x_n\}$ , for all  $i \in \{1, \dots, n\}$ ,  $M_i \in s$  and  $s \in \text{VAR} \cup \text{SAT} \cup \text{ABS}$  then  $\bar{M}[x_1 := M_1, \dots, x_n := M_n] \in s$ .*

**Proof.** By induction on the structure of  $\bar{M}$ .  $\square$

Using Lemma 3.1, we are now able to prove that each term in  $\Lambda_c^\beta$  has  $\beta$ -CR.

**Corollary 3.2**  $\Lambda_c^\beta \subseteq \text{CR}$ .

**Proof.** Let  $\bar{M} \in \Lambda_c^\beta$  and  $\text{fv}(\bar{M}) \setminus \{c\} = \{x_1, \dots, x_n\}$ . By Lemma 2.2.viii.(a), 2.2.viii.(b) and 2.2.viii.(c),  $\text{CR} \in \text{SAT} \cup \text{VAR} \cup \text{ABS}$  and  $x_1, \dots, x_n \in \text{CR}$ . So by Lemma 3.1,  $\bar{M} \in \text{CR}$ .  $\square$

Lemma 3.3 states that the freezing function associates to each term of the untyped  $\lambda$ -calculus (which does not contain  $c$ ) a term in the language  $\Lambda_c^\beta$ .

**Lemma 3.3** *If  $c \notin \text{fv}(M)$  then  $\Psi_c(M) \in \Lambda_c^\beta$ .*

**Proof.** By induction on the structure of  $M$ .  $\square$

Here is another lemma containing needed technicalities:

**Lemma 3.4** *Let  $\bar{M}, \bar{N} \in \Lambda_c^\beta$  and  $\bar{x} \in \text{Var}_c$ .*

- (i)  $\bar{M}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$ .
- (ii) *If  $\bar{M} \rightarrow_\beta^* N$  then  $N \in \Lambda_c^\beta$ .*
- (iii) *If  $\bar{M} \rightarrow_c^* N$  and  $c \notin \text{fv}(N)$  then  $\bar{M} \rightarrow_c^* \Psi_c(N)$ .*
- (iv) *There exists  $N$  such that  $c \notin \text{fv}(N)$  and  $\bar{M} \rightarrow_c^* N$ .*

**Proof.** Items i, iii and iv are by induction on the structure of  $\bar{M}$ . Item ii is by induction on the length of the derivation  $\bar{M} \rightarrow_\beta^* N$ .  $\square$

The second property (3.4.ii) states that the terms in  $\Lambda_c^\beta$   $\beta$ -reduce only to terms in  $\Lambda_c^\beta$ . The fourth property (3.4.iv) expresses that a term in  $\Lambda_c^\beta$  can always be completely reduced w.r.t.  $\rightarrow_c^*$ .

Lemma 3.5 states that we can simulate the  $\beta$ -reduction of a term in  $\Lambda_c^\beta$  from any of its (partially or totally) “unfrozen” versions.

**Lemma 3.5** (i) *If  $\bar{M}_1 \in \Lambda_c^\beta$ ,  $\bar{M}_1 \rightarrow_\beta N_1$  and  $\bar{M}_1 \rightarrow_c^* M_2$  then there exists  $N_2$  such that  $M_2 \rightarrow_\beta N_2$  and  $N_1 \rightarrow_c^* N_2$ .*

(ii) *If  $\bar{M}_1 \in \Lambda_c^\beta$ ,  $\bar{M}_1 \rightarrow_\beta^* N_1$  and  $\bar{M}_1 \rightarrow_c^* M_2$  then there exists  $N_2$  such that  $M_2 \rightarrow_\beta^* N_2$  and  $N_1 \rightarrow_c^* N_2$ .*

**Proof.**

i By induction on the structure of  $\bar{M}_1$ .

ii By induction on the length of the reduction  $\bar{M}_1 \rightarrow_\beta^* N_1$  using Lemma 3.5.i.  $\square$



Lemma 3.6 is a key lemma of the simulation method of a reduction by some developments. It states that the reflexive and transitive closure of  $\rightarrow_\beta$  is equal to the reflexive and transitive closure of  $\rightarrow_1$ .

**Lemma 3.6** *Let  $c \notin \text{fv}(M) \cup \text{fv}(N)$ , then  $M \rightarrow_\beta^* N \iff M \rightarrow_1^* N$ .*

**Proof.**

$\Rightarrow$ ) Let  $M \rightarrow_\beta^* N$ . We prove that  $M \rightarrow_1^* N$  by induction on the size of the reduction  $M \rightarrow_\beta^* N$ .

$\Leftarrow$ ) Let  $M \rightarrow_1^* N$ . We prove that  $M \rightarrow_\beta^* N$  by induction on the size of the derivation  $M \rightarrow_1^* N$ . □

Lemma 3.7 states the confluence of the  $\beta$ -developments.

**Lemma 3.7** (i) *If  $M \rightarrow_1 M_1$  and  $M \rightarrow_1 M_2$  then there exists  $M_3$  such that  $M_1 \rightarrow_1 M_3$  and  $M_2 \rightarrow_1 M_3$ .*

(ii) *If  $M \rightarrow_1^* M_1$  and  $M \rightarrow_1^* M_2$  then there exists  $M_3$  such that  $M_1 \rightarrow_1^* M_3$  and  $M_2 \rightarrow_1^* M_3$ .*

**Proof.**

i By definition, there exist  $P_1, P_2$  such that  $\Psi_c(M) \rightarrow_\beta^* P_1$ ,  $\Psi_c(M) \rightarrow_\beta^* P_2$ ,  $P_1 \rightarrow_c^* M_1$ ,  $P_2 \rightarrow_c^* M_2$  and  $c \notin \text{fv}(M) \cup \text{fv}(M_1) \cup \text{fv}(M_2)$ . By Lemma 3.3,  $\Psi_c(M) \in \Lambda_c^\beta$ . So by Corollary 3.2, there exists  $P_3$  such that  $P_1 \rightarrow_\beta^* P_3$  and  $P_2 \rightarrow_\beta^* P_3$ . By Lemma 3.4.ii,  $P_1, P_2, P_3 \in \Lambda_c^\beta$ . By lemma 3.4.iv, there exists  $M_3$  such that  $P_3 \rightarrow_c^* M_3$  and  $c \notin \text{fv}(M_3)$ . By Lemma 3.4.iii,  $P_1 \rightarrow_c^* \Psi_c(M_1)$  and  $P_2 \rightarrow_c^* \Psi_c(M_2)$ . By Lemma 3.5.ii, there exist  $Q_1, Q_2$  such that  $P_3 \rightarrow_c^* Q_1$ ,  $P_3 \rightarrow_c^* Q_2$ ,  $\Psi_c(M_1) \rightarrow_\beta^* Q_1$  and  $\Psi_c(M_2) \rightarrow_\beta^* Q_2$ . By Lemma 2.9.x,  $Q_1 \rightarrow_c^* M_3$  and  $Q_2 \rightarrow_c^* M_3$ . So  $M_1 \rightarrow_1 M_3$  and  $M_2 \rightarrow_1 M_3$ .

ii By Lemma 3.7.i □

The confluence of the untyped  $\lambda$ -calculus w.r.t.  $\beta$ -reduction is now proved using the confluence of the  $\beta$ -developments and the equality between  $\rightarrow_\beta^*$  and  $\rightarrow_1^*$ .

**Theorem 3.8**  $\Lambda = \text{CR}$ .

**Proof.**  $\text{CR} \subseteq \Lambda$  is trivial, we only prove  $\Lambda \subseteq \text{CR}$ . Let  $M, M_1, M_2 \in \Lambda$  such that  $M \rightarrow_\beta^* M_1$  and  $M \rightarrow_\beta^* M_2$  and  $c \notin \text{fv}(M)$ . By Lemma 2.2.iii,  $c \notin \text{fv}(M_1) \cup \text{fv}(M_2)$ . By Lemma 3.6,  $M \rightarrow_1^* M_1$  and  $M \rightarrow_1^* M_2$ . By Lemma 3.5.ii, there exists  $M_3$  such that  $M_1 \rightarrow_1^* M_3$  and  $M_2 \rightarrow_1^* M_3$ . By definition  $c \notin \text{fv}(M_3)$ . By Lemma 3.6,  $M_1 \rightarrow_\beta^* M_3$  and  $M_2 \rightarrow_\beta^* M_3$ . □

## 4 A simple Church-Rosser proof for $\beta\eta$ -reduction

Now that we have stated the principal steps of the method of the Church-Rosser property of the untyped  $\lambda$ -calculus w.r.t.  $\beta$ -reduction, we will generalise it to  $\beta\eta$ -reduction following exactly the same steps and using the  $\Lambda_c^{\beta\eta}$  language. This generalisation can be regarded both as a simplification and an extension of methods by

for example Ghilezan and Kunčak [6], Kamareddine and Rahli [8] and Barendregt et al. [1,2] (Section 11.2).

As part of our simplification of a reducibility method, Lemma 4.1 states the “soundness” of our simple calculus based on the set of term  $\Lambda_c^{\beta\eta}$  w.r.t. the remains of our type interpretation based on a set  $s$  satisfying the saturation, variable and abstraction properties (see Definition 2.1).

**Lemma 4.1 (Soundness)** *If  $\bar{M} \in \Lambda_c^{\beta\eta}$ ,  $\text{fv}(\bar{M}) \setminus \{c\} = \{x_1, \dots, x_n\}$ , for all  $i \in \{1, \dots, n\}$ ,  $M_i \in s$  and  $s \in \text{SAT} \cup \text{VAR} \cup \text{ABS}$  then  $\bar{M}[x_1 := M_1, \dots, x_n := M_n] \in s$ .*

**Proof.** By induction on the structure of  $\bar{M}$ . □

Using lemma 4.1, we are now able to prove that each term in  $\Lambda_c^{\beta\eta}$  has  $\beta\eta$ -CR.

**Corollary 4.2**  $\Lambda_c^{\beta\eta} \subseteq \text{CR}^{\beta\eta}$ .

**Proof.** Let  $\bar{M} \in \Lambda_c^{\beta\eta}$  and  $\text{fv}(\bar{M}) \setminus \{c\} = \{x_1, \dots, x_n\}$ . By Lemma 2.2.viii.(a), 2.2.viii.(b) and 2.2.viii.(c),  $\text{CR}^{\beta\eta} \in \text{SAT} \cup \text{VAR} \cup \text{ABS}$  and  $x_1, \dots, x_n \in \text{CR}^{\beta\eta}$ . So by Lemma 4.1,  $\bar{M} \in \text{CR}^{\beta\eta}$ . □

Lemma 4.3 states that the freezing function associates to each term of the untyped  $\lambda$ -calculus (which does not contain the variable  $c$ ) a term in  $\Lambda_c^{\beta\eta}$ . This result is trivial because  $\Lambda_c^\beta \subset \Lambda_c^{\beta\eta}$ .

**Lemma 4.3** *If  $c \notin \text{fv}(M)$  then  $\Psi_c(M) \in \Lambda_c^{\beta\eta}$ .*

**Proof.** By Lemma 3.3,  $\Psi_c(M) \in \Lambda_c^\beta$ . Since  $\Lambda_c^\beta \subset \Lambda_c^{\beta\eta}$  then  $\Psi_c(M) \in \Lambda_c^{\beta\eta}$ . □

Here is another lemma containing needed technicalities:

**Lemma 4.4** *Let  $\bar{M}, \bar{N} \in \Lambda_c^{\beta\eta}$  and  $\bar{x} \in \text{Var}_c$ .*

- (i)  $\bar{M}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$ .
- (ii) *If  $\bar{M} \rightarrow_{\beta\eta}^* N$  then  $N \in \Lambda_c^{\beta\eta}$ .*
- (iii) *If  $\bar{M} \rightarrow_c^* N$  and  $c \notin \text{fv}(N)$  then  $\bar{M} \rightarrow_c^* \Psi_c(N)$ .*
- (iv) *There exists  $N$  such that  $c \notin \text{fv}(N)$  and  $\bar{M} \rightarrow_c^* N$ .*

**Proof.** Items i, iii and iv are by induction on the structure of  $\bar{M}$ . Item ii is by induction on the length of the derivation  $\bar{M} \rightarrow_{\beta\eta}^* N$ . □

The second property (4.4.ii) states that the terms in  $\Lambda_c^{\beta\eta}$   $\beta$ -reduce only terms in  $\Lambda_c^{\beta\eta}$ . The fourth property (4.4.iv) expresses that a term in  $\Lambda_c^{\beta\eta}$  can always be completely reduced w.r.t.  $\rightarrow_c^*$ .

Lemma 4.5 states that we can simulate the reduction of a term in  $\Lambda_c^{\beta\eta}$  from any of its (partially or totally) “unfrozen” versions.

- Lemma 4.5** (i) *If  $\bar{M}_1 \in \Lambda_c^{\beta\eta}$ ,  $\bar{M}_1 \rightarrow_{\beta\eta} N_1$  and  $\bar{M}_1 \rightarrow_c^* M_2$  then there exists  $N_2$  such that  $M_2 \rightarrow_{\beta\eta} N_2$  and  $N_1 \rightarrow_c^* N_2$ .*
- (ii) *If  $\bar{M}_1 \in \Lambda_c^{\beta\eta}$  such that  $\bar{M}_1 \rightarrow_{\beta\eta}^* N_1$  and  $\bar{M}_1 \rightarrow_c^* M_2$  then there exists  $N_2$  such that  $M_2 \rightarrow_{\beta\eta}^* N_2$  and  $N_1 \rightarrow_c^* N_2$ .*

**Proof.** *i.* By induction on the structure of  $M_1$ . *ii.* By Lemma 4.5.i. □

Lemma 4.6 is a key lemma of the simulation method of a reduction by some developments. It states that the reflexive and transitive closure of  $\rightarrow_{\beta\eta}$  is equal to the reflexive and transitive closure of  $\rightarrow_2$ .

**Lemma 4.6** *Let  $c \notin \text{fv}(M) \cup \text{fv}(N)$ , then  $M \rightarrow_{\beta\eta}^* N \iff M \rightarrow_2^* N$ .*

**Proof.**

$\Rightarrow$ ) Let  $M \rightarrow_{\beta\eta}^* N$ . We prove that  $M \rightarrow_2^* N$  by induction on the size of the reduction  $M \rightarrow_{\beta\eta}^* N$ .

$\Leftarrow$ ) Let  $M \rightarrow_2^* N$ . We prove that  $M \rightarrow_{\beta\eta}^* N$  by induction on the size of the derivation  $M \rightarrow_2^* N$ . □

It is then easy to deduce the confluence of the  $\beta\eta$ -developments.

**Lemma 4.7** (i) *If  $M \rightarrow_2 M_1$  and  $M \rightarrow_2 M_2$  then there exists  $M_3$  such that  $M_1 \rightarrow_2 M_3$  and  $M_2 \rightarrow_2 M_3$ .*

(ii) *If  $M \rightarrow_2^* M_1$  and  $M \rightarrow_2^* M_2$  then there exists  $M_3$  such that  $M_1 \rightarrow_2^* M_3$  and  $M_2 \rightarrow_2^* M_3$ .*

**Proof.**

*i* By definition, there exist  $P_1, P_2$  such that  $\Psi_c(M) \rightarrow_{\beta\eta}^* P_1$ ,  $\Psi_c(M) \rightarrow_{\beta\eta}^* P_2$ ,  $P_1 \rightarrow_c^* M_1$ ,  $P_2 \rightarrow_c^* M_2$  and  $c \notin \text{fv}(M) \cup \text{fv}(M_1) \cup \text{fv}(M_2)$ . By Lemma 4.3,  $\Psi_c(M) \in \Lambda_c^{\beta\eta}$ . So by Corollary 4.2, there exists  $P_3$  such that  $P_1 \rightarrow_{\beta\eta}^* P_3$  and  $P_2 \rightarrow_{\beta\eta}^* P_3$ . By Lemma 4.4.ii,  $P_1, P_2, P_3 \in \Lambda_c^{\beta\eta}$ . By lemma 4.4.iv, there exists  $M_3$  such that  $P_3 \rightarrow_c^* M_3$  and  $c \notin \text{fv}(M_3)$ . By Lemma 4.4.iii,  $P_1 \rightarrow_c^* \Psi_c(M_1)$  and  $P_2 \rightarrow_c^* \Psi_c(M_2)$ . By Lemma 4.5.ii, there exist  $Q_1, Q_2$  such that  $P_3 \rightarrow_c^* Q_1$ ,  $P_3 \rightarrow_c^* Q_2$ ,  $\Psi_c(M_1) \rightarrow_{\beta\eta}^* Q_1$  and  $\Psi_c(M_2) \rightarrow_{\beta\eta}^* Q_2$ . By Lemma 2.9.x,  $Q_1 \rightarrow_c^* M_3$  and  $Q_2 \rightarrow_c^* M_3$ . So  $M_1 \rightarrow_2 M_3$  and  $M_2 \rightarrow_2 M_3$ .

*ii* Easy by Lemma 4.7.i. □

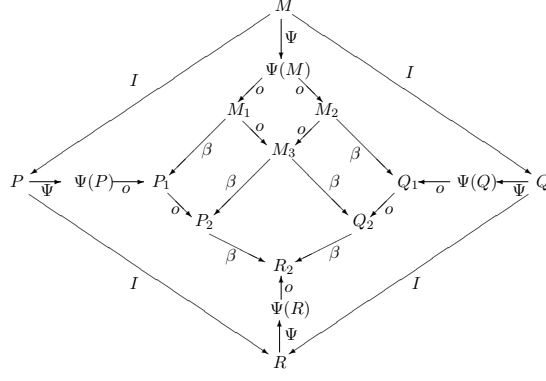
The confluence of the untyped  $\lambda$ -calculus w.r.t.  $\beta\eta$ -reduction is then proved using the confluence of the  $\beta\eta$ -developments and the equality between  $\rightarrow_{\beta\eta}^*$  and  $\rightarrow_2^*$ .

**Theorem 4.8**  $\Lambda = \text{CR}^{\beta\eta}$ .

**Proof.**  $\text{CR}^{\beta\eta} \subseteq \Lambda$  is trivial, we only prove  $\Lambda \subseteq \text{CR}^{\beta\eta}$ . Let  $M, M_1, M_2 \in \Lambda$  and  $c \notin \text{fv}(M)$  such that  $M \rightarrow_{\beta\eta}^* M_1$  and  $M \rightarrow_{\beta\eta}^* M_2$ . By Lemma 2.2.iii,  $c \notin \text{fv}(M_1) \cup \text{fv}(M_2)$ . By Lemma 4.6,  $M \rightarrow_2^* M_1$  and  $M \rightarrow_2^* M_2$ . By Lemma 4.7.ii, there exists  $M_3$  such that  $M_1 \rightarrow_2^* M_3$  and  $M_2 \rightarrow_2^* M_3$ . By definition  $c \notin \text{fv}(M_3)$ . By Lemma 4.6,  $M_1 \rightarrow_{\beta\eta}^* M_3$  and  $M_2 \rightarrow_{\beta\eta}^* M_3$ . □

## 5 Comparison and Related Work

In this section we compare our proposal to two semantic methods to prove confluence [6,11]. We also compare our developments to those of Tait and Martin-Löf. In


 Fig. 2. The method of Ghilezan and Kunčák for the confluence of  $\rightarrow_I$ 

this section and only in this section, we consider the confluence property w.r.t.  $\beta$ -reduction. In the Figures 2 and 3, an arrow labelled with  $c$ ,  $o$  or  $\beta$  stands for  $\rightarrow_c^*$ ,  $\rightarrow_o^*$  or  $\rightarrow_\beta^*$  respectively. An arrow labelled with  $\Psi$  stands for the application of the function with the same name to the term at the start of the arrow.

### 5.1 Comparison to Ghilezan+Kunčák and to Koletsos+Stravinos

In Figure 2 we recall the proof of Ghilezan and Kunčák [6] for the confluence of the untyped  $\lambda$ -calculus w.r.t.  $\beta$ -reduction. This proof, based on the embedding of the developments into  $\lambda_{\rightarrow}$ , uses the confluence w.r.t. another reduction  $\rightarrow_I$  (a development) whose transitive closure is equal to  $\rightarrow_\beta^*$ . The reduction  $\rightarrow_I$  is defined as  $\tau^{-1} \circ \rightarrow_\beta^* \circ \tau$  where:

- $\tau = \rightarrow_o^* \circ \Psi$
- $\rightarrow_o$  is the compatible closure of the rule  $(o) : f(g(\lambda x.M))N \rightarrow_o (\lambda x.M)N$
- $\Psi$  is defined on the  $\lambda$ -calculus by:  $\Psi(x) = x$ ,  $\Psi(\lambda x.M) = g(\lambda x.\Psi(M))$  and  $\Psi(MN) = f\Psi(M)\Psi(N)$ , where  $f$  and  $g$  are two constants (see Remark 2.3).

The relation  $\tau$  enables to “freeze” some  $\beta$ -redexes and the potential  $\beta$ -redexes (the other applications) of a term. (In fact,  $\tau$  does more, because  $\Psi$  does more by encapsulating the  $\lambda$ -abstractions using  $g$ . This technicality is needed by Ghilezan and Kunčák to prove the typability of a defined set of terms in  $\lambda_{\rightarrow}$ .) The reduction  $\tau^{-1}$  is similar to our erasure relation  $\rightarrow_c$  (see Definition 2.6) and to Krivine’s erasure function [12], which “unfreeze” the redexes in a term. By definition of  $M \rightarrow_I P$ , there exist  $M_1$  and  $P_1$  such that  $\Psi(M) \rightarrow_o^* M_1 \rightarrow_\beta^* P_1$  and  $\Psi(P) \rightarrow_o^* P_1$  (the left part of the figure). By definition of  $M \rightarrow_I Q$ , there exist  $M_2$  and  $Q_1$  such that  $\Psi(M) \rightarrow_o^* M_2 \rightarrow_\beta^* Q_1$  and  $\Psi(Q) \rightarrow_o^* Q_1$  (the right part of the figure). Because  $M_1$  can be different from  $M_2$ , a confluence lemma for the  $\rightarrow_o$  reduction and a commutation lemma for the reductions  $\rightarrow_o^*$  and  $\rightarrow_\beta^*$  are needed. The central part of the figure is due to the confluence of the terms typable in  $\lambda_{\rightarrow}$ . For example, as cited by Ghilezan and Kunčák, Koletsos [10] proved this result using a reducibility method. Hence, when combined with Koletsos’s proof of the confluence of  $\lambda_{\rightarrow}$ , Ghilezan and Kunčák’s method can be regarded as the combination of a reducibility method and a simulation method.

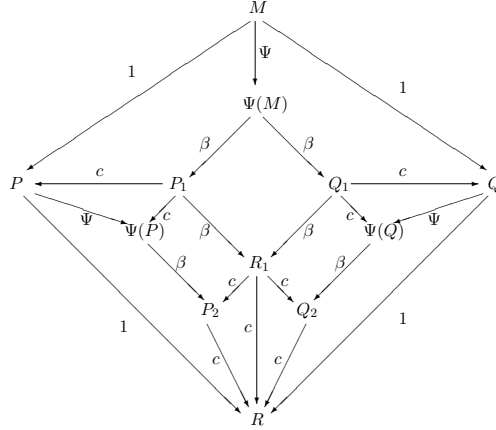
The reduction  $\rightarrow_I$  designed by Ghilezan and Kunčák [6] defines a development without specifying explicitly the set of redexes which are allowed to be reduced (as done for example by Barendregt et al. [1] which differs from approaches like those of Barendregt [2] or Hindley [7]). Let us consider the reduction  $M \rightarrow_I P$  (unfolded above). First, the function  $\Psi$  blocks all the redexes in  $M$ . Then  $\rightarrow_o^*$  enables to select the set of redexes which are allowed to be reduced in  $M$  without explicitly naming them, by unblocking some redexes in  $\Psi(M)$ . The reduction  $M_1 \rightarrow_\beta^* P_1$  reduces the allowed redexes and their residuals. And finally in  $\Psi(P) \rightarrow_o^* P_1$ , the reduction  $\rightarrow_o^*$  selects the set of residuals of the set of redexes in  $M_1$  without naming them.

This way of dealing with occurrences of redexes is simple and sufficient enough to prove the Church-Rosser property. However, in some other works, occurrences of redexes are handled in a complicated way, as for example in the work of Krivine [12] or Koletsos and Stavrinou [11]. In these works, occurrences are treated intuitively and not formally. So, the work turns out to be much more complicated than it seems when one wants to “formally” prove the results (see [8]), or even just formally defining the developments. Ghilezan and Kunčák [6] do not face the same problem. The reduction  $\rightarrow_o^*$  enables to unblock a certain set of redexes without explicitly specifying the set of unblocked redexes. In the work of Ghilezan and Kunčák, as in the work of Barendregt et al. [1] for example, a development of a term is defined without explicit control on the set of occurrences of reduced redexes, which is not needed.

Although Ghilezan and Kunčák [6] consider a simpler definition of developments than the “common” one, the scheme of their proof method is exactly the one followed by Koletsos and Stavrinou [11]. Koletsos and Stavrinou consider the following “common” definition of developments: there exists a development from  $M$  to  $N$  iff  $\langle M, s_1 \rangle \rightarrow_d^* \langle N, s_2 \rangle$  where  $s_1$  is a set of redexes in  $M$  and  $s_2$  is the set of residuals of  $s_1$  in  $N$  (where  $\rightarrow_d^*$  is a new (complex) reduction relation based on  $\rightarrow_\beta^*$ ). Their proof of the confluence of developments uses, among other things, the following claim: if  $\langle M, s_1 \rangle \rightarrow_d^* \langle N, s_2 \rangle$  then there exists  $s_4$  such that  $\langle M, s_1 \cup s_3 \rangle \rightarrow_d^* \langle N, s_2 \cup s_4 \rangle$ , where  $s_3$  is a set of redexes of  $M$ . It is useful to prove that if  $\langle M, s_1 \rangle \rightarrow_d^* \langle M_1, s'_1 \rangle$  and  $\langle M, s_2 \rangle \rightarrow_d^* \langle M_2, s'_2 \rangle$  then there exist  $s''_1$  and  $s''_2$  such that  $\langle M, s_1 \cup s_2 \rangle \rightarrow_d^* \langle M_1, s'_1 \cup s''_2 \rangle$  and  $\langle M, s_2 \cup s_1 \rangle \rightarrow_d^* \langle M_2, s'_2 \cup s''_1 \rangle$ . This corresponds to the proof of the confluence of  $\rightarrow_o^*$  of Ghilezan and Kunčák, which is useful to get the reductions  $(\Psi(M) \rightarrow_o^* M_1 \rightarrow_o^* M_3 \rightarrow_\beta^* P_2$  and  $\Psi(P) \rightarrow_o^* P_1 \rightarrow_o^* P_2)$  and  $(\Psi(M) \rightarrow_o^* M_2 \rightarrow_o^* M_3 \rightarrow_\beta^* Q_2$  and  $\Psi(Q) \rightarrow_o^* Q_1 \rightarrow_o^* Q_2)$ . Ghilezan and Kunčák emphasise this more strongly than Koletsos and Stavrinou.

The differences between the method of Ghilezan and Kunčák and that of Barendregt et al include:

- Ghilezan and Kunčák do not use the finiteness of developments when Barendregt et al. do;
- Ghilezan and Kunčák base their result on a well known result (the confluence of the simply typed  $\lambda$ -terms) to give a simple proof of the confluence of developments when Barendregt et al. have to prove everything;
- Ghilezan and Kunčák do not really introduce new terms when Barendregt et al. do (the underlined terms are introduced to prove the confluence of developments).


 Fig. 3. Our method for the confluence of  $\rightarrow_1$ 

Barendregt et al. also give a definition of developments without explicitly naming an occurrence of a redex (no set of occurrences is defined), introducing among other things, a second abstraction  $\underline{\lambda}$ . There exists a simple correspondence between the calculus with this second abstraction and the marked calculus introduced by Krivine and reused in this paper and in many other works [12,6,11,8].

In Figure 3 we draw the diagram of our method to prove the confluence of the  $\lambda$ -calculus. By definition of  $M \rightarrow_1 P$  (Definition 2.7), there exists  $P_1$  such that  $\Psi(M) \rightarrow_{\beta}^* P_1$  and  $P_1 \rightarrow_c^* P$  (the left part of the figure). By definition of  $M \rightarrow_1 Q$ , there exists  $Q_1$  such that  $\Psi(M) \rightarrow_{\beta}^* Q_1$  and  $Q_1 \rightarrow_c^* Q$  (the right part of the figure). Moreover  $P_1 \rightarrow_c^* \Psi_c(P)$  and  $Q_1 \rightarrow_c^* \Psi_c(Q)$ . So, because  $P_1$  and  $\Psi_c(P)$  might be different (as for  $Q_1$  and  $\Psi_c(Q)$ ), as Ghilezan and Kunčák [6], we need a commutation result for the reductions  $\rightarrow_{\beta}^*$  and  $\rightarrow_c^*$ . Then, the whole diagram commutes because  $P_2$ ,  $R_1$  and  $Q_2$  reduce to the same term. Like in Figure 2, the central part is due to the confluence of a defined set of terms (typable in  $\lambda_{\rightarrow}$  for Ghilezan and Kunčák and typable in  $\mathcal{D}$  in our case even though we do not use this fact).

Our method is also based on some kind of simple developments, where first, all the  $\beta$ -redexes are left unblocked and where all the potential  $\beta$ -redexes (all the other applications) are blocked. In this paper we define two simple developments:  $\rightarrow_1$  for the  $\beta$  case and  $\rightarrow_2$  for the  $\beta\eta$  case. In that way, we do not need the reduction  $\rightarrow_o^*$  to unblock some redexes in order to perform some reductions. But, it does not seem possible to get rid of the work done by this reduction. Indeed, our choice implies the introduction of some other material which turns out to be similar to the reduction  $\rightarrow_o^*$ . Both methods need the introduction of some similar material, but not at the same places. The reduction  $\rightarrow_o^*$  is used by Ghilezan and Kunčák to unblock some redexes in order to enable some reductions whereas we use the reduction  $\rightarrow_c^*$  to unblock some redexes which become blocked after some reductions.

As we can see in these two figures, because the occurrences of redexes are not explicitly taken into consideration, the function  $\Psi$  (which enables to embed a term in a typed term, by blocking redexes or potential redexes) needs to either block all the redexes of a term or to leave them all unblocked. If all the redexes are blocked by

$\Psi$ , a reduction such as  $\rightarrow_o$  is needed before being able to perform some reductions (see Figure 2). In this case some technical results are needed such as the confluence of  $\rightarrow_o$ . In the other case ( $\Psi$  leaves all the redexes unlocked), because a term whose redexes are all unblocked does not necessarily reduce to a term whose redexes are all unblocked, some technical results on a reduction such as  $\rightarrow_o$  are also needed as we explained above (see Figure 3).

Finally, although our work derives from the one done by Koletsos and Stavrinou [11] and Kamareddine and Rahli [8], it turned out that it is also a simplification and generalisation of the work done by Ghilezan and Kunčák [6] and Barendregt et al. [1]. Because the work we achieved is more similar to the one of Ghilezan and Kunčák, we adapted some of our notations to theirs and focused our comparisons with the related work to their work.

Thereby, the two improvements of the present article can be regarded as the simplification of the work done by Ghilezan and Kunčák [6] by getting rid of all the type machinery and the extension of the defined method to the  $\beta\eta$ -reduction.

As explained above, the main lines of our proof are:

- the definition of some simple developments;
- the proof of the confluence of a simple calculus w.r.t. the considered reduction ( $\beta$  and  $\beta\eta$ ) using a method based on a simplification of a reducibility method;
- the proof of the confluence of the defined developments;
- the proof of the equality between the reflexive and transitive closure of the developments and the reflexive and transitive closure of the considered reduction;
- the proof of CR of the untyped  $\lambda$ -calculus w.r.t. a given reduction using a simulation of the considered reduction by developments.

## 5.2 Comparison to Tait and Martin-Löf

Tait and Martin-Löf's syntactic proof [13,2] (and its extensions, by for example Takahashi [15]) that the untyped  $\lambda$ -calculus satisfies the Church-Rosser property is the simplest so far. Our method started from the semantic framework when we attempted to simplify and generalise existing semantic proofs. To our surprise, our simplification and generalisation of such semantic proofs led to our method which did not need types anymore. Hence, the type interpretation and the reducibility argument are no longer used in our article. This way our method moved from the semantic camp to the syntactic one. Nonetheless, our method can still be projected into a semantic argument method (something that does not hold for methods like those of Tait and Martin-Löf and Takahashi). In this way, we consider our work to be a bridge between the syntactic and semantic methods. There is another notable difference to our method: our developments allow (strictly) more reductions than those of Takahashi (for both the  $\beta$  and  $\beta\eta$  cases) as we establish in this section.

**Definition 5.1** [[15]] Let  $r \in \{\beta, \beta\eta\}$ . We define  $\Rightarrow_r$  as follows:

- $M \Rightarrow_r M$
- $\lambda x.M \Rightarrow_r \lambda x.N$  if  $M \Rightarrow_r N$
- $MN \Rightarrow_r M'N'$  if  $M \Rightarrow_r M'$  and  $N \Rightarrow_r N'$

- $(\lambda x.M)N \Rightarrow_r M'[x := N']$  if  $M \Rightarrow_r M'$  and  $N \Rightarrow_r N'$
- $\lambda x.Mx \Rightarrow_{\beta\eta} N$  if  $x \notin \text{fv}(M)$  and  $M \Rightarrow_{\beta\eta} N$

**Lemma 5.2** (i) If  $M \Rightarrow_{\beta} N$  or  $M \Rightarrow_{\beta\eta} N$  then  $\text{fv}(N) \subseteq \text{fv}(M)$ .

(ii) Let  $M, N$  such that  $c \notin \text{fv}(M) \cup \text{fv}(N)$ . If  $M \Rightarrow_{\beta} N$  then  $M \rightarrow_1 N$ .

(iii) Let  $M, N$  such that  $c \notin \text{fv}(M) \cup \text{fv}(N)$ . If  $M \Rightarrow_{\beta\eta} N$  then  $M \rightarrow_2 N$ .

**Proof.** ii. Let  $M \Rightarrow_{\beta} N$ . The proof is by induction on the size of the derivation of  $M \Rightarrow_{\beta} N$  and then by case on the last rule of the derivation.

iii. Let  $M \Rightarrow_{\beta\eta} N$ . The proof is by induction on the size of the derivation of  $M \Rightarrow_{\beta\eta} N$  and then by case on the last rule of the derivation.  $\square$

**Remark 5.3** (i) We have  $M = (\lambda x.xx)((\lambda z.z)y) \rightarrow_1 y((\lambda z.z)y)$  (where  $c \notin \{x, y, z\}$ ) because  $\Psi_c(M) = (\lambda x.cxx)((\lambda z.z)y) \rightarrow_{\beta} c((\lambda z.z)y)((\lambda z.z)y) \rightarrow_{\beta} cy((\lambda z.z)y) \rightarrow_c y((\lambda z.z)y)$ . But, we do not have  $M \Rightarrow_{\beta} y((\lambda z.z)y)$ .

(ii) We have  $M = \lambda x.y((\lambda z.z)x) \rightarrow_2 y$  (where  $c \notin \{x, y, z\}$ ) because  $\Psi_c(M) = \lambda x.cy((\lambda z.z)x) \rightarrow_{\beta} \lambda x.cyx \rightarrow_{\eta} cy \rightarrow_c y$ . But, we do not have  $M \Rightarrow_{\beta\eta} y$ .

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