Refining the Barendregt Cube with Parameters

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The Low Level approach of functions

- Historically, functions have long been treated as a kind of meta-objects.
- Function *values* have always been important, but abstract functions have not been recognised in their own right until the third of the 20th century.
- In the *low level approach* or *operational* view on functions, there are no functions as such, but only function values.
- E.g., the sine-function, is always expressed together with a value: $\sin(\pi)$, $\sin(x)$ and properties like: $\sin(2x) = 2\sin(x)\cos(x)$.
- It has long been usual to call $f(x)$—and not f—the function and this is still the case in many introductory mathematics courses.

The revolution of treating functions as first class citizens

- In the nowadays accepted view on functions, they are 'first class citizens'.
- Abstraction and application form the basis of the λ-calculus and type theory.
- This is rigid and does not represent the development of logic in 20th century.
- Frege and Russell's conceptions of functional abstraction, instantiation and application do not fit well with the λ-calculus approach.
- In *Principia Mathematica* [Whitehead and Russell, 1910^1, 1927^2]: If, for some a, there is a proposition ϕa, then there is a function $\phi\hat{x}$, and vice versa.
- The function ϕ is not a separate entity but always has an argument.

λ-calculus does not fully represent functionalisation

1. Abstraction from a subexpression $2 + 3 \mapsto x + 3$

2. Function construction $x + 3 \mapsto \lambda.x + 3$

3. Application construction $(\lambda x.(x + 3))2$

4. Concretisation to a subexpression $(\lambda x.(x + 3))2 \to 2 + 3$

- Cannot identify the original term from which a function has been abstracted.

$$\mathtt{let}\ \mathtt{add}_2 = (\lambda x.x + 2)\ \mathtt{in}\ \mathtt{add}_2(x) + \mathtt{add}_2(y)$$

- cannot abstract only half way: $x + 3$ is not a function, $\lambda x.x + 3$ is.

- cannot apply $x + 3$ to an argument: $(x + 3)2$ does not evaluate to 2+3.

Parameters: What and Why

- we speak about *functions with parameters* when referring to functions with variable values in the *low-level* approach. The x in $f(x)$ is a parameter.
- Parameters enable the same expressive power as the high-level case, while allowing us to stay at a lower order. E.g. first-order with parameters versus second-order without [Laan and Franssen, 2001].
- Desirable properties of the lower order theory (decidability, easiness of calculations, typability) can be maintained, without losing the flexibility of the higher-order aspects.
- This low-level approach is still worthwile for many exact disciplines. In fact, both in logic and in computer science it has certainly not been wiped out, and for good reasons.

Automath

- The first tool for mechanical representation and verification of mathematical proofs, AUTOMATH, has a parameter mechanism.

- The representation of a mathematical text in AUTOMATH consists of a finite list of *lines* where every line has the following format:

$$x_1 : A_1, \ldots, x_n : A_n \vdash g(x_1, \ldots, x_n) = t : T.$$

  Here g is a new name, an abbreviation for the expression t of type T and $x_1, \ldots, x_n$ are the parameters of g, with respective types $A_1, \ldots, A_n$.

- Each line introduces a new definition which is inherently parametrised by the variables occurring in the context needed for it.

- Developments of ordinary mathematical theory in AUTOMATH [Benthem Jutting, 1977] revealed that this combined definition and parameter mechanism is vital for keeping proofs manageable and sufficiently readable for humans.

The Barendregt Cube

- $\mathcal{T}_P ::= \mathcal{V} \mid \boldsymbol{S} \mid \mathcal{T}_P\mathcal{T}_P \mid \lambda\mathcal{V}{:}\mathcal{T}_P.\mathcal{T}_P \mid \Pi\mathcal{V}{:}\mathcal{T}_P.\mathcal{T}_P$
- $\mathcal{V}$ is a set of variables and $\boldsymbol{S} = \{*, \Box\}$.

(axiom) $$\langle\rangle \vdash * : \Box$$

(start) $$\frac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A} \quad x \notin \mathrm{DOM}(\Gamma)$$

(weak) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x{:}C \vdash A : B} \quad x \notin \mathrm{DOM}(\Gamma)$$

(Π) $$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.B) : s_2} \quad (s_1, s_2) \in \boldsymbol{R}$$

(λ) $$\frac{\Gamma, x{:}A \vdash b : B \qquad \Gamma \vdash (\Pi x{:}A.B) : s}{\Gamma \vdash (\lambda x{:}A.b) : (\Pi x{:}A.B)}$$

(appl) $$\frac{\Gamma \vdash F : (\Pi x{:}A.B) \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x{:=}a]}$$

(conv) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s \qquad B =_\beta B'}{\Gamma \vdash A : B'}$$

Different type formation conditions

- $$(\Pi) \qquad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.B) : s_2} \; (s_1, s_2) \in \boldsymbol{R}$$

- $(\Box, *)$ takes care of polymorphism. $\lambda 2$ is weakest on cube satisfying this.

- $(\Box, \Box)$ takes care of type constructors. $\lambda\underline{\omega}$ is weakest on cube satisfying this.

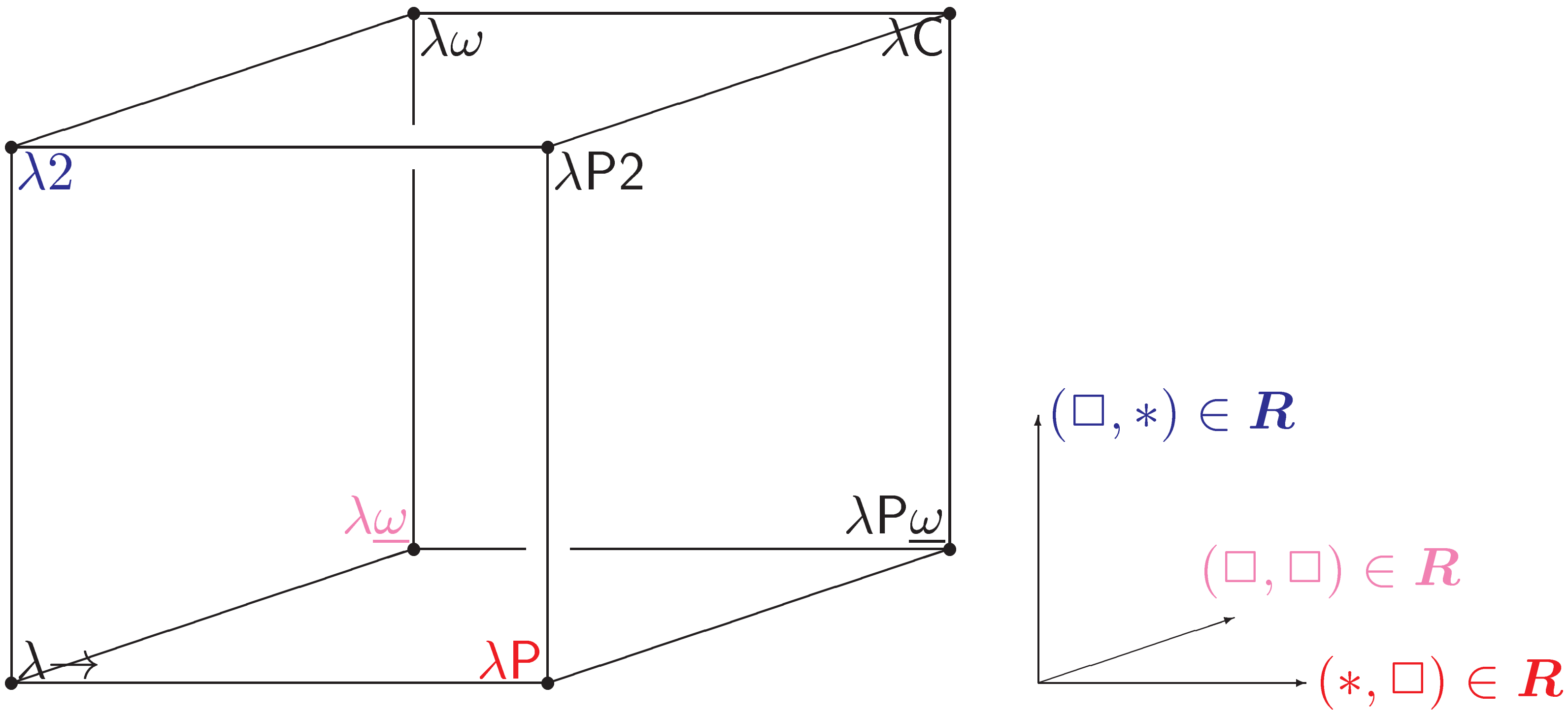
- $(*, \Box)$ takes care of term dependent types. λP is weakest on cube satisfying this.

$\lambda\!\to$	$(*, *)$			
$\lambda 2$	$(*, *)$	$(\Box, *)$		
λP	$(*, *)$		$(*, \Box)$	
$\lambda\underline{\omega}$	$(*, *)$			$(\Box, \Box)$
λP2	$(*, *)$	$(\Box, *)$	$(*, \Box)$	
$\lambda\omega$	$(*, *)$	$(\Box, *)$		$(\Box, \Box)$
λP$\underline{\omega}$	$(*, *)$		$(*, \Box)$	$(\Box, \Box)$
λC	$(*, *)$	$(\Box, *)$	$(*, \Box)$	$(\Box, \Box)$

Systems of the Barendregt Cube

System	Rel. system	Names, references
$\lambda{\to}$	λ^τ	simply typed λ-calculus; [Church, 1940], [Barendregt, 1984] (Appendix A), [Hindley and Seldin, 1986] (Chapter 14)
$\lambda 2$	F	second order typed λ-calculus; [Girard, 1972], [Reynolds, 1974]
λP	AUT-QE	[Bruijn, 1968]
	LF	[Harper et al., 1987]
$\lambda P2$		[Longo and Moggi, 1988]
$\lambda\underline{\omega}$	POLYREC	[Renardel de Lavalette, 1991]
$\lambda\omega$	Fω	[Girard, 1972]
λC	CC	Calculus of Constructions; [Coquand and Huet, 1988]

The Barendregt Cube



LF

- LF (see [Harper et al., 1987]) is often described as λP of the Barendregt Cube.
- [Geuvers, 1993] shows that the use of the Π-formation rule $(*, \Box)$ is very restricted in the practical use of LF.
- This use is in fact based on a parametric construct rather than on Π-formation.
- We will find a more precise position of LF on the Cube (between $\lambda{\to}$ and λP).

ML

- We only consider an explicit version of a subset of ML.
- In ML, One can define the polymorphic identity by:

$$\mathtt{Id}(\alpha{:}*) = (\lambda x{:}\alpha.x) : (\alpha \to \alpha) \tag{1}$$

- But in ML, it is not possible to make an explicit λ-abstraction over $\alpha : *$ by:

$$\mathtt{Id} = (\lambda\alpha{:}*.\lambda x{:}\alpha.x) : (\Pi\alpha{:}*.\alpha \to \alpha) \tag{2}$$

- The type $\Pi\alpha{:}*.\alpha \to \alpha$ does not belong to the language of ML and hence the λ-abstraction of equation (2) is not possible in ML.

ML

- Therefore, we can state that ML does not have a Π-formation rule $(\Box, *)$.
- Nevertheless, ML has some parameter mechanism (α parameter of `Id`)
- ML has limited access to the rule $(\Box, *)$ enabling equation (1) to be defined.
- ML's type system is none of those of the eight systems of the Cube.
- We place the type system of ML on our refined Cube (between $\lambda 2$ and $\lambda \underline{\omega}$).

Extending the Cube with parametric constructs

- Parametric constructs are $c(b_1, \ldots, b_n)$ with $b_1, \ldots, b_n$ terms of certain types.

- $\mathcal{T}_P \ ::= \ \mathcal{V} \mid \boldsymbol{S} \mid \mathcal{C}(\underbrace{\mathcal{T}_{P_1}, \ldots \mathcal{T}_{P_n}}_{n \geq 0}) \mid \mathcal{T}_P \mathcal{T}_P \mid \lambda \mathcal{V}{:}\mathcal{T}_P.\mathcal{T}_P \mid \Pi \mathcal{V}{:}\mathcal{T}_P.\mathcal{T}_P$

  $\mathcal{C}$ is a set of constants, $b_1, \ldots, b_n$ are called the *parameters* of $c(b_1, \ldots, b_n)$.

- $\mathbf{R}$ allows several kinds of Π-constructs. We also use a set $\mathbf{P}$ of (s_1, s_2) where $s_1, s_2 \in \{*, \Box\}$ to allow several kinds of parametric constructs.

- $(s_1, s_2) \in \mathbf{P}$ means that we allow parametric constructs $c(b_1, \ldots, b_n) : A$ where $b_1, \ldots, b_n$ have types $B_1, \ldots, B_n$ of sort s_1, and A is of type s_2.

- If both $(*, s_2) \in \mathbf{P}$ and $(\Box, s_2) \in \mathbf{P}$ then combinations of parameters allowed. For example, it is allowed that B_1 has type $*$, whilst B_2 has type $\Box$.

The Cube with parametric constants

- Let $\mathbf{R}$, $\mathbf{P} \subseteq \{(*,*),(*,\Box),(\Box,*),(\Box,\Box)\}$ containing $(*,*)$.

- $\lambda\mathbf{RP} = \lambda\mathbf{R}$ and the two rules **($\vec{\mathsf{C}}$-weak)** and **($\vec{\mathsf{C}}$-app)**:

$$\frac{\Gamma \vdash b : B \quad \Gamma, \Delta_i \vdash B_i : s_i \quad \Gamma, \Delta \vdash A : s}{\Gamma, c(\Delta) : A \vdash b : B} \ (s_i, s) \in \mathbf{P}, c \text{ is } \Gamma\text{-fresh}$$

$$\frac{\begin{array}{lcll} \Gamma_1, c(\Delta){:}A, \Gamma_2 & \vdash & b_i{:}B_i[x_j{:=}b_j]_{j=1}^{i-1} & (i = 1, \ldots, n) \\ \Gamma_1, c(\Delta){:}A, \Gamma_2 & \vdash & A : s & (\text{if } n = 0) \end{array}}{\Gamma_1, c(\Delta){:}A, \Gamma_2 \vdash c(b_1, \ldots, b_n) : A[x_j{:=}b_j]_{j=1}^{n}}$$

$\Delta \equiv x_1{:}B_1, \ldots, x_n{:}B_n$.
$\Delta_i \equiv x_1{:}B_1, \ldots, x_{i-1}{:}B_{i-1}$

Properties of the Refined Cube

- Correctness of types) If $\Gamma \vdash A : B$ then ($B \equiv \Box$ or $\Gamma \vdash B : S$ for some sort S).
- (Subject Reduction SR) If $\Gamma \vdash A : B$ and $A \twoheadrightarrow_\beta A'$ then $\Gamma \vdash A' : B$
- (Strong Normalisation) For all $\vdash$-legal terms M, we have $\mathrm{SN}_{\twoheadrightarrow_\beta}(M)$. I.e. M is strongly normalising with respect to $\twoheadrightarrow_\beta$.
- Other properties such as Uniqueness of types and typability of subterms hold.
- $\lambda\mathbf{RP}$ is the system which has Π-formation rules $\boldsymbol{R}$ and parameter rules $\boldsymbol{P}$.
- Let $\lambda\mathbf{RP}$ parametrically conservative (i.e., $(s_1, s_2) \in \boldsymbol{P}$ implies $(s_1, s_2) \in \boldsymbol{R}$).
  - The parameter-free system $\lambda\mathbf{R}$ is at least as powerful as $\lambda\mathbf{RP}$.
  - If $\Gamma \vdash_{\mathbf{RP}} a : A$ then $\{\Gamma\} \vdash_{\mathbf{R}} \{a\} : \{A\}$.

Example

- $\boldsymbol{R} = \{(*,*)\}$

  $\boldsymbol{P}_1 = \emptyset \qquad \boldsymbol{P}_2 = \{(*,*)\} \qquad \boldsymbol{P}_3 = \{(*,\Box)\} \qquad \boldsymbol{P}_4 = \{(*,*),(*,\Box)\}$

  All $\lambda\boldsymbol{R}\boldsymbol{P}_i$ for $1 \leq i \leq 4$ with the above specifications are all equal in power.

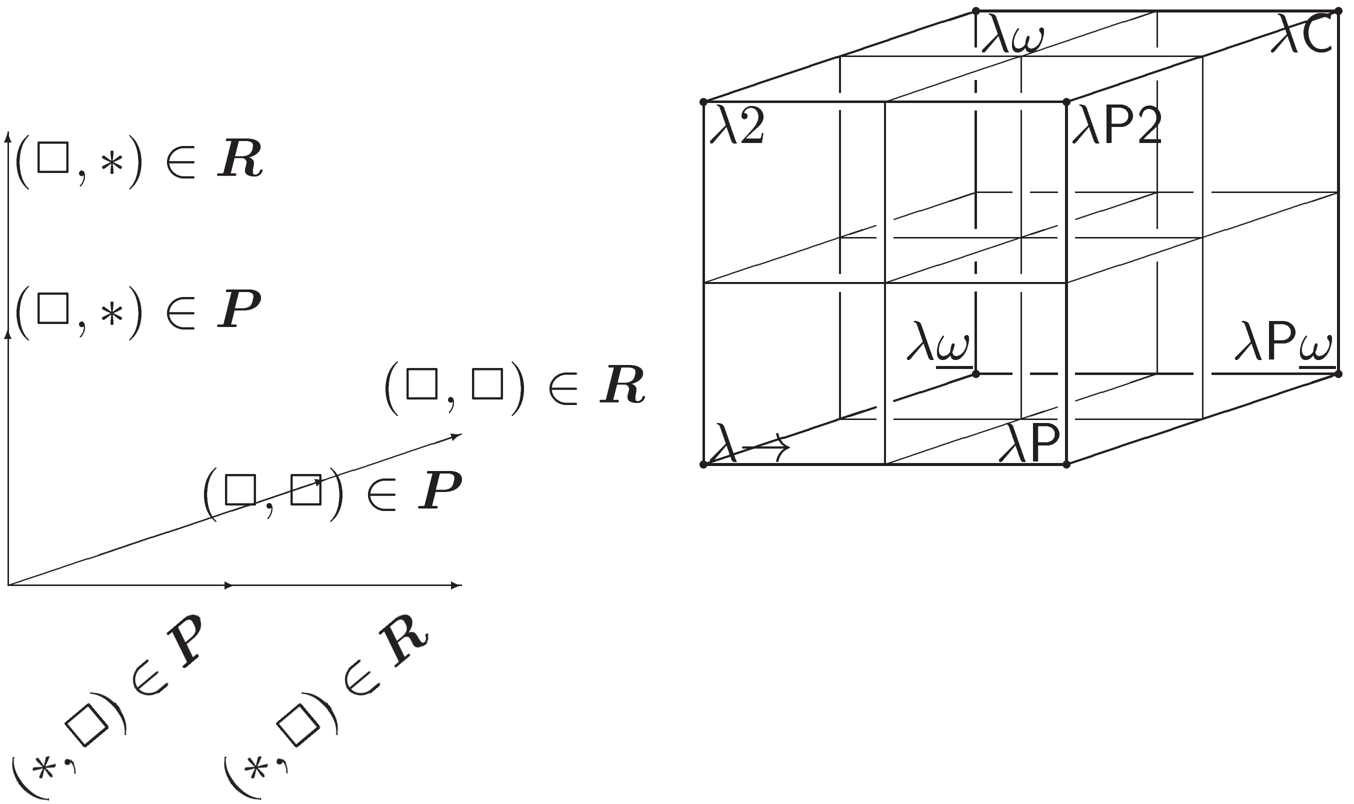
- $\boldsymbol{R}_5 = \{(*,*)\} \qquad \boldsymbol{P}_5 = \{(*,*),(*,\Box)\}$.

  $\lambda\!\to\; < \; \lambda\boldsymbol{R}_5\boldsymbol{P}_5 \; < \; \lambda$P: we can to talk about predicates:

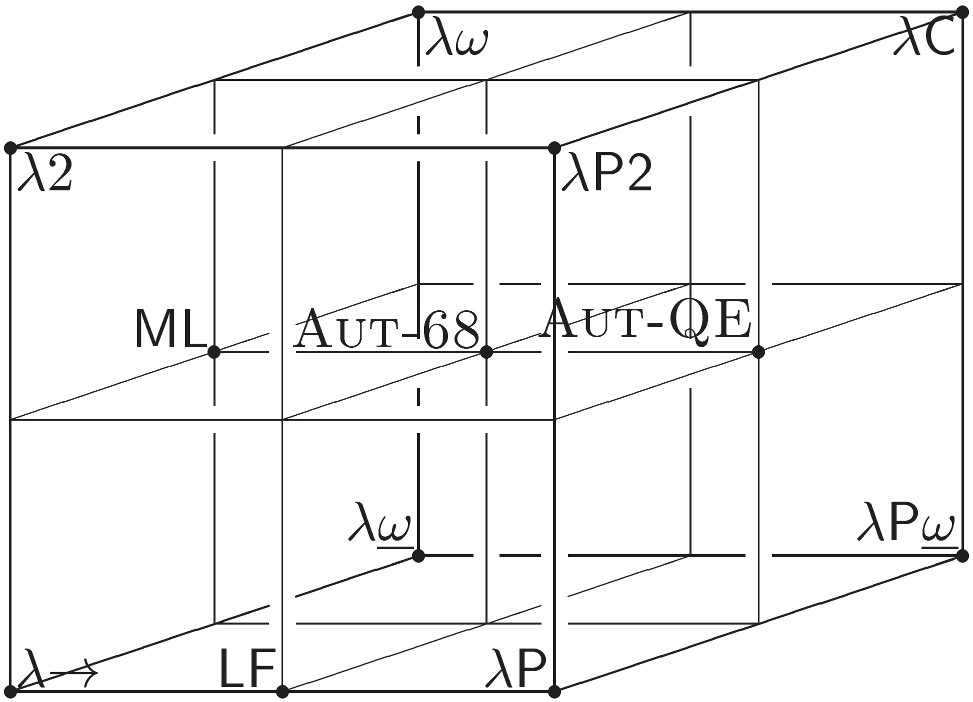
$$
\begin{array}{rcl}
\alpha & : & *, \\
\mathtt{eq}(\mathtt{x}{:}\alpha, \mathtt{y}{:}\alpha) & : & *, \\
\mathtt{refl}(\mathtt{x}{:}\alpha) & : & \mathtt{eq}(\mathtt{x},\mathtt{x}), \\
\mathtt{symm}(\mathtt{x}{:}\alpha, \mathtt{y}{:}\alpha, \mathtt{p}{:}\mathtt{eq}(\mathtt{x},\mathtt{y})) & : & \mathtt{eq}(\mathtt{y},\mathtt{x}), \\
\mathtt{trans}(\mathtt{x}{:}\alpha, \mathtt{y}{:}\alpha, \mathtt{z}{:}\alpha, \mathtt{p}{:}\mathtt{eq}(\mathtt{x},\mathtt{y}), \mathtt{q}{:}\mathtt{eq}(\mathtt{y},\mathtt{z})) & : & \mathtt{eq}(\mathtt{x},\mathtt{z})
\end{array} .
$$

  eq not possible in $\lambda\!\to$.

The refined Barendregt Cube



LF, ML, AUT-68, and AUT-QE in the refined Cube



LF

- [Geuvers, 1993] initially described LF as the system λP of the Cube.
  However, the Π-formation rule $(*, \Box)$ is restricted in most applications of LF.

- [Geuvers, 1993] splits λ-formation in two (LF – (λ_P) is called LF$^-$):

$$(\lambda_0)\frac{\Gamma, x{:}A \vdash M : B \qquad \Gamma \vdash \Pi x{:}A.B : *}{\Gamma \vdash \lambda_0 x{:}A.M : \Pi x{:}A.B} \qquad (\lambda_0 x{:}A.M)N \to_{\beta_0} M[x{:=}N]$$

$$(\lambda_P)\frac{\Gamma, x{:}A \vdash M : B \qquad \Gamma \vdash \Pi x{:}A.B : \Box}{\Gamma \vdash \lambda_P x{:}A.M : \Pi x{:}A.B} \qquad (\lambda_P x{:}A.M)N \to_{\beta_P} M[x{:=}N]$$

- If $M : *$ or $M : A : *$ in LF, then the β_P-normal form of M contains no λ_P;

- If $\Gamma \vdash_{\mathsf{LF}} M : A$, and Γ, M, A do not contain a λ_P, then $\Gamma \vdash_{\mathsf{LF}^-} M : A$;

- If $\Gamma \vdash_{\mathsf{LF}} M : A(: *)$, all in β_P-normal form, then $\Gamma \vdash_{\mathsf{LF}^-} M : A(: *)$.

LF

- Hence: the only need for a type $\Pi x{:}A.B : \Box$ is to declare a variable in it.

- This is only done when the Propositions-As-Types principle PAT is applied during the construction of the type of the operator Prf as follows:

$$\frac{\texttt{prop}{:}* \vdash \texttt{prop}{:}\, * \qquad \texttt{prop}{:}*, \alpha{:}\texttt{prop} \vdash *{:}\Box}{\texttt{prop}{:}* \vdash (\Pi\alpha{:}\texttt{prop}.*) : \Box}.$$

- In LF, this is the only point where the Π-formation rule $(*, \Box)$ is used.

- No λ_P-abstractions are used. Prf is only used when applied to term $p{:}\texttt{prop}$.

- Hence, the practical use of LF would not be restricted if we present Prf in a parametric form, and use $(*, \Box)$ as a parameter instead of a Π-formation rule.

- This puts LF in between $\lambda\!\rightarrow$ and λP in the Refined Cube.

Logicians versus mathematicians and induction over numbers

- Logician uses `ind: Ind` as proof term for an application of the induction axiom. The type `Ind` can only be described in $\lambda\boldsymbol{R}$ where $\boldsymbol{R} = \{(*,*),(*,\Box),(\Box,*)\}$:

$$\texttt{Ind} = \Pi p{:}(\mathbb{N}{\to}*).p0{\to}(\Pi n{:}\mathbb{N}.\Pi m{:}\mathbb{N}.pn{\to}Snm{\to}pm){\to}\Pi n{:}\mathbb{N}.pn \qquad (3)$$

- Mathematician uses `ind` only with $P : \mathbb{N}{\to}*$, $Q : P0$ and $R : (\Pi n{:}\mathbb{N}.\Pi m{:}\mathbb{N}.Pn{\to}Snm{\to}Pm)$ to form a term $(\texttt{ind}PQR){:}(\Pi n{:}\mathbb{N}.Pn)$.

- The use of the induction axiom by the mathematician is better described by the parametric scheme (p, q and r are the *parameters* of the scheme):

$$\texttt{ind}(p{:}\mathbb{N}{\to}*, q{:}p0, r{:}(\Pi n{:}\mathbb{N}.\Pi m{:}\mathbb{N}.pn{\to}Snm{\to}pm)) : \Pi n{:}\mathbb{N}.pn \qquad (4)$$

- The logician's type `Ind` is not needed by the mathematician and the types that occur in 4 can all be constructed in $\lambda\boldsymbol{R}$ with $\boldsymbol{R} = \{(*,*)(*,\Box)\}$.

Logicians versus mathematicians and induction over numbers

- Mathematician: only *applies* the induction axiom and doesn't need to know the proof-theoretical backgrounds.

- A logician develops the induction axiom (or studies its properties).

- $(\Box, *)$ is not needed by the mathematician. It is needed in logician's approach in order to form the Π-abstraction $\Pi p{:}(\mathbb{N} \to *). \cdots)$.

- Consequently, the type system that is used to describe the mathematician's use of the induction axiom can be weaker than the one for the logician.

- Nevertheless, the parameter mechanism gives the mathematician limited (but for his purposes sufficient) access to the induction scheme.

Conclusions

- Parameters enable the same expressive power as the high-level case, while allowing us to stay at a lower order. E.g. first-order with parameters versus second-order without [Laan and Franssen, 2001].
- Desirable properties of the lower order theory (decidability, easiness of calculations, typability) can be maintained, without losing the flexibility of the higher-order aspects.
- Parameters enable us to find an exact position of type systems in the generalised framework of type systems.
- Parameters describe the difference between *developers* and *users* of systems.

Future Work

- The above only explained the extension of the Cube with parametric constants.
- A larger extension can be made to the more generalised Pure Type Systems.
- We can add definitions and parametric definitions to the Cube and Pure Type systems. This can be found in [Laan, 1997].

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