Simplified Reducibility Proofs of Church-Rosser for β- and $\beta\eta$-reduction

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Abstract

The simplest proofs of the Church Rosser Property are usually done by the syntactic method of parallel reduction. On the other hand, reducibility is a semantic method which has been used to prove a number of properties in the λ-calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we concentrate on simplifying a semantic method based on reducibility for proving Church-Rosser for both β- and $\beta\eta$-reduction. Interestingly, this simplification results in a syntactic method (so the semantic aspect disappears) which is nonetheless projectable into a semantic method. Our contributions are as follows:

- We give a simplification of a semantic proof of CR for β-reduction which unlike some common proofs in the literature, avoids any type machinery and is solely carried out in a completely untyped setting.
- We give a new proof of CR for $\beta\eta$-reduction which is a generalisation of our simple proof for β-reduction.
- Our simplification of the semantic proof results into a syntactic proof which is projectable into a semantic method and can hence be used as a bridge between syntactic and semantic methods.

Keywords: Church-Rosser, Reducibility, Parallel reductions

1 Introduction

Reducibility is a method based on realisability semantics [9], developed by Tait [14] in order to prove normalisation of some functional theories. The idea is to interpret types by sets of λ-terms closed under some properties. Since its introduction, this method has gone through a number of improvements and generalisations. In particular, Krivine [12] uses reducibility to prove the strong normalisation (SN) of his intersection type system called system $\mathcal{D}$. Koletsos [10] uses reducibility to prove that the set of simply typed λ-terms holds the Church-Rosser property (CR, also known as confluence property) w.r.t. β-reduction. Although it is well known that β-reduction satisfies CR, reducibility proofs of CR are in line with proofs of SN and hence, one can establish both SN and CR for some calculus using the same method. Moreover, CR proofs can be quite involved. So, reducibility proofs can help within

[1] http://www.macs.hw.ac.uk/ultra/

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the same machinery to prove the most important properties of a λ-calculus (such as SN, CR or standardisation).

In this paper we simplify a reducibility method for proving CR for β-reduction and we generalise our proof to proving CR for $\beta\eta$-reduction. Our proof of CR for β-reduction is simpler that the ones given by Ghilezan and Kunčak [6], by Koletsos and Stavrinos [11] and by Barendregt, Bergstra, Klop and Volken [1,2]. Furthermore, our proof is generalisable into a new proof of CR for $\beta\eta$-reduction. The CR theorem is a strong form of a theorem stated by Church and Rosser [5] proving the consistency of the λ-calculus. The Church-Rosser property of the untyped λ-calculus w.r.t. β-reduction can be stated as follows: for any λ-terms M, M_1, M_2 such that $M \rightarrow^*_\beta M_1$ and $M \rightarrow^*_\beta M_2$ there exists M_3 such that $M_1 \rightarrow^*_\beta M_3$ and $M_2 \rightarrow^*_\beta M_3$, where $\rightarrow^*_\beta$ is the β-reduction relation.

As in a number of other works [11,8], our method to prove CR for a given set of terms w.r.t. a reduction relation (we consider both β-reduction and $\beta\eta$-reduction) consists of two main steps:

- Our first step is based on a simplification of a reducibility method used to prove the confluence of developments. We started the construction of our method with a "common" reducibility method. We observed that not all the types were actually needed in the method and that the interpretations of the few needed types corresponded to simple sets of terms satisfying simple closure properties (such as saturation). This led us to the removal of the whole type machinery and we obtained as a result a syntactic method. Interestingly, although our method has turned into a syntactic one, it can still be embedded into a semantic method based on a type system such as Krivine's system $\mathcal{D}$ [12]. [2] As it is crucial to a reducibility method to use a soundness result, we call soudness the corresponding result in our method.
- The second step of our method consists in reducing the problem of the confluence of the λ-calculus w.r.t. the considered reduction relation to the problem of the confluence of the defined set of terms w.r.t. the defined reduction. This second step is achieved through a common simulation method used in most proofs of CR. The simulation consists in defining a new simple confluent reduction whose transitive closure is equal to the considered reduction.

To achieve their goals, all of [1,6,11,8] use the notion of developments. Both Koletsos and Stavrinos [11] as well as Kamareddine and Rahli [8] use a complicated handling of developments. On the other hand, Barendregt et al. [1], Ghilezan and Kunčak [6] as well as this article are based on some simpler and sufficient notions of developments. These notions of developments are simpler because, as in the so called method of parallel reductions [13,15], they do not deal with residuals. Because this article does not make use of a type system and does not deal with residuals, it is a simplification of the work done by [11] and by [8]. However, this article can also be regarded as a simplification and a generalisation of the work done by:

- Barendregt et al. [1], because we do not introduce a new calculus and we do not use the finiteness of developments,

[2] The methods of [6] and [11] use the Simply Typed Lambda Calculus $\lambda_\rightarrow$ and the type system $\mathcal{D}$ [4,3].

- Ghilezan and Kunčak [6], because we do not make use of a type system.

In Section 2 we introduce the needed machinery about the λ-calculus and our developments. In Section 3 we give our proof of the Church-Rosser property of the untyped λ-calculus w.r.t. β-reduction. In Section 4 we give our proof of the Church-Rosser property of the untyped λ-calculus w.r.t. $\beta\eta$-reduction. Finally, in Section 5, we compare our solution to the related work in the literature, especially to the work of Ghilezan and Kunčak [6], of Koletsos and Stavrinos [11] and to the developments of Tait and Martin-Löf [13,2]. Omitted proofs can be found in Appendix A.

2 The Formal Machinery

In this section we provide some known formal machinery and introduce new definitions and lemmas that are necessary for the paper. Let n, m be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We take as convention that if a metavariable v ranges over a set s then the metavariables v_i such that $i \geq 0$ and the metavariables $v', v'', etc.$ also range over s.

2.1 Background on the λ-calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the λ-calculus and one basic lemma.

Definition 2.1 (i) Let x, y, z range over Var, a countable infinite set of term variables (or just variables). The set of terms of the λ-calculus is defined by:
$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$
We let M, N, P, Q, R range over Λ. We assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $MN_0 \cdots N_n$ instead of $(\cdots((MN_0)N_1)\cdots N_{n-1})N_n$. We also assume the usual definition of subterms and write $N \subseteq M$ if N is a subterm of M (note that $M \subseteq M$). We call a term of the form $\lambda x.M$, a λ-abstraction (or just abstraction) and a term of the form $M_1 M_2$ an application.

We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $\mathrm{fv}(M)$ for the set of the free variables of term M.

(ii) We define as usual the substitution $M[x := N]$ of N for all free occurrences of x in M. We let $M[x_1 := N_1, \ldots, x_n := N_n]$ be the simultaneous substitution of N_i for all free occurrences of x_i in M for $1 \leq i \leq n$.

(iii) We assume the usual definition of compatibility (see the last line of Figure 1). For $r \in \{\beta, \beta\eta\}$, we define the reduction relation $\rightarrow_r$ on Λ as the least compatible relation closed under rule $(r) : L \rightarrow_r R$ below, and we call L an r*-redex* and R the r*-contractum* of L (or the L r-contractum).

- (β): $(\lambda x.M)N \rightarrow_\beta M[x := N]$.
- (η): $\lambda x.Mx \rightarrow_\eta M$ where $x \notin \mathrm{fv}(M)$.

We define $\rightarrow_{\beta\eta} = \rightarrow_\beta \cup \rightarrow_\eta$.

(iv) Let $r \in \{\beta, \eta, \beta\eta\}$. We use $\rightarrow_r^*$ to denote the reflexive transitive closure (see let $\mathcal{R}$ be a binary relation on Λ.

$$\frac{}{M \; \mathcal{R} \; M}\;(\mathit{refl}) \qquad \frac{M_1 \; \mathcal{R} \; M_2 \quad M_2 \; \mathcal{R} \; M_3}{M_1 \; \mathcal{R} \; M_3}\;(\mathit{tr})$$

$$\frac{P \; \mathcal{R} \; Q}{\lambda x.P \; \mathcal{R} \; \lambda x.Q}\;(\mathit{abs}) \qquad \frac{Q \; \mathcal{R} \; Q'}{PQ \; \mathcal{R} \; PQ'}\;(\mathit{app_1}) \qquad \frac{P \; \mathcal{R} \; P'}{PQ \; \mathcal{R} \; P'Q}\;(\mathit{app_2})$$

Fig. 1. Closure rules

the rules (*refl*) and (*tr*) of Figure 1) of $\rightarrow_r$. We let $\simeq_r$ denote the equivalence relation induced by $\rightarrow_r$. If the r-reduction from M to N is in k steps, we write $M \rightarrow_r^k N$.

(v) Let $r \in \{\beta, \beta\eta\}$. A term $(\lambda x.M')N'$ is a *direct r-reduct* of $(\lambda x.M)N$ iff $M \rightarrow_r^* M'$ and $N \rightarrow_r^* N'$.

(vi) Let $r \in \{\beta, \beta\eta\}$. We say that M has the Church-Rosser property for r (has r-CR) if whenever $M \rightarrow_r^* M_1$ and $M \rightarrow_r^* M_2$ then there exists M_3 such that $M_1 \rightarrow_r^* M_3$ and $M_2 \rightarrow_r^* M_3$. We define $\mathsf{CR}^r = \{M \mid M \text{ has } r\text{-CR}\}$. We use CR to denote CR^β.

(vii) We define the set SAT of the sets satisfying the saturation property as follows: $\mathsf{SAT} = \{s \subseteq \Lambda \mid M[x := N] \in s \Rightarrow (\lambda x.M)N \in s\}$.

(viii) We define the set VAR of the sets satisfying the variable property as follows: $\mathsf{VAR} = \{s \subseteq \Lambda \mid n \geq 0 \wedge (\forall i \in \{1, \ldots, n\}.\ M_i \in s) \Rightarrow x M_1 \cdots M_n \in s\}$.

(ix) We define the set ABS of the sets satisfying the abstraction property as follows: $\mathsf{ABS} = \{s \subseteq \Lambda \mid M \in s \Rightarrow \lambda x.M \in s\}$.

Lemma 2.2 *Let $r \in \{\beta, \beta\eta\}$. The following hold:*

(i) *If $M \rightarrow_r^* N$ and $P \rightarrow_r^* Q$ then $M[x := P] \rightarrow_r^* N[x := Q]$.*

(ii) $\mathrm{fv}(M[x := N]) \subseteq \mathrm{fv}((\lambda x.M)N)$.

(iii) *If $M \rightarrow_r^* N$ then $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$.*

(iv) *If $\lambda x.M \rightarrow_{\beta\eta}^* N$ then either $N = \lambda x.M'$ such that $M \rightarrow_{\beta\eta}^* M'$ or $M \rightarrow_{\beta\eta}^* Nx$ such that $x \notin \mathrm{fv}(N)$.*

(v) *If $x \notin \mathrm{fv}(M)$ and $Mx \rightarrow_{\beta\eta}^* N$ then $M \rightarrow_{\beta\eta}^* P$ and either $N = Px$ or $P = \lambda x.N$.*

(vi) *If $n \geq 0$, $Q = (\lambda x.M)N \rightarrow_r^k P$ and P is not a direct r-reduct of Q then (a) $k \geq 1$, (b) if $k = 1$ then $P = M[x := N]$ and (c) there exists a direct r-reduct $(\lambda x.M')N'$ of Q such that $M'[x := N'] \rightarrow_r^* P$.*

(vii) *Let $n \geq 0$ and $(\lambda x.M)N \rightarrow_r^* P$. There exists P' such that $P \rightarrow_r^* P'$ and $M[x := N] \rightarrow_r^* P'$.*

(viii) *a)* $\mathsf{CR}^r \in \mathsf{SAT}$ *b)* $\mathsf{CR}^r \in \mathsf{VAR}$ *c)* $\mathsf{CR}^r \in \mathsf{ABS}$

2.2 Pseudo Development Definitions

Throughout, we take c to be a distinct metavariables ranging over Var.

Remark 2.3 Such a distinct variable is usually given not as metavariable, but as a new variable or constant [6,11,12]. We noted that this usual way leads to problems.

For example, Ghilezan and Kunčak [6] use two of these distinct variables (called f and g and introduced as "predefined constants" not belonging to the untyped λ-calculus). But the "freezing" function Ψ defined by Ghilezan and Kunčak (similar to our function Ψ_c) is proved to be a function from Λ to Λ_0 where Λ_0 is defined as follows: $\Lambda_0 = \{M \in \Lambda \mid (\exists x_1, \ldots, x_n)\Gamma_0, x_1 : 0, \ldots, x_n : 0 \vdash M : 0\}$, which is the set of terms in Λ which are typable in $\lambda_\rightarrow$ (0 is a ground type and Γ_0 is a predefined type environment). Hence, by their definition, $\Lambda_0 \subset \Lambda$. So, it is obvious that their function Ψ does not associate a term in Λ_0 to each term in Λ since Ψ adds some f and g to the terms (for example $\Psi(xx) = fxx$, but $fxx \notin \Lambda$, so $fxx \notin \Lambda_0$).

Moreover, typing environments (contexts) are defined as sets of type assignments of the form $x : \varphi$ where x is a term variable and φ is a simple type. Later, some contexts are built with type assignments of the form $f : \varphi$, but f is not defined as a term variable. More generally, the introduction of a new variable or a new constant implies that the considered type system has to be defined on the new calculus.

Koletsos and Stavrinos [11], define two sets $\mathsf{CR}, \mathsf{CR}_0 \subseteq \Lambda$ which turn out to be equal to ours. Koletsos and Stavrinos prove that each term typable in the type system $\mathcal{D}$ has the Church-Rosser property. The proof of this statement fails, for example, for terms with free variables not belonging to the set of variables of the initial λ-calculus (c is defined as a variable not belonging to this set), since the proof uses the fact that the free variables of the term belong to the set CR_0. But, further, this statement is used for a term which may contain some c.

We call *current redex* the occurrence of a redex in a given term M. We call *potential redex* an application which is not an occurrence of a redex in a given term M but which is the occurrence of a redex in the term obtained after at least one reduction step from M. As done by Krivine [12] and followed by many others [6,11,8], we use c to "freeze" some current or potential redexes in a term. The parametrised calculi (with parameter c) presented in Definition 2.4 are the "frozen" calculi based on the λ-calculus where some reductions are blocked by the use of c. For example $(\lambda x.xy)(\lambda z.z) \rightarrow_\beta (\lambda z.z)y \rightarrow_\beta y$, but $(\lambda x.cxy)(\lambda z.z) \rightarrow_\beta c(\lambda z.z)y$ which does not reduce further. It is easy to see that $\Lambda_c^\beta \subset \Lambda_c^{\beta\eta} \subset \Lambda$.

Definition 2.4 $[\Lambda_c^\beta,\, \Lambda_c^{\beta\eta}]$ Let $\bar{x}, \bar{y} \in \mathsf{Var}_c = \mathsf{Var} \setminus \{c\}$.

$$\bar{M} \in \Lambda_c^\beta ::= \bar{x} \mid \lambda\bar{x}.\bar{M} \mid (\lambda\bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2$$

$$\bar{M} \in \Lambda_c^{\beta\eta} ::= \bar{x} \mid \lambda\bar{x}.\bar{M} \mid (\lambda\bar{x}.\bar{M}_1)\bar{M}_2 \mid c\bar{M}_1\bar{M}_2 \mid c\bar{M}$$

We let $\bar{M}, \bar{N}, \bar{P}, \bar{Q}, \bar{R}$ range over both Λ_c^β and $\Lambda_c^{\beta\eta}$.

Definition 2.5, introduces the "freezing" function which allows to "freeze" the potential redexes of a term. Unlike definitions in the literature [6,11,12,8], with our definition (the third clause below), the current β-redexes of a term are all "unfrozen". Furthermore, our definition does not freeze any of the current or potential η-redexes. For example, $M = \lambda x.(\lambda y.czx)z$ does not contain any η-redex but contains a potential η-redex, since $M \rightarrow_\beta \lambda x.czx = N$ and N contains a η-redex. As we will see in this paper, it is not necessary to "freeze" the potential η-redexes.

Definition 2.5 [$\Psi_c(-)$] $\Psi_c(-)$ is defined as follows:

(i) $\Psi_c(x) = x$

(ii) $\Psi_c(\lambda x.N) = \lambda x.\Psi_c(N)$, where $x \notin \{c\}$

(iii) If P is a λ-abstraction then $\Psi_c(PQ) = \Psi_c(P)\Psi_c(Q)$

(iv) If P is not a λ-abstraction then $\Psi_c(PQ) = c\Psi_c(P)\Psi_c(Q)$.

Definition 2.6 introduces the reduction $\to_c$ which allows to remove the c's from a term. This reduction can be regarded as a simplification of the reduction $\to_o$ defined by Ghilezan and Kunčak [6].

Definition 2.6 [$\to_c$] Let the c-reduction relation $\to_c$ be the least compatible relation on Λ closed under the rule: $(c) : cM \to_c M$
As usual $\to_c^*$ is the reflexive and transitive closure of $\to_c$.

In Definition 2.7, we introduce our β-developments ($\to_1$) as well as our $\beta\eta$-developments ($\to_2$).

Definition 2.7 [Developments: $\to_1$, $\to_2$] Let $\langle r, s\rangle \in \{\langle 1, \beta\rangle, \langle 2, \beta\eta\rangle\}$.

$$M \to_r N \iff \exists P.\ \Psi_c(M) \to_s^* P \wedge P \to_c^* N \wedge c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$$

As usual, $\to_r^*$ is the reflexive and transitive closure of $\to_r$. (Note that $\to_r$ is reflexive, but in order not to have to introduce a new symbol for its transitive closure, we consider $\to_r^*$.)

Definition 2.8 defines the parametric set of terms A_c built over the parameter c using application. Such terms contain only c's and no abstraction. This set of terms is especially needed to state Lemma 2.9.vii. The particularity of such terms being that they can be completely erased by the c-reduction (see Lemma 2.9.v).

Definition 2.8 $d \in \mathsf{A}_c ::= c \mid dd$

Lemma 2.9 is informative about the reduction relation $\to_c$.

Lemma 2.9 (i) $\Psi_c(M) \to_c^* M$.

(ii) *If* $M \to_c^* N$ *then* $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(N) \setminus \{c\}$.

(iii) $\mathrm{fv}(M) \setminus \{c\} = \mathrm{fv}(\Psi_c(M)) \setminus \{c\}$.

(iv) $\Lambda_c^{\beta} \cap \mathsf{A}_c = \varnothing = \Lambda_c^{\beta\eta} \cap \mathsf{A}_c$.

(v) $dM \to_c^* M$.

(vi) *If* $M \to_c^* N$ *then* $M \in \mathsf{A}_c$ *iff* $N \in \mathsf{A}_c$.

(vii) *Let* $M \to_c^* N$.
- *If* $M = x$ *then* $N = x$.
- *If* $M = \lambda x.M_1$ *then* $N = \lambda x.N_1$ *such that* $M_1 \to_c^* N_1$.
- *If* $M = M_1M_2$ *then either* $M_1 \in \mathsf{A}_c$ *and* $M_2 \to_c^* N$ *or* $N = N_1N_2$ *and* $M_1 \to_c^* N_1$ *and* $M_2 \to_c^* N_2$.

(viii) *If* $M \to_c^* M'$, $N \to_c^* N'$ *and* $x \neq c$ *then* $M[x := N] \to_c^* M'[x := N']$.

(ix) *If* $c \notin \mathrm{fv}(M)$ *and* $M \to_c^* N$ *then* $M = N$.

(x) *If* $M \to_c^* N$, $M \to_c^* P$ *and* $c \notin \mathrm{fv}(N)$ *then* $P \to_c^* N$.

Proof.

i,viii,x By induction on the structure of M.

iii Corollary of Lemma 2.9.i and Lemma 2.9.ii.

iv Let $\bar{M} \in \Lambda_c^{\beta\eta}$. By induction on the structure of $\bar{M}$ that $\bar{M} \notin \mathsf{A}_c$.

v By induction on the structure of d.

vi ⇒) By induction on the length of the reduction $M \to_c^* d$.
⇐) By induction on the reduction $d \to_c^* N$.

vii,ix By induction on the length of the reduction $M \to_c^* N$.

□

Lemma 2.9.i stresses the relation between $\to_c$ and the freezing function.
Lemma 2.9.vii presents the shape of a term w.r.t. a c-reduction.
Lemma 2.9.x is a sort of weak confluence property w.r.t. $\to_c^*$.

3 A simple Church-Rosser proof for β-reduction

Koletsos and Stavrinos [11] gave a proof of the Church-Rosser property for the set of terms typable in an intersection type system called system $\mathcal{D}$ [12] w.r.t. β-reduction and showed that this can be used to establish the confluence of β-developments without using strong normalisation. Ghilezan and Kunčak [6] gave a proof of the Church-Rosser property for the set of terms typable in $\lambda_\to$ w.r.t. β-reduction and showed that this can be used to establish the confluence of a weak form of β-developments without using strong normalisation.

The first aim of this section, was to simplify the proof of Koletsos and Stavrinos [11]. During this simplification, we obtained a proof that bore some resemblance to the proof of Ghilezan and Kunčak [6]. A second simplification of our proof started with the observation that in both proofs of [6,11] only few types were really needed and that one can actually completely get rid of the type system. We considered two type interpretations based on the sets CR^β and $\mathsf{CR}^{\beta\eta}$ and interpreted the few needed types by sets of terms satisfying simple properties: saturation, variable and abstraction (see Definition 2.1). Since the calculus used by Koletsos and Stavrinos to prove the confluence of developments is simpler than the one used by Ghilezan and Kunčak, a third simplification which led to our actual simple proof has been to come back to the use of a calculus similar to the one used by Koletsos and Stavrinos as well as Krivine [12] before them (see Definition 2.4). As mentioned above, our proof is carried out in an untyped setting but one can relate the first part of the method to the reducibility proof in a semantic method using, for example, the type system $\mathcal{D}$. Interestingly, our proof can also be related to the proof given in [1,2].

The second aim of this section is to provide a framework for our main result: the extension of our proof to $\beta\eta$-reduction where we give a non semantic proof of Church-Rosser for $\beta\eta$-reduction which can be projectable into a semantic proof (Section 4).

As part of our simplification of a reducibility method, Lemma 3.1 states the "soundness" of our simple calculus (based on the set of terms Λ_c^β w.r.t. the remains of our type interpretation using a set s satisfying the saturation, variable and abstraction properties given in Definition 2.1).

Lemma 3.1 (Soundness) *If $\bar{M} \in \Lambda_c^\beta$, $\text{fv}(\bar{M}) \setminus \{c\} = \{x_1, \ldots, x_n\}$, for all $i \in \{1, \ldots, n\}$, $M_i \in s$ and $s \in \mathsf{VAR} \cup \mathsf{SAT} \cup \mathsf{ABS}$ then $\bar{M}[x_1 := M_1, \ldots, x_n := M_n] \in s$.*

Proof. By induction on the structure of $\bar{M}$. □

Using Lemma 3.1, we are now able to prove that each term in Λ_c^β has β-CR.

Corollary 3.2 $\Lambda_c^\beta \subseteq \mathsf{CR}$.

Proof. Let $\bar{M} \in \Lambda_c^\beta$ and $\text{fv}(\bar{M}) \setminus \{c\} = \{x_1, \ldots, x_n\}$. By Lemma 2.2.viii, 2.2.viii and 2.2.viii, $\mathsf{CR} \in \mathsf{SAT} \cup \mathsf{VAR} \cup \mathsf{ABS}$ and $x_1, \ldots, x_n \in \mathsf{CR}$. So by Lemma 3.1, $\bar{M} \in \mathsf{CR}$. □

Lemma 3.3 states that the freezing function associates to each term of the untyped λ-calculus (which does not contain c) a term in the language Λ_c^β.

Lemma 3.3 *If $c \notin \text{fv}(M)$ then $\Psi_c(M) \in \Lambda_c^\beta$.*

Proof. By induction on the structure of M. □

Here is another lemma containing needed technicalities:

Lemma 3.4 *Let $\bar{M}, \bar{N} \in \Lambda_c^\beta$ and $\bar{x} \in \mathsf{Var}_c$.*

(i) $\bar{M}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$.

(ii) *If $\bar{M} \rightarrow_\beta^* N$ then $N \in \Lambda_c^\beta$.*

(iii) *If $\bar{M} \rightarrow_c^* N$ and $c \notin \text{fv}(N)$ then $\bar{M} \rightarrow_c^* \Psi_c(N)$.*

(iv) *There exists N such that $c \notin \text{fv}(N)$ and $\bar{M} \rightarrow_c^* N$.*

Proof. Items i, iii and iv are by induction on the structure of $\bar{M}$. Item ii is by induction on the length of the derivation $\bar{M} \rightarrow_\beta^* N$. □

The second property (3.4.ii) states that the terms in Λ_c^β β-reduce only to terms in Λ_c^β. The fourth property (3.4.iv) expresses that a term in Λ_c^β can always be completely reduced w.r.t. $\rightarrow_c^*$.

Lemma 3.5 states that we can simulate the β-reduction of a term in Λ_c^β from any of its (partially or totally) "unfrozen" versions.

Lemma 3.5 (i) *If $\bar{M}_1 \in \Lambda_c^\beta$, $\bar{M}_1 \rightarrow_\beta N_1$ and $\bar{M}_1 \rightarrow_c^* M_2$ then there exists N_2 such that $M_2 \rightarrow_\beta N_2$ and $N_1 \rightarrow_c^* N_2$.*

(ii) *If $\bar{M}_1 \in \Lambda_c^\beta$, $\bar{M}_1 \rightarrow_\beta^* N_1$ and $\bar{M}_1 \rightarrow_c^* M_2$ then there exists N_2 such that $M_2 \rightarrow_\beta^* N_2$ and $N_1 \rightarrow_c^* N_2$.*

Proof.

i By induction on the structure of $\bar{M}_1$.

ii By induction on the length of the reduction $M_1 \rightarrow_\beta^* N_1$ using Lemma 3.5.i.

□

Lemma 3.6 is a key lemma of the simulation method of a reduction by some developments. It states that the reflexive and transitive closure of $\to_\beta$ is equal to the reflexive and transitive closure of $\to_1$.

Lemma 3.6 *Let* $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$*, then* $M \to_\beta^* N \iff M \to_1^* N$*.*

Proof.

$\Rightarrow$) Let $M \to_\beta^* N$. We prove that $M \to_1^* N$ by induction on the size of the reduction $M \to_\beta^* N$.

$\Leftarrow$) Let $M \to_1^* N$. We prove that $M \to_\beta^* N$ by induction on the size of the derivation $M \to_1^* N$.

□

Lemma 3.7 states the confluence of the β-developments.

Lemma 3.7 (i) *If* $M \to_1 M_1$ *and* $M \to_1 M_2$ *then there exists* M_3 *such that* $M_1 \to_1 M_3$ *and* $M_2 \to_1 M_3$*.*

(ii) *If* $M \to_1^* M_1$ *and* $M \to_1^* M_2$ *then there exists* M_3 *such that* $M_1 \to_1^* M_3$ *and* $M_2 \to_1^* M_3$*.*

Proof.

i By definition, there exist P_1, P_2 such that $\Psi_c(M) \to_\beta^* P_1$, $\Psi_c(M) \to_\beta^* P_2$, $P_1 \to_c^* M_1$, $P_2 \to_c^* M_2$ and $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2)$. By Lemma 3.3, $\Psi_c(M) \in \Lambda_c^\beta$. So by Corollary 3.2, there exists P_3 such that $P_1 \to_\beta^* P_3$ and $P_2 \to_\beta^* P_3$. By Lemma 3.4.ii, $P_1, P_2, P_3 \in \Lambda_c^\beta$. By lemma 3.4.iv, there exists M_3 such that $P_3 \to_c^* M_3$ and $c \notin \mathrm{fv}(M_3)$. By Lemma 3.4.iii, $P_1 \to_c^* \Psi_c(M_1)$ and $P_2 \to_c^* \Psi_c(M_2)$. By Lemma 3.5.ii, there exist Q_1, Q_2 such that $P_3 \to_c^* Q_1$, $P_3 \to_c^* Q_2$, $\Psi_c(M_1) \to_\beta^* Q_1$ and $\Psi_c(M_2) \to_\beta^* Q_2$. By Lemma 2.9.x, $Q_1 \to_c^* M_3$ and $Q_2 \to_c^* M_3$. So $M_1 \to_1 M_3$ and $M_2 \to_1 M_3$.

ii By Lemma 3.7.i

□

The confluence of the untyped λ-calculus w.r.t. β-reduction is now proved using the confluence of the β-developments and the equality between $\to_\beta^*$ and $\to_1^*$.

Theorem 3.8 $\Lambda = \mathsf{CR}$.

Proof. $\mathsf{CR} \subseteq \Lambda$ is trivial, we only prove $\Lambda \subseteq \mathsf{CR}$. Let $M, M_1, M_2 \in \Lambda$ such that $M \to_\beta^* M_1$ and $M \to_\beta^* M_2$ and $c \notin \mathrm{fv}(M)$. By Lemma 2.2.iii, $c \notin \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2.)$. By Lemma 3.6, $M \to_1^* M_1$ and $M \to_1^* M_2$. By Lemma 3.5.ii, there exists M_3 such that $M_1 \to_1^* M_3$ and $M_2 \to_1^* M_3$. By definition $c \notin \mathrm{fv}(M_3)$. By Lemma 3.6, $M_1 \to_\beta^* M_3$ and $M_2 \to_\beta^* M_3$. □

4 A simple Church-Rosser proof for $\beta\eta$-reduction

Now that we have stated the principal steps of the method of the Church-Rosser property of the untyped λ-calculus w.r.t. β-reduction, we will generalise it to $\beta\eta$-

reduction following exactly the same steps and using the $\Lambda_c^{\beta\eta}$ language. This generalisation can be regarded both as a simplification and an extension of methods by for example Ghilezan and Kunčak [6], Kamareddine and Rahli [8] and Barendregt et al. [1,2] (Section 11.2).

As part of our simplification of a reducibility method, Lemma 4.1 states the "soundness" of our simple calculus based on the set of term $\Lambda_c^{\beta\eta}$ w.r.t. the remains of our type interpretation based on a set s satisfying the saturation, variable and abstraction properties (see Definition 2.1).

Lemma 4.1 (Soundness) *If $\bar{M} \in \Lambda_c^{\beta\eta}$, $\mathrm{fv}(\bar{M}) \setminus \{c\} = \{x_1, \dots, x_n\}$, for all $i \in \{1, \dots, n\}$, $M_i \in s$ and $s \in \mathsf{SAT} \cup \mathsf{VAR} \cup \mathsf{ABS}$ then $\bar{M}[x_1 := M_1, \dots, x_n := M_n] \in s$.*

Proof. By induction on the structure of $\bar{M}$. □

Using lemma 4.1, we are now able to prove that each term in $\Lambda_c^{\beta\eta}$ has $\beta\eta$-CR.

Corollary 4.2 $\Lambda_c^{\beta\eta} \subseteq \mathsf{CR}^{\beta\eta}$.

Proof. Let $\bar{M} \in \Lambda_c^{\beta\eta}$ and $\mathrm{fv}(\bar{M}) \setminus \{c\} = \{x_1, \dots, x_n\}$. By Lemma 2.2.viii, 2.2.viii and 2.2.viii, $\mathsf{CR}^{\beta\eta} \in \mathsf{SAT} \cup \mathsf{VAR} \cup \mathsf{ABS}$ and $x_1, \dots, x_n \in \mathsf{CR}^{\beta\eta}$. So by Lemma 4.1, $\bar{M} \in \mathsf{CR}^{\beta\eta}$. □

Lemma 4.3 states that the freezing function associates to each term of the untyped λ-calculus (which does not contain the variable c) a term in $\Lambda_c^{\beta\eta}$. This result is trivial because $\Lambda_c^{\beta} \subset \Lambda_c^{\beta\eta}$.

Lemma 4.3 *If $c \notin \mathrm{fv}(M)$ then $\Psi_c(M) \in \Lambda_c^{\beta\eta}$.*

Proof. By Lemma 3.3, $\Psi_c(M) \in \Lambda_c^{\beta}$. Since $\Lambda_c^{\beta} \subset \Lambda_c^{\beta\eta}$ then $\Psi_c(M) \in \Lambda_c^{\beta\eta}$. □

Here is another lemma containing needed technicalities:

Lemma 4.4 *Let $\bar{M}, \bar{N} \in \Lambda_c^{\beta\eta}$ and $\bar{x} \in \mathsf{Var}_c$.*

(i) *$\bar{M}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$.*

(ii) *If $\bar{M} \to_{\beta\eta}^* N$ then $N \in \Lambda_c^{\beta\eta}$.*

(iii) *If $\bar{M} \to_c^* N$ and $c \notin \mathrm{fv}(N)$ then $\bar{M} \to_c^* \Psi_c(N)$.*

(iv) *There exists N such that $c \notin \mathrm{fv}(N)$ and $\bar{M} \to_c^* N$.*

Proof. Items i, iii and iv are by induction on the structure of $\bar{M}$. Item ii is by induction on the length of the derivation $\bar{M} \to_{\beta\eta}^* N$. □

The second property (4.4.ii) states that the terms in $\Lambda_c^{\beta\eta}$ β-reduce only terms in $\Lambda_c^{\beta\eta}$. The fourth property (4.4.iv) expresses that a term in $\Lambda_c^{\beta\eta}$ can always be completely reduced w.r.t. $\to_c^*$.

Lemma 4.5 states that we can simulate the reduction of a term in $\Lambda_c^{\beta\eta}$ from any of its (partially or totally) "unfrozen" versions.

Lemma 4.5 (i) *If $\bar{M}_1 \in \Lambda_c^{\beta\eta}$, $\bar{M}_1 \to_{\beta\eta} N_1$ and $\bar{M}_1 \to_c^* M_2$ then there exists N_2 such that $M_2 \to_{\beta\eta} N_2$ and $N_1 \to_c^* N_2$.*

(ii) *If $\bar{M}_1 \in \Lambda_c^{\beta\eta}$ such that $\bar{M}_1 \to_{\beta\eta}^* N_1$ and $\bar{M}_1 \to_c^* M_2$ then there exists N_2 such that $M_2 \to_{\beta\eta}^* N_2$ and $N_1 \to_c^* N_2$.*

Proof. i. By induction on the structure of M_1. ii. By Lemma 4.5.i. □

Lemma 4.6 is a key lemma of the simulation method of a reduction by some developments. It states that the reflexive and transitive closure of $\to_{\beta\eta}$ is equal to the reflexive and transitive closure of $\to_2$.

Lemma 4.6 *Let $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$, then $M \to^*_{\beta\eta} N \iff M \to^*_2 N$.*

Proof.

$\Rightarrow$) Let $M \to^*_{\beta\eta} N$. We prove that $M \to^*_2 N$ by induction on the size of the reduction $M \to^*_{\beta\eta} N$.

$\Leftarrow$) Let $M \to^*_2 N$. We prove that $M \to^*_{\beta\eta} N$ by induction on the size of the derivation $M \to^*_2 N$.

□

It is then easy to deduce the confluence of the $\beta\eta$-developments.

Lemma 4.7 (i) *If $M \to_2 M_1$ and $M \to_2 M_2$ then there exists M_3 such that $M_1 \to_2 M_3$ and $M_2 \to_2 M_3$.*

(ii) *If $M \to^*_2 M_1$ and $M \to^*_2 M_2$ then there exists M_3 such that $M_1 \to^*_2 M_3$ and $M_2 \to^*_2 M_3$.*

Proof.

i By definition, there exist P_1, P_2 such that $\Psi_c(M) \to^*_{\beta\eta} P_1$, $\Psi_c(M) \to^*_{\beta\eta} P_2$, $P_1 \to^*_c M_1$, $P_2 \to^*_c M_2$ and $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2)$. By Lemma 4.3, $\Psi_c(M) \in \Lambda^{\beta\eta}_c$. So by Corollary 4.2, there exists P_3 such that $P_1 \to^*_{\beta\eta} P_3$ and $P_2 \to^*_{\beta\eta} P_3$. By Lemma 4.4.ii, $P_1, P_2, P_3 \in \Lambda^{\beta\eta}_c$. By lemma 4.4.iv, there exists M_3 such that $P_3 \to^*_c M_3$ and $c \notin \mathrm{fv}(M_3)$. By Lemma 4.4.iii, $P_1 \to^*_c \Psi_c(M_1)$ and $P_2 \to^*_c \Psi_c(M_2)$. By Lemma 4.5.ii, there exist Q_1, Q_2 such that $P_3 \to^*_c Q_1$, $P_3 \to^*_c Q_2$, $\Psi_c(M_1) \to^*_{\beta\eta} Q_1$ and $\Psi_c(M_2) \to^*_{\beta\eta} Q_2$. By Lemma 2.9.x, $Q_1 \to^*_c M_3$ and $Q_2 \to^*_c M_3$. So $M_1 \to_2 M_3$ and $M_2 \to_2 M_3$.

ii Easy by Lemma 4.7.i.

□

The confluence of the untyped λ-calculus w.r.t. $\beta\eta$-reduction is then proved using the confluence of the $\beta\eta$-developments and the equality between $\to^*_{\beta\eta}$ and $\to^*_2$.

Theorem 4.8 $\Lambda = \mathsf{CR}^{\beta\eta}$.

Proof. $\mathsf{CR}^{\beta\eta} \subseteq \Lambda$ is trivial, we only prove $\Lambda \subseteq \mathsf{CR}^{\beta\eta}$. Let $M, M_1, M_2 \in \Lambda$ and $c \notin \mathrm{fv}(M)$ such that $M \to^*_{\beta\eta} M_1$ and $M \to^*_{\beta\eta} M_2$. By Lemma 2.2.iii, $c \notin \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2)$. By Lemma 4.6, $M \to^*_2 M_1$ and $M \to^*_2 M_2$. By Lemma 4.7.ii, there exists M_3 such that $M_1 \to^*_2 M_3$ and $M_2 \to^*_2 M_3$. By definition $c \notin \mathrm{fv}(M_3)$. By Lemma 4.6, $M_1 \to^*_{\beta\eta} M_3$ and $M_2 \to^*_{\beta\eta} M_3$. □

5 Comparison and Related Work

In this section we compare our proposal to two semantic methods to prove confluence [6,11]. We also compare our developments to those of Tait and Martin-Löf. In

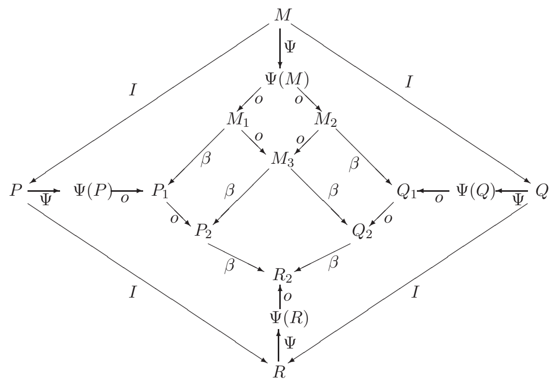

Fig. 2. The method of Ghilezan and Kunčak for the confluence of $\rightarrow_I$

this section and only in this section, we consider the confluence property w.r.t. β-reduction. In the Figures 2 and 3, an arrow labelled with c, o or β stands for $\rightarrow_c^*$, $\rightarrow_o^*$ or $\rightarrow_\beta^*$ respectively. An arrow labelled with Ψ stands for the application of the function with the same name to the term at the start of the arrow.

5.1 Comparison to Ghilezan+Kuncak and to Koletsos+Stravinos

In Figure 2 we recall the proof of Ghilezan and Kunčak [6] for the confluence of the untyped λ-calculus w.r.t. β-reduction. This proof, based on the embedding of the developments into $\lambda_\rightarrow$, uses the confluence w.r.t. another reduction $\rightarrow_I$ (a development) whose transitive closure is equal to $\rightarrow_\beta^*$. The reduction $\rightarrow_I$ is defined as $\tau^{-1} \circ \rightarrow_\beta^* \circ \tau$ where:

- $\tau = \rightarrow_o^* \circ \Psi$
- $\rightarrow_o$ is the compatible closure of the rule $(o) : f(g(\lambda x.M))N \rightarrow_o (\lambda x.M)N$
- Ψ is defined on the λ-calculus by: $\Psi(x) = x$, $\Psi(\lambda x.M) = g(\lambda x.\Psi(M))$ and $\Psi(MN) = f\Psi(M)\Psi(N)$, where f and g are two constants (see Remark 2.3).

The relation τ enables to "freeze" some β-redexes and the potential β-redexes (the other applications) of a term. (In fact, τ does more, because Ψ does more by encapsulating the λ-abstractions using g. This technicality is needed by Ghilezan and Kunčak to prove the typability of a defined set of terms in $\lambda_\rightarrow$.) The reduction τ^{-1} is similar to our erasure relation $\rightarrow_c$ (see Definition 2.6) and to Krivine's erasure function [12], which "unfreeze" the redexes in a term. By definition of $M \rightarrow_I P$, there exist M_1 and P_1 such that $\Psi(M) \rightarrow_o^* M_1 \rightarrow_\beta^* P_1$ and $\Psi(P) \rightarrow_o^* P_1$ (the left part of the figure). By definition of $M \rightarrow_I Q$, there exist M_2 and Q_1 such that $\Psi(M) \rightarrow_o^* M_2 \rightarrow_\beta^* Q_1$ and $\Psi(Q) \rightarrow_o^* Q_1$ (the right part of the figure). Because M_1 can be different from M_2, a confluence lemma for the $\rightarrow_o$ reduction and a commutation lemma for the reductions $\rightarrow_o^*$ and $\rightarrow_\beta^*$ are needed. The central part of the figure is due to the confluence of the terms typable in $\lambda_\rightarrow$. For example, as cited by Ghilezan and Kunčak, Koletsos [10] proved this result using a reducibility method. Hence, when combined with Koletsos's proof of the confluence of $\lambda_\rightarrow$, Ghilezan and Kunčak's method can be regarded as the combination of a reducibility method and a simulation method.

The reduction $\to_I$ designed by Ghilezan and Kunčak [6] defines a development without specifying explicitly the set of redexes which are allowed to be reduced (as done for example by Barendregt al. [1] which differs from approaches like those of Barendregt [2] or Hindley [7]). Let us consider the reduction $M \to_I P$ (unfolded above). First, the function Ψ blocks all the redexes in M. Then $\to_o^*$ enables to select the set of redexes which are allowed to be reduced in M without explicitly naming them, by unblocking some redexes in $\Psi(M)$. The reduction $M_1 \to_\beta^* P_1$ reduces the allowed redexes and their residuals. And finally in $\Psi(P) \to_o^* P_1$, the reduction $\to_o^*$ selects the set of residuals of the set of redexes in M_1 without naming them.

This way of dealing with occurrences of redexes is simple and sufficient enough to prove the Church-Rosser property. However, in some other works, occurrences of redexes are handled in a complicated way, as for example in the work of Krivine [12] or Koletsos and Stavrinos [11]. In these works, occurrences are treated intuitively and not formally. So, the work turns out to be much more complicated than it seems when one wants to "formally" prove the results (see [8]), or even just formally defining the developments. Ghilezan and Kunčak [6] do not face the same problem. The reduction $\to_o^*$ enables to unblock a certain set of redexes without explicitly specifying the set of unblocked redexes. In the work of Ghilezan and Kunčak, as in the work of Barendregt et al. [1] for example, a development of a term is defined without explicit control on the set of occurrences of reduced redexes, which is not needed.

Although Ghilezan and Kunčak [6] consider a simpler definition of developments than the "common" one, the scheme of their proof method is exactly the one followed by Koletsos and Stavrinos [11]. Koletsos and Stavrinos consider the following "common" definition of developments: there exists a development from M to N iff $\langle M, s_1\rangle \to_d^* \langle N, s_2\rangle$ where s_1 is a set of redexes in M and s_2 is the set of residuals of s_1 in N (where $\to_d^*$ is a new (complex) reduction relation based on $\to_\beta^*$). Their proof of the confluence of developments uses, among other things, the following claim: if $\langle M, s_1\rangle \to_d^* \langle N, s_2\rangle$ then there exists s_4 such that $\langle M, s_1 \cup s_3\rangle \to_d^* \langle N, s_2 \cup s_4\rangle$, where s_3 is a set of redexes of M. It is useful to prove that if $\langle M, s_1\rangle \to_d^* \langle M_1, s'_1\rangle$ and $\langle M, s_2\rangle \to_d^* \langle M_2, s'_2\rangle$ then there exist s''_1 and s''_2 such that $\langle M, s_1 \cup s_2\rangle \to_d^* \langle M_1, s'_1 \cup s''_2\rangle$ and $\langle M, s_2 \cup s_1\rangle \to_d^* \langle M_2, s'_2 \cup s''_1\rangle$. This corresponds to the proof of the confluence of $\to_o^*$ of Ghilezan and Kunčak, which is useful to get the reductions ($\Psi(M) \to_o^* M_1 \to_o^* M_3 \to_\beta^* P_2$ and $\Psi(P) \to_o^* P_1 \to_o^* P_2$) and ($\Psi(M) \to_o^* M_2 \to_o^* M_3 \to_\beta^* Q_2$ and $\Psi(Q) \to_o^* Q_1 \to_o^* Q_2$). Ghilezan and Kunčak emphasise this more strongly than Koletsos and Stavrinos.

The differences between the method of Ghilezan and Kunčak and that of Barendregt et al include:

- Ghilezan and Kunčak do not use the finiteness of developments when Barendregt et al. do;
- Ghilezan and Kunčak base their result on a well known result (the confluence of the simply typed λ-terms) to give a simple proof of the confluence of developments when Barendregt et al. have to prove everything;
- Ghilezan and Kunčak do not really introduce new terms when Barendregt et al. do (the underlined terms are introduced to prove the confluence of developments).

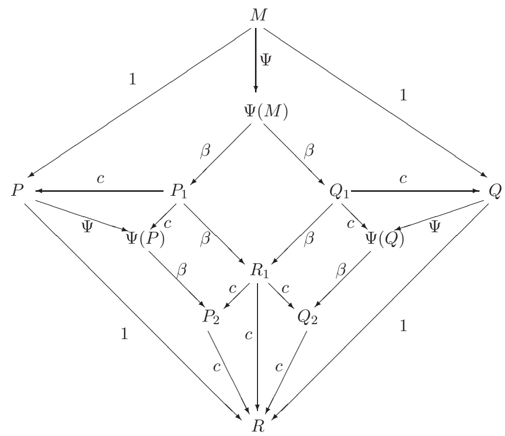

Fig. 3. Our method for the confluence of $\rightarrow_1$

Barendregt et al. also give a definition of developments without explicitly naming an occurrence of a redex (no set of occurrences is defined), introducing among other things, a second abstraction $\underline{\lambda}$. There exists a simple correspondence between the calculus with this second abstraction and the marked calculus introduced by Krivine and reused in this paper and in many other works [12,6,11,8].

In Figure 3 we draw the diagram of our method to prove the confluence of the λ-calculus. By definition of $M \rightarrow_1 P$ (Definition 2.7), there exists P_1 such that $\Psi(M) \rightarrow_\beta^* P_1$ and $P_1 \rightarrow_c^* P$ (the left part of the figure). By definition of $M \rightarrow_1 Q$, there exists Q_1 such that $\Psi(M) \rightarrow_\beta^* Q_1$ and $Q_1 \rightarrow_c^* Q$ (the right part of the figure). Moreover $P_1 \rightarrow_c^* \Psi_c(P)$ and $Q_1 \rightarrow_c^* \Psi_c(Q)$. So, because P_1 and $\Psi_c(P)$ might be different (as for Q_1 and $\Psi_c(Q)$), as Ghilezan and Kunčak [6], we need a commutation result for the reductions $\rightarrow_\beta^*$ and $\rightarrow_c^*$. Then, the whole diagram commutes because P_2, R_1 and Q_2 reduce to the same term. Like in Figure 2, the central part is due to the confluence of a defined set of terms (typable in $\lambda_\rightarrow$ for Ghilezan and Kunčak and typable in $\mathcal{D}$ in our case even though we do not use this fact).

Our method is also based on some kind of simple developments, where first, all the β-redexes are left unblocked and where all the potential β-redexes (all the other applications) are blocked. In this paper we define two simple developments: $\rightarrow_1$ for the β case and $\rightarrow_2$ for the $\beta\eta$ case. In that way, we do not need the reduction $\rightarrow_o^*$ to unblock some redexes in order to perform some reductions. But, it does not seem possible to get rid of the work done by this reduction. Indeed, our choice implies the introduction of some other material which turns out to be similar to the reduction $\rightarrow_o^*$. Both methods need the introduction of some similar material, but not at the same places. The reduction $\rightarrow_o^*$ is used by Ghilezan and Kunčak to unblock some redexes in order to enable some reductions whereas we use the reduction $\rightarrow_c^*$ to unblock some redexes which become blocked after some reductions.

As we can see in these two figures, because the occurrences of redexes are not explicitly taken into consideration, the function Ψ (which enables to embed a term in a typed term, by blocking redexes or potential redexes) needs to either block all the redexes of a term or to leave them all unblocked. If all the redexes are blocked by

Ψ, a reduction such as $\to_o$ is needed before being able to perform some reductions (see Figure 2). In this case some technical results are needed such as the confluence of $\to_o$. In the other case (Ψ leaves all the redexes unlocked), because a term whose redexes are all unblocked does not necessarily reduce to a term whose redexes are all unblocked, some technical results on a reduction such as $\to_o$ are also needed as we explained above (see Figure 3).

Finally, although our work derives from the one done by Koletsos and Stavrinos [11] and Kamareddine and Rahli [8], it turned out that it is also a simplification and generalisation of the work done by Ghilezan and Kunčak [6] and Barendregt et al. [1]. Because the work we achieved is more similar to the one of Ghilezan and Kunčak, we adapted some of our notations to theirs and focused our comparisons with the related work to their work.

Thereby, the two improvements of the present article can be regarded as the simplification of the work done by Ghilezan and Kunčak [6] by getting rid of all the type machinery and the extension of the defined method to the $\beta\eta$-reduction.

As explained above, the main lines of our proof are:

- the definition of some simple developments;
- the proof of the confluence of a simple calculus w.r.t. the considered reduction (β and $\beta\eta$) using a method based on a simplification of a reducibility method;
- the proof of the confluence of the defined developments;
- the proof of the equality between the reflexive and transitive closure of the developments and the reflexive and transitive closure of the considered reduction;
- the proof of CR of the untyped λ-calculus w.r.t. a given reduction using a simulation of the considered reduction by developments.

5.2 Comparison to Tait and Martin-Löf

Tait and Martin-Löf's syntactic proof [13,2] (and its extensions, by for example Takahashi [15]) that the untyped λ-calculus satisfies the Church-Rosser property is the simplest so far. Our method started from the semantic framework when we attempted to simplify and generalise existing semantic proofs. To our surprise, our simplification and generalisation of such semantic proofs led to our method which did not need types anymore. Hence, the type interpretation and the reducibility argument are no longer used in our article. This way our method moved from the semantic camp to the syntactic one. Nonetheless, our method can still be projected into a semantic argument method (something that does not hold for methods like those of Tait and Martin-Löf and Takahashi). In this way, we consider our work to be a bridge between the syntactic and semantic methods. There is another notable difference to our method: our developments allow (strictly) more reductions than those of Takahashi (for both the β and $\beta\eta$ cases) as we establish in this section.

Definition 5.1 [[15]] Let $r \in \{\beta, \beta\eta\}$. We define $\Rightarrow_r$ as follows:

- $M \Rightarrow_r M$
- $\lambda x.M \Rightarrow_r \lambda x.N$ if $M \Rightarrow_r N$
- $MN \Rightarrow_r M'N'$ if $M \Rightarrow_r M'$ and $N \Rightarrow_r N'$

- $(\lambda x.M)N \Rightarrow_r M'[x := N']$ if $M \Rightarrow_r M'$ and $N \Rightarrow_r N'$
- $\lambda x.Mx \Rightarrow_{\beta\eta} N$ if $x \notin \mathrm{fv}(M)$ and $M \Rightarrow_{\beta\eta} N$

Lemma 5.2 (i) *If $M \Rightarrow_\beta N$ or $M \Rightarrow_{\beta\eta} N$ then $\mathrm{fv}(N) \subseteq \mathrm{fv}(M)$.*

(ii) *Let M, N such that $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$. If $M \Rightarrow_\beta N$ then $M \to_1 N$.*

(iii) *Let M, N such that $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(N)$. If $M \Rightarrow_{\beta\eta} N$ then $M \to_2 N$.*

Proof. ii. Let $M \Rightarrow_\beta N$. The proof is by induction on the size of the derivation of $M \Rightarrow_\beta N$ and then by case on the last rule of the derivation.
iii. Let $M \Rightarrow_{\beta\eta} N$. The proof is by induction on the size of the derivation of $M \Rightarrow_{\beta\eta} N$ and then by case on the last rule of the derivation. $\square$

Remark 5.3 (i) We have $M = (\lambda x.xx)((\lambda z.z)y) \to_1 y((\lambda z.z)y)$ (where $c \notin \{x, y, z\}$) because $\Psi_c(M) = (\lambda x.cxx)((\lambda z.z)y) \to_\beta c((\lambda z.z)y))((\lambda z.z)y) \to_\beta cy((\lambda z.z)y) \to_c y((\lambda z.z)y)$. But, we do not have $M \Rightarrow_\beta y((\lambda z.z)y)$.

(ii) We have $M = \lambda x.y((\lambda z.z)x) \to_2 y$ (where $c \notin \{x, y, z\}$) because $\Psi_c(M) = \lambda x.cy((\lambda z.z)x) \to_\beta \lambda x.cyx \to_\eta cy \to_c y$. But, we do not have $M \Rightarrow_{\beta\eta} y$.

References

[1] H. Barendregt, J. A. Bergstra, J. W. Klop, H. Volken. Degrees, reductions and representability in the lambda calculus. Technical Report Preprint no. 22, University of Utrecht, Department of Mathematics, 1976.

[2] H. P. Barendregt. *The Lambda Calculus: Its Syntax and Semantics.* North-Holland, revised edition, 1984.

[3] H. P. Barendregt. Lambda calculi with types. In *Handbook of Logic in Computer Science, Volumes 1 (Background: Mathematical Structures) and 2 (Background: Computational Structures), Abramsky & Gabbay & Maibaum (Eds.), Clarendon*, vol. 2. 1992.

[4] A. Church. A formulation of the simple theory of types. *The Journal of Symbolic Logic*, 5(2), 1940.

[5] A. Church, J. B. Rosser. Some properties of conversion. *Transactions of the American Mathematical Society*, 39(3), 1936.

[6] S. Ghilezan, V. Kunčak. Confluence of untyped lambda calculus via simple types. *Lecture Notes in Computer Science*, 2202, 2001.

[7] R. Hindley. Reductions of residuals are finite. *Transactions of the American Mathematical Society*, 240, 1978.

[8] F. Kamareddine, V. Rahli, J. B. Wells. Reducibility proofs in the λ-calculus. Presented to ITRS '08, 4th Workshop on Intersection Types and Related Systems, Turin, Italy, 25 March 2008, 2007.

[9] S. C. Kleene. On the interpretation of intuitionistic number theory. *The Journal of Symbolic Logic*, 10(4), 1945.

[10] G. Koletsos. Church-Rosser theorem for typed functional systems. *Journal of Symbolic Logic*, 50(3), 1985.

[11] G. Koletsos, G. Stavrinos. Church-Rosser property and intersection types. *Australasian Journal of Logic*, 2007.

[12] J. L. Krivine. *Lambda-calcul, types et modèles.* Dunod, 1990.

[13] J.-J. Lévy. An algebraic interpretation of the *lambda beta* k-calculus; and an application of a labelled *lambda* -calculus. *Theoretical Compututer Science*, 2(1), 1976.

[14] W. W. Tait. Intensional interpretations of functionals of finite type I. *The Journal of Symbolic Logic*, 32(2), 1967.

[15] M. Takahashi. Parallel reductions in lambda-calculus. *Journal of Symbolic Computation*, 7(2), 1989.

A Proofs

Proof. [of Lemma 2.2]

i. If $r = \beta\eta$, the proof is by induction on the length of the reduction $M \to^*_{\beta\eta} N$.
- If $M = N$ then $M[x := P] = N[x := P]$. We prove that $N[x := P] \to^*_{\beta\eta} N[x := Q]$ by induction on the structure of N.
  - · Let $N \in \mathsf{Var}$. If $N = x$ then $N[x := P] = P \to^*_{\beta\eta} Q = N[x := Q]$, else $N[x := P] = N = N[x := Q]$.
  - · Let $N = \lambda y.N'$. By IH, $N[x := P] = \lambda y.N'[x := P] \to^*_{\beta\eta} \lambda y.N'[x := Q] = N[x := Q]$ such that $y \notin \text{fv}(PQx)$.
  - · Let $N = N_1N_2$. By IH, $N[x := P] = N_1[x := P]N_2[x := P] \to^*_{\beta\eta} N_1[x := Q]N_2[x := Q] = N[x := Q]$.
- Let $M \to^*_{\beta\eta} M' \to_{\beta\eta} N$. By IH, $M[x := P] \to^*_{\beta\eta} M'[x := Q]$. We prove that $M'[x := Q] \to_{\beta\eta} N[x := Q]$ by induction on the structure of M'.
  - · Let $M' \in \mathsf{Var}$ then nothing to prove since M' does not reduce.
  - · Let $M' = \lambda y.M'_1$.
    Either $N = \lambda y.M'_2$ such that $M'_1 \to_{\beta\eta} M'_2$. By IH, $M'_1[x := Q] \to_{\beta\eta} M'_2[x := Q]$. So $M'[x := Q] = \lambda y.M'_1[x := Q] \to_{\beta\eta} \lambda y.M'_2[x := Q] = N[x := Q]$ such that $y \notin \text{fv}(Qx)$.
    Or $M'_1 = Ny$ such that $y \notin \text{fv}(N)$. So $M'[x := Q] = \lambda y.N[x := Q]y \to_\eta N[x := Q]$ such that $y \notin \text{fv}(Qx)$.
  - · Let $M' = M_1M_2$.
    Either $N = M'_1M_2$ such that $M_1 \to_{\beta\eta} M'_1$. By IH, $M_1[x := Q] \to_{\beta\eta} M'_1[x := Q]$. So $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_{\beta\eta} M'_1[x := Q]M_2[x := Q] = N[x := Q]$.
    Or $N = M_1M'_2$ such that $M_2 \to_{\beta\eta} M'_2$. By IH, $M_2[x := Q] \to_{\beta\eta} M'_2[x := Q]$, so $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_{\beta\eta} M_1[x := Q]M'_2[x := Q] = N[x := Q]$.
    Or $M_1 = \lambda y.M'_1$ and $N = M'_1[y := M_2]$. So, $M'[x := Q] = (\lambda y.M'_1[x := Q])M_2[x := Q] \to_\beta M'_1[x := Q][y := M_2[x := Q]] = N[x := Q]$ by the well known substitution lemma and such that $y \notin \text{fv}(Qx)$.

If $r = \beta$, the proof is by induction on the length of the reduction $M \to^*_\beta N$.
- If $M = N$ then $M[x := P] = N[x := P]$. We prove that $N[x := P] \to^*_\beta N[x := Q]$ by induction on the structure of N.
  - · Let $N \in \mathsf{Var}$. If $N = x$ then $N[x := P] = P \to^*_\beta Q = N[x := Q]$, else $N[x := P] = N = N[x := Q]$.
  - · Let $N = \lambda y.N'$. By IH, $N[x := P] = \lambda y.N'[x := P] \to^*_\beta \lambda y.N'[x := Q] = N[x := Q]$ such that $y \notin \text{fv}(PQx)$.
  - · Let $N = N_1N_2$. By IH, $N[x := P] = N_1[x := P]N_2[x := P] \to^*_\beta N_1[x := Q]N_2[x := Q] = N[x := Q]$.
- Let $M \to^*_\beta M' \to_\beta N$. By IH, $M[x := P] \to^*_\beta M'[x := Q]$. We prove that $M'[x := Q] \to_\beta N[x := Q]$ by induction on the structure of M'.
  - · Let $M' \in \mathsf{Var}$ then nothing to prove since M' does not reduce.
  - · Let $M' = \lambda y.M'_1$. Then $N = \lambda y.M'_2$ such that $M'_1 \to_\beta M'_2$. By IH, $M'_1[x := Q] \to_\beta M'_2[x := Q]$, so $M'[x := Q] = \lambda y.M'_1[x := Q] \to_\beta \lambda y.M'_2[x := Q] = N[x := Q]$ such that $y \notin \text{fv}(Qx)$.

- Let $M' = M_1M_2$.
  Either $N = M_1'M_2$ such that $M_1 \to_\beta M_1'$. By IH, $M_1[x := Q] \to_\beta M_1'[x := Q]$, so $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_\beta M_1'[x := Q]M_2[x := Q] = N[x := Q]$.
  Or $N = M_1M_2'$ such that $M_2 \to_\beta M_2'$. By IH, $M_2[x := Q] \to_\beta M_2'[x := Q]$, so $M'[x := Q] = M_1[x := Q]M_2[x := Q] \to_\beta M_1[x := Q]M_2'[x := Q] = N[x := Q]$.
  Or $M_1 = \lambda y.M_1'$ and $N = M_1'[y := M_2]$. So, $M'[x := Q] = (\lambda y.M_1'[x := Q])M_2[x := Q] \to_\beta M_1'[x := Q][y := M_2[x := Q]] = N[x := Q]$ by the well known substitution lemma and such that $y \notin \text{fv}(Qx)$.

ii. We prove this lemma by induction on the structure of M.
- Let $M \in \mathsf{Var}$ then either $M = x$ and so $\text{fv}(M[x := N]) = \text{fv}(N) = \text{fv}((\lambda x.M)N)$. Or $M \neq x$ and so $\text{fv}(M[x := N]) = \text{fv}(M) \subseteq \text{fv}(M) \cup \text{fv}(N) = \text{fv}((\lambda x.M)N)$.
- Let $M = \lambda y.P$ then $\text{fv}(M[x := N]) = \text{fv}(\lambda y.P[x := N]) = \text{fv}(P[x := N]) \setminus \{y\} \subseteq^{IH} \text{fv}((\lambda x.P)N) \setminus \{y\} = \text{fv}((\lambda x.M)N)$ such that $y \notin \text{fv}(Nx)$.
- let $M = P_1P_2$ then $\text{fv}(M[x := N]) = \text{fv}(P_1[x := N]) \cup \text{fv}(P_2[x := N]) \subseteq^{IH} \text{fv}((\lambda x.P_1)N) \cup \text{fv}((\lambda x.P_2)N) = \text{fv}((\lambda x.M)N)$.

iii. We prove this lemma by induction on the length of the reduction $M \to^*_{\beta\eta} N$.
- Let $M = N$ then $\text{fv}(M) = \text{fv}(N)$.
- Let $M \to^*_{\beta\eta} M' \to_{\beta\eta} N$. By IH, $\text{fv}(M') \subseteq \text{fv}(M)$. We prove that $\text{fv}(N) \subseteq \text{fv}(M')$ by induction on the structure of M'.
  - Let $M' \in \mathsf{Var}$ then nothing to prove since M' does not reduce.
  - Let $M' = \lambda x.P$.
    Either $N = \lambda x.Q$ such that $P \to_{\beta\eta} Q$. By IH, $\text{fv}(Q) \subseteq \text{fv}(P)$. So $\text{fv}(N) \subseteq \text{fv}(M')$.
    Or $P = Nx$ such that $x \notin \text{fv}(N)$. So $\text{fv}(N) = \text{fv}(M')$.
  - Let $M' = P_1P_2$.
    Either $N = P_1'P_2$ such that $P_1 \to_{\beta\eta} P_1'$. By IH, $\text{fv}(P_1') \subseteq \text{fv}(P_1)$, so $\text{fv}(N) \subseteq \text{fv}(M')$.
    Or $N = P_1P_2'$ such that $P_2 \to_{\beta\eta} P_2'$. By IH, $\text{fv}(P_2') \subseteq \text{fv}(P_2)$, so $\text{fv}(N) \subseteq \text{fv}(M')$.
    Or $P_1 = \lambda x.P_0$ and $N = P_0[x := P_2]$. By Lemma 2.2.ii, $\text{fv}(N) \subseteq \text{fv}(M')$.

  A corollary of this result is that if $M \to^*_\beta N$ then $\text{fv}(N) \subseteq \text{fv}(M)$.

iv By induction on the length of the reduction $\lambda x.M \to^*_{\beta\eta} N$.
- Let $\lambda x.M = N$ then it is done.
- Let $\lambda x.M \to^*_{\beta\eta} P \to_{\beta\eta} N$. By IH:
  - Either $P = \lambda x.Q$ such that $M \to^*_{\beta\eta} Q$. Then, by compatibility:
    Either $Q = Nx$ such that $x \notin \text{fv}(N)$. So it is done since $M \to^*_{\beta\eta} Nx$.
    Or $N = \lambda x.M'$ such that $Q \to_{\beta\eta} M'$. So it is done since $M \to^*_{\beta\eta} M'$.
  - Or $M \to^*_{\beta\eta} Px$ such that $x \notin \text{fv}(P)$. So $M \to^*_{\beta\eta} Nx$ and it is done since by Lemma 2.2.iii, $x \notin \text{fv}(N)$.

v By induction on the length of the reduction $Mx \to^*_{\beta\eta} N$.
- Let $N = Mx$ then it is done.
- Let $Mx \to^*_{\beta\eta} P \to_{\beta\eta} N$. Then by IH, $M \to^*_{\beta\eta} Q$ (by Lemma 2.2.iii, $x \notin$

fv(Q)) and:

· Either $P = Qx$. Then, by compatibility:
  Either $N = Q'x$ such that $Q \rightarrow_{\beta\eta} Q'$. So it is done since $M \rightarrow^*_{\beta\eta} Q'$.
  Or $Q = \lambda y.Q'$ and $N = Q'[y := x]$. So $M \rightarrow^*_{\beta\eta} \lambda y.Q' = \lambda x.N$.

· Or $Q = \lambda x.P$. So it is done since $M \rightarrow^*_{\beta\eta} Q = \lambda x.P \rightarrow_{\beta\eta} \lambda x.N$.

vi. (a) If $k = 0$ then $P = Q$ is a direct r-reduct of Q, absurd.

(b) Assume $k = 1$, we prove $P = M[x := N]$ by case on r.

- Let $r = \beta$. The proof is by case on $Q = (\lambda x.M)N \rightarrow_\beta P$.
  · If $(\lambda x.M)N \rightarrow_\beta M[x := N]$ then we are done.
  · If $(\lambda x.M)N \rightarrow_\beta (\lambda x.M')N = P$ such that $M \rightarrow_\beta M'$ then P is a direct β-reduct of $(\lambda x.M)N$, absurd.
  · If $(\lambda x.M)N \rightarrow_\beta (\lambda x.M)N' = P$ such that $N \rightarrow_\beta N'$ then P is a direct β-reduct of $(\lambda x.M)N$, absurd.
- Let $r = \beta\eta$. The proof is by case on $Q = (\lambda x.M)N \rightarrow_{\beta\eta} P$.
  · If $(\lambda x.M)N \rightarrow_\beta M[x := N]$, then we are done.
  · If $\lambda x.M \rightarrow_{\beta\eta} R$ and $P = RN$ then:
    Either $R = \lambda x.M'$ such that $M \rightarrow_{\beta\eta} M'$. So P is a direct $\beta\eta$-reduct of $(\lambda x.M)N$, absurd.
    Or $M = Rx$ and $x \notin FV(R)$. Hence, $P = RN = M[x := N]$ and we are done.
  · If $N \rightarrow_{\beta\eta} N'$ and $P = (\lambda x.M)N'$ then P is a direct $\beta\eta$-reduct of $(\lambda x.M)N$, absurd.

(c) We prove the statement by induction on $k \geq 1$.

- If $k = 1$ then it is done since by (b) $P = M[x := N]$.
- Else, let $k \geq 1$ and $Q = (\lambda x.M)N \rightarrow^k_r R \rightarrow_r P$.
  · If R is a direct r-reduct of Q, then $R = (\lambda x.M')N'$, such that $M \rightarrow^*_r M'$ and $N \rightarrow^*_r N'$. Since P is not a direct r-reduct of Q, P is not a direct r-reduct of R. Hence by (b), $P = M'[x := N']$.
  · Else, by IH, there exists a direct r-reduct $(\lambda x.M')N'$ of Q such that $M'[x := N'] \rightarrow^*_r R \rightarrow_r P$.

vii. If P is a direct r-reduct of $(\lambda x.M)N$ then $P = (\lambda x.M')N'$ such that $M \rightarrow^*_r M'$ and $N \rightarrow^*_r N'$. So $P \rightarrow_r M'[x := N']$ and $M[x := N] \rightarrow^*_r M'[x := N']$, by Lemma 2.2.i. If P is not a direct r-reduct of $(\lambda x.M)N$ then by Lemma 2.2.vi, there exists a direct r-reduct, $(\lambda x.M')N'$ of $(\lambda x.M)N$ such that $M \rightarrow^*_r M'$, $N \rightarrow^*_r N'$ and $M'[x := N'] \rightarrow^*_r P$. Finally, by Lemma 2.2.i, $M[x := N] \rightarrow^*_r M'[x := N'] \rightarrow^*_r P$, .

viii.a) Let $n \geq 0$, $M[x := N] \in \mathsf{CR}^r$, $(\lambda x.M)N \rightarrow^*_r M_1$ and $(\lambda x.M)N \rightarrow^*_r M_2$. By Lemma 2.2.vii, there exist M'_1 and M'_2 such that $M_1 \rightarrow^*_r M'_1$, $M[x := N] \rightarrow^*_r M'_1$, $M_2 \rightarrow^*_r M'_2$ and $M[x := N] \rightarrow^*_r M'_2$. Then we conclude using $M[x := N] \in \mathsf{CR}^r$.

viii.b) Let $n \geq 0$ and for all $i \in \{1, \dots, n\}$, $M_i \in \mathsf{CR}^r$. First we prove that if $xM_1 \cdots M_n \rightarrow^*_r N$ then $N = xM'_1 \cdots M'_n$ such that for all $i \in \{1, \dots, n\}$, $M_i \rightarrow^*_r M'_i$. We prove the result by induction on the length of the reduction $xM_1 \cdots M_n \rightarrow^*_r N$.

- Let $xM_1 \cdots M_n = N$ then it is done

- Let $xM_1 \cdots M_n \to_r^* N' \to_r N$. By IH, $N' = xM'_1 \cdots M'_n$ such that for all $i \in \{1, \ldots, n\}$, $M_i \to_r^* M'_i$. We prove the result by induction on n.
  - Let $n = 0$ then it is done since x does not reduce by $\to_r$.
  - Let $n = m + 1$ such that $m \geq 0$. By compatibility:
    Either $N = PM'_n$ such that $xM'_1 \cdots M'_m \to_r P$ Then by IH $P = xM''_1 \cdots M''_m$ such that for all $i \in \{1, \ldots, m\}$, $M'_i \to_r^* M''_i$. So it is done.
    Or $N = xM'_1 \cdots M'_m M''_n$ such that $M'_n \to_r M''_n$ then it is done.

viii.c) Case β: Let $\lambda x.M \to_\beta^* P_1$ and $\lambda x.M \to_\beta^* P_2$ then $P_1 = \lambda x.M_1$ and $P_2 = \lambda x.M_2$ such that $M \to_\beta^* M_1$ and $M \to_\beta^* M_2$. By hypothesis, there exists M_3 such that $M_1 \to_\beta^* M_3$ and $M_2 \to_\beta^* M_3$. So $P_1 \to_\beta^* \lambda x.M_3$ and $P_2 \to_\beta^* \lambda x.M_3$.

Case $\beta\eta$: Let $\lambda x.M \to_{\beta\eta}^* P_1$ and $\lambda x.M \to_{\beta\eta}^* P_2$. By Lemma 2.2.iv:

- Either $P_1 = \lambda x.Q_1$ such that $M \to_{\beta\eta}^* Q_1$ and $P_2 = \lambda x.Q_2$ such that $M \to_{\beta\eta}^* Q_2$. So by hypothesis there exists Q_3 such that $Q_1 \to_{\beta\eta}^* Q_3$ and $Q_2 \to_{\beta\eta}^* Q_3$, hence, $P_1 \to_{\beta\eta}^* \lambda x.Q_3$ and $P_2 \to_{\beta\eta}^* \lambda x.Q_3$.
- Or $P_1 = \lambda x.Q_1$ such that $M \to_{\beta\eta}^* Q_1$ and $M \to_{\beta\eta}^* P_2 x$ such that $x \notin \text{fv}(P_2)$. By hypothesis there exists Q_3 such that $Q_1 \to_{\beta\eta}^* Q_3$ and $P_2 x \to_{\beta\eta}^* Q_3$. So, by Lemma 2.2.v $P_2 \to_{\beta\eta}^* Q_2$ (by Lemma 2.2.iii, $x \notin \text{fv}(Q_2)$) and:
  - Either $Q_3 = Q_2 x$. So $P_1 = \lambda x.Q_1 \to_{\beta\eta}^* \lambda x.Q_3 = \lambda x.Q_2 x \to_\eta Q_2$.
  - Or $Q_2 = \lambda x.Q_3$. So it is done since $P_1 = \lambda x.Q_1 \to_{\beta\eta}^* \lambda x.Q_3$.
- Or $M \to_{\beta\eta}^* P_1 x$ such that $x \notin \text{fv}(P_1)$ and $P_2 = \lambda x.Q_2$ such that $M \to_{\beta\eta}^* Q_2$. This case is similar to the previous one.
- Or $M \to_{\beta\eta}^* P_1 x$ such that $x \notin \text{fv}(P_1)$ and $M \to_{\beta\eta}^* P_2 x$ such that $x \notin \text{fv}(P_2)$. So by hypothesis, there exists Q_3 such that $P_1 x \to_{\beta\eta}^* Q_3$ and $P_2 x \to_{\beta\eta}^* Q_3$. By Lemma 2.2.v, $P_1 \to_{\beta\eta}^* Q_1$, $P_2 \to_{\beta\eta}^* Q_2$ (by Lemma 2.2.iii, $x \notin \text{fv}(Q_1) \cup \text{fv}(Q_2)$) and:
  - Either $Q_3 = Q_1 x$ and $Q_3 = Q_2 x$ so $Q_1 = Q_2$.
  - Or $Q_3 = Q_1 x$ and $Q_2 = \lambda x.Q_3$ so $Q_2 \to_\eta Q_1$.
  - Or $Q_1 = \lambda x.Q_3$ and $Q_3 = Q_2 x$ so $Q_1 \to_\eta Q_2$.
  - Or $Q_1 = \lambda x.Q_3$ and $Q_2 = \lambda x.Q_3$ so $Q_1 = Q_2$.

□

Proof. [of Lemma 2.9]

i By induction on the structure of M.
- Let $M = x$ then $\Psi_c(M) = M$.
- Let $M = \lambda x.N$. Let $x \neq c$. By IH, $\Psi_c(N) \to_c^* N$. Then, $\Psi_c(M) = \lambda x.\Psi_c(N) \to_c^* \lambda x.N = M$.
- Let $M = M_1 M_2$. By IH, for $i \in \{1, 2\}$, $\Psi_c(M_i) \to_c^* M_i$.
  - If M_1 is a λ-abstraction, then $\Psi_c(M) = \Psi_c(M_1)\Psi_c(M_2) \to_c^* M_1 M_2 = M$.
  - Else $\Psi_c(M) = c\Psi_c(M_1)\Psi_c(M_2) \to_c \Psi_c(M_1)\Psi_c(M_2) \to_c^* M_1 M_2 = M$.

ii By induction on the length of the reduction $M \to_c^* N$. The basic case ($M = N$) is trivial. Let us prove the induction case. Let $M \to_c M' \to_c^* N$. By IH, $\text{fv}(M') \setminus \{c\} = \text{fv}(N) \setminus \{c\}$. We prove that $\text{fv}(M) \setminus \{c\} = \text{fv}(M') \setminus \{c\}$ by induction on the size of the derivation of $M \to_c M'$ and then by case on the last rule of the derivation.
- Let $M = cM' \to_c M'$ then it is done.
- Let $M = \lambda x.P \to_c \lambda x.P' = M'$ such that $P \to_c P'$ then it is done by IH.

- Let $M = PQ \to_c P'Q = M'$ such that $P \to_c P'$ then it is done by IH.
- Let $M = PQ \to_c PQ' = M'$ such that $Q \to_c Q'$ then it is done by IH.

iii Corollary of Lemma 2.9.i and Lemma 2.9.ii.

iv Let $\bar{M} \in \Lambda_c^{\beta\eta}$. We prove by induction on the structure of $\bar{M}$ that $\bar{M} \notin \mathsf{A}_c$.
- Let $\bar{M} \in \mathsf{Var}_c$ then $\bar{M} \notin \mathsf{A}_c$.
- Let $\bar{M} = \lambda\bar{x}.\bar{M}_1$ then $\bar{M} \notin \mathsf{A}_c$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{M}_1)\bar{M}_2$ then because $\lambda\bar{x}.\bar{M}_1 \notin \mathsf{A}_c$ then $\bar{M} \notin \mathsf{A}_c$.
- Let $\bar{M} = c\bar{M}_1\bar{M}_2$. By IH, $\bar{M}_2 \notin \mathsf{A}_c$ so $\bar{M} \notin \mathsf{A}_c$.
- Let $\bar{M} = c\bar{M}_1$. By IH, $\bar{M}_1 \notin \mathsf{A}_c$ so $\bar{M} \notin \mathsf{A}_c$.

v We prove this lemma by induction on the structure of d.
- Let $d = c$ then $cM \to_c M$.
- Let $d = d_1d_2$ then by IH, $d = d_1d_2 \to_c^* d_2$ and again by IH, $d_2M \to_c^* M$, so by compatibility $dM \to_c^* M$.

vi $\Rightarrow$) We prove this lemma by induction on the length of the reduction $M \to_c^* d$.
- Let $M = d$ then it is done.
- Let $M \to_c^* M' \to_c d$. We prove the lemma by induction on the length of the derivation of $M' \to_c d$ and then by case on the last rule.
  - · Let $M' = cd \to_c d$ then $M' \in \mathsf{A}_c$ and by IH, $M \in \mathsf{A}_c$.
  - · Let $M' = \lambda x.M_1 \to_c \lambda x.M_2 = d$ such that $M_1 \to_c M_2$, then it is done because by case on d, $d \neq \lambda x.M_2$.
  - · Let $M' = M_1M_2 \to_c M_1'M_2 = d$ such that $M_1 \to_c M_1'$. By case on d, $M_1', M_2 \in \mathsf{A}_c$, so by IH, $M_1 \in \mathsf{A}_c$. Hence, $M' \in \mathsf{A}_c$ and by IH, $M \in \mathsf{A}_c$.
  - · Let $M' = M_1M_2 \to_c M_1M_2' = d$ such that $M_2 \to_c M_2'$. By case on d, $M_1, M_2' \in \mathsf{A}_c$ so by IH, $M_2 \in \mathsf{A}_c$. Hence $M' \in \mathsf{A}_c$ and by IH, $M \in \mathsf{A}_c$.

$\Leftarrow$) We prove this lemma by induction on the reduction $d \to_c^* N$.
- Let $d = N$ then it is done.
- Let $d \to_c^* N' \to_c N$. By IH, $N' \in \mathsf{A}_c$. We prove that $N \in \mathsf{A}_c$ by induction on the size of the derivation of $N' \to_c N$ and then by case on the last rule.
  - · Let $N' = cN \to_c N$ then $N \in \mathsf{A}_c$.
  - · Let $N' = \lambda x.P \to_c \lambda x.P' = N$ such that $P \to_c P'$ then it is done because by case on N', $N' \neq \lambda x.P$.
  - · Let $N' = PQ \to_c P'Q = N$ such that $P \to_c P'$. Then $P, Q \in \mathsf{A}_c$, by IH $P' \in \mathsf{A}_c$, so $N \in \mathsf{A}_c$.
  - · Let $N' = PQ \to_c PQ' = N$ such that $Q \to_c Q'$. Then $P, Q \in \mathsf{A}_c$, by IH $Q' \in \mathsf{A}_c$, so $N \in \mathsf{A}_c$.

vii We prove this lemma by induction on the length of the reduction $M \to_c^* N$. The basic case is trivial. Let us prove the induction case. Let $M \to_c M' \to_c^* N$. We prove the lemma by induction on the structure of M.
- Let $M = x$ then it is done since $M \to_c M'$ is wrong.
- Let $M = \lambda x.M_1$ then by compatibility $M' = \lambda x.M_1'$ such that $M_1 \to_c M_1'$. By IH, $N = \lambda x.N_1$ such that $M_1' \to_c^* N_1$. Hence, $M_1 \to_c^* N_1$.
- Let $M = M_1M_2$. By compatibility:
  - · Either $M' = M_1'M_2$ such that $M_1 \to_c M_1'$. By IH, either $M_1' \in \mathsf{A}_c$ and $M_2 \to_c^* N$ or $N = N_1N_2$ and $M_1' \to_c^* N_1$ and $M_2 \to_c^* N_2$. In the first case,

by Lemma 2.9.vi, $M_1 \in \mathsf{A}_c$. In the second case, $M_1 \to_c^* N_1$.

- Or $M' = M_1 M_2'$ such that $M_2 \to_c^* M_2'$. By IH, either $M_1 \in \mathsf{A}_c$ and $M_2' \to_c^* N$ or $N = N_1 N_2$ and $M_1 \to_c^* N_1$ and $M_2' \to_c^* N_2$. In the first case, $M_2 \to_c^* N$. In the second case, $M_2 \to_c^* N_2$.
- Or $M_1 = c$ and $M = cM_2 \to_c M_2 = M'$. Then it is done because $M = cM_2 \to_c M_2 = M' \to_c^* N$.

viii We prove this lemma by induction on the structure of M.

- Let $M = y$. By Lemma 2.9.vii, $M' = y$. If $y = x$ then $M[x := N] = N \to_c^* N' = M'[x := N']$. Else $y \neq x$ and $M[x := N] = M = M' = M'[x := N']$.
- Let $M = \lambda y.M_1$. Let $y \notin \text{fv}(N) \cup \text{fv}(N') \cup \{x\}$. Then by Lemma 2.9.vii, $M' = \lambda y.M_1'$ such that $M_1 \to_c^* M_1'$. Hence, by IH, $M[x := N] = \lambda y.M_1[x := N] \to_c^* \lambda y.M_1'[x := N'] = M'[x := N']$.
- Let $M = M_1 M_2$. By Lemma 2.9.vii, either $M_1 \in \mathsf{A}_c$ and $M_2 \to_c^* M'$ or $M' = M_1' M_2'$ and $M_1 \to_c^* M_1'$ and $M_2 \to_c^* M_2'$.
  - If $M_1 \in \mathsf{A}_c$ and $M_2 \to_c^* M'$ then by IH and Lemma 2.9.v, $M[x := N] = (M_1 M_2)[x := N] = M_1(M_2[x := N]) \to_c^* M_2[x := N] \to_c^* M'[x := N']$.
  - If $M' = M_1' M_2'$ and $M_1 \to_c^* M_1'$ and $M_2 \to_c^* M_2'$ then by IH, $M[x := N] = (M_1 M_2)[x := N] = M_1[x := N] M_2[x := N] \to_c^* M_1'[x := N'] M_2'[x := N'] = M'[x := N']$.

ix We prove this lemma by induction on the length of the reduction $M \to_c^* N$. The basic case is trivial. Let $M \to_c M' \to_c^* N$. We prove that $M \to_c M'$ is false by first proving that if $M \to_c M'$ then $c \in \text{fv}(M)$ by induction on the size of the derivation $M \to_c M'$ and then by case on the last rule of the derivation:

- Let $M = cM' \to_c M'$ then $c \in \text{fv}(M)$.
- Let $M = \lambda x.M_1 \to_c \lambda x.M_1' = M'$ such that $M_1 \to_c M_1'$. Let $x \neq c$. By IH, $c \in \text{fv}(M_1)$, hence $c \in \text{fv}(M)$.
- Let $M = M_1 M_1 \to_c M_1' M_2 = M'$ such that $M_1 \to_c M_1'$. By IH, $c \in \text{fv}(M_1) \subseteq \text{fv}(M)$.
- Let $M = M_1 M_2 \to_c M_1 M_2'$ such that $M_2 \to_c M_2'$. By IH, $c \in \text{fv}(M_2) \subseteq \text{fv}(M)$.

x We prove this lemma by induction on the structure of M.

- Let $M = x$ then by lemma 2.9.vii it is done because $M = P = N$.
- Let $M = \lambda x.M'$. Let $x \neq c$. By lemma 2.9.vii, $N = \lambda x.N'$ and $P = \lambda x.P'$ such that $M' \to_c^* N'$ and $M' \to_c^* N'$. By IH, $P' \to_c^* N'$, hence $P \to_c^* N$.
- Let $M = M_1 M_2$. By lemma 2.9.vii:
  - Either $M_2 \to_c^* P$, $M_2 \to_c^* N$ and $M_1 \in \mathsf{A}_c$. By IH, $P \to_c^* N$.
  - Or $M_2 \to_c^* P$, $M_1 \in \mathsf{A}_c$, $N = N_1 N_2$, $M_1 \to_c^* N_1$ and $M_2 \to_c^* N_2$. By lemma 2.9.vi, $N_1 \in \mathsf{A}_c$, so $c \in \text{fv}(N_1) \subseteq \text{fv}(N)$. We get a contradiction.
  - Or $P = P_1 P_2$, $M_1 \to_c^* P_1$, $M_2 \to_c^* P_2$, $M_1 \in \mathsf{A}_c$ and $M_2 \to_c^* N$. By IH, $P_2 \to_c^* N$. By lemma 2.9.vi, $P_1 \in \mathsf{A}_c$. By lemma 2.9.v, $P \to_c^* P_2 \to_c^* N$.
  - Or $P = P_1 P_2$, $N = N_1 N_2$, $M_1 \to_c^* P_1$, $M_1 \to_c^* N_1$, $M_2 \to_c^* P_2$, $M_2 \to_c^* N_2$. By IH, $P_1 \to_c^* N_1$ and $P_2 \to_c^* N_2$. Hence, $P \to_c^* N$.

□

Proof. [of Lemma 3.1] We prove the result by induction on the structure of $\bar{M}$:

- Let $\bar{M} = x \in \mathsf{Var}_c$ and $M \in s$ then $x[x := M] = M \in s$.

- Let $\bar{M} = \lambda\bar{x}.\bar{N}$. Let $\mathrm{fv}(\bar{N}) \setminus \{c, \bar{x}\} = \{x_1, \dots, x_n\}$ and $M_1, \dots, M_n \in s$. Let $\bar{x} \notin \mathrm{fv}(M_1) \cup \dots \cup \mathrm{fv}(M_n)$. Because $s \in \mathsf{VAR}$ then $\bar{x} \in s$. By IH, $\bar{N}[x_1 := M_1, \dots, x_n := M_n] \in s$. Because $s \in \mathsf{ABS}$ then $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \dots, x_n := M_n] \in s$.
- Let $\bar{M} = c\bar{P}\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c\} = \{x_1, \dots, x_n, x'_1, \dots, x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c\} = \{x_1, \dots, x_n, x''_1, \dots, x''_{n_2}\}$, $\{x'_1, \dots, x'_{n_1}\} \cap \{x''_1, \dots, x''_{n_2}\} = \varnothing$ and $M_1, \dots, M_n, M'_1, \dots, M'_{n_1}, M''_1, \dots, M''_{n_2} \in s$. By IH, $\bar{P}[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}]$, $\bar{Q}[x_1 := M_1, \dots, x_n := M_n, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}] \in s$. Because $s \in \mathsf{VAR}$ then $(c\bar{P}\bar{Q})[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}] \in s$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c, \bar{x}\} = \{x_1, \dots, x_n, x'_1, \dots, x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c\} = \{x_1, \dots, x_n, x''_1, \dots, x''_{n_2}\}$ and $M_1, \dots, M_n, M'_1, \dots, M'_{n_1}, M''_1, \dots, M''_{n_2} \in s$ and $\{x'_1, \dots, x'_{n_1}\} \cap \{x''_1, \dots, x''_{n_2}\} = \varnothing$. Let $\bar{x} \notin \mathrm{fv}(M_1) \cup \dots \cup \mathrm{fv}(M_n) \cup \mathrm{fv}(M'_1) \cup \dots \cup \mathrm{fv}(M'_{n_1}) \cup \mathrm{fv}(M''_1) \cup \dots \cup \mathrm{fv}(M''_{n_2})$. By IH, $Q' = \bar{Q}[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}] \in s$. By IH, $\bar{P}[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}, \bar{x} := Q'] \in s$. Because $s \in \mathsf{SAT}$, $((\lambda\bar{x}.\bar{P})\bar{Q})[x_1 := M_1, \dots, x_n := M_n, x'_1 := M'_1, \dots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \dots, x''_{n_2} := M''_{n_2}] \in s$.

□

Proof. [of Lemma 3.3] By induction on the structure of M.

- Let $M \in \mathsf{Var}$, so $\Psi_c(M) = M \in \mathsf{Var}_c$, since $M \neq c$.
- Let $M = \lambda x.N$. Let $x \neq c$. By IH, $\Psi_c(N) \in \Lambda_c^\beta$, so $\Psi_c(M) = \lambda x.\Psi_c(N) \in \Lambda_c^\beta$.
- Let $M = PQ$.
  · If $P = \lambda x.N$ such that $x \neq c$ then $\Psi_c(M) = (\lambda x.\Psi_c(N))\Psi_c(Q)$. By IH, $\Psi_c(N), \Psi_c(Q) \in \Lambda_c^\beta$, so $\Psi_c(M) \in \Lambda_c^\beta$.
  · Else $\Psi_c(M) = c\Psi_c(P)\Psi_c(Q)$. By IH, $\Psi_c(P), \Psi_c(Q) \in \Lambda_c^\beta$, so $\Psi_c(M) \in \Lambda_c^\beta$.

□

Proof. [of Lemma 3.4]

i By induction on the structure of $\bar{M}$.
  - Let $\bar{M} \in \mathsf{Var}_c$. Either $\bar{M} = \bar{x}$, then $\bar{M}[\bar{x} := \bar{N}] = \bar{N} \in \Lambda_c^\beta$. Or, $\bar{M} \neq \bar{x}$ and so $\bar{M}[\bar{x} := \bar{N}] = \bar{M} \in \Lambda_c^\beta$.
  - Let $\bar{M} = \lambda\bar{y}.\bar{P}$ such that $\bar{y} \in \mathsf{Var}_c$ and $\bar{P} \in \Lambda_c^\beta$. By IH, $\bar{P}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$. Then, $\bar{M}[\bar{x} := \bar{N}] = \lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
  - Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{y} \in \mathsf{Var}_c$ and $\bar{P}, \bar{Q} \in \Lambda_c^\beta$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$. Then, $\bar{M}[\bar{x} := \bar{N}] = (\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$ such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
  - Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda_c^\beta$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$. Then, $\bar{M}[\bar{x} := \bar{N}] = c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^\beta$.

ii We prove the lemma by induction on the length of the derivation $\bar{M} \to_\beta^* N$.
  - let $\bar{M} = N$ then it is done.
  - Let $\bar{M} \to_\beta^* M' \to_\beta N$. By IH, $M' \in \Lambda_c^\beta$. We prove that $N \in \Lambda_c^\beta$ by induction on the structure of M'.
    · Let $M' \in \mathsf{Var}_c$ then it is done because M' does not reduce.
    · Let $M' = \lambda x.P$ such that $x \in \mathsf{Var}_c$ and $P \in \Lambda_c^\beta$, so by compatibility $N =$

$\lambda x.P'$ such that $P \to_\beta P'$. By IH, $P' \in \Lambda_c^\beta$ so $N \in \Lambda_c^\beta$.

· Let $M' = (\lambda x.P)Q$ such that $x \in \mathsf{Var}_c$ and $P, Q \in \Lambda_c^\beta$. By compatibility:
  Either $N = (\lambda x.P')Q$ such that $P \to_\beta P'$. By IH, $P' \in \Lambda_c^\beta$ so $N \in \Lambda_c^\beta$.
  Or $N = (\lambda x.P)Q'$ such that $Q \to_\beta Q'$. By IH, $Q' \in \Lambda_c^\beta$ so $N \in \Lambda_c^\beta$.
  Or $N = P[x := Q]$, so by Lemma 3.4.i, $N \in \Lambda_c^\beta$

· Let $M' = cPQ$ such that $P, Q \in \Lambda_c^\beta$. By compatibility:
  Either $N = cP'Q$ such that $P \to_\beta P'$. By IH, $P' \in \Lambda_c^\beta$ so $N \in \Lambda_c^\beta$.
  Or $N = cPQ'$ such that $Q \to_\beta Q'$. By IH, $Q' \in \Lambda_c^\beta$ so $N \in \Lambda_c^\beta$.

iii We prove this lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_c$ then it is done because by lemma 2.9.vii, $N = \bar{M}$ and $\Psi_c(N) = \bar{M}$.
- Let $\bar{M} = \lambda\bar{x}.\bar{M}'$. By lemma 2.9.vii, $N = \lambda\bar{x}.N'$ such that $\bar{M}' \to_c^* N'$. By IH, $\bar{M}' \to_c^* \Psi_c(N')$. Hence, $\bar{M} \to_c^* \lambda\bar{x}.\Psi_c(N') = N$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{M}_1)\bar{M}_2$. By lemma 2.9.vii, $N = (\lambda\bar{x}.N_1)N_2$ such that $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. By IH, $\bar{M}_1 \to_c^* \Psi_c(N_1)$ and $\bar{M}_2\Psi_c(N_2)$, so $\bar{M} \to_c^* (\lambda\bar{x}.\Psi_c(N_1))\Psi_c(N_2) = \Psi_c(N)$.
- Let $\bar{M} = c\bar{M}_1\bar{M}_2$. By lemma 2.9.vii and lemma 2.9.iv:
  · Either $N = N_1N_2$ such that $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. By IH, $\bar{M}_1 \to_c^* \Psi_c(N_1)$ and $\bar{M}_2 \to_c^* \Psi_c(N_2)$. If N_1 is a λ-abstraction then $\bar{M} \to_c^* c\Psi_c(N_1)\Psi_c(N_2) \to_c \Psi_c(N_1)\Psi_c(N_2) = \Psi_c(N)$ else $\bar{M} \to_c^* c\Psi_c(N_1)\Psi_c(N_2) = \Psi_c(N)$.
  · Or $N = cN_1N_2$ such that $\bar{M}_2 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. We obtain a contradiction because by IH, $c \notin \mathrm{fv}(N)$.

iv We prove this lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_c$ then it is done with $N = \bar{M}$.
- Let $\bar{M} = \lambda\bar{x}.\bar{M}'$. By IH there exists N' such that $c \notin \mathrm{fv}(N')$ and $\bar{M}' \to_c^* N'$. So, $\bar{M} \to_c^* \lambda\bar{x}.N' = N$ and $c \notin \mathrm{fv}(N)$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{M}_1)\bar{M}_2$. By IH, there exists N_1, N_2 such that $c \notin \mathrm{fv}(N_1) \cup \mathrm{fv}(N_2)$, $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. So, $\bar{M} \to_c^* (\lambda\bar{x}.N_1)N_2 = N$ and $c \notin \mathrm{fv}(N)$.
- Let $\bar{M} = c\bar{M}_1\bar{M}_2$. By IH, there exists N_1, N_2 such that $c \notin \mathrm{fv}(N_1) \cup \mathrm{fv}(N_2)$, $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. So, $\bar{M} \to_c^* cN_1N_2 \to_c N_1N_2 = N$ and $c \notin \mathrm{fv}(N)$.

□

Proof. [of Lemma 3.5]

i By induction on the structure of $\bar{M}_1$.

- Let $\bar{M}_1 \in \mathsf{Var}_c$ then it is done because $\bar{M}_1$ does not reduce.
- Let $\bar{M}_1 = \lambda\bar{x}.\bar{P}_1$ such that $\bar{P}_1 \in \Lambda_c^\beta$ and $\bar{x} \in \mathsf{Var}_c$, then by Lemma 2.9.vii, $M_2 = \lambda\bar{x}.P_2$ such that $\bar{P}_1 \to_c^* P_2$ and by compatibility $N_1 = \lambda\bar{x}.Q_1$ such that $\bar{P}_1 \to_\beta Q_1$. By IH, there exists Q_2 such that $P_2 \to_\beta Q_2$ and $Q_1 \to_c^* Q_2$. So it is done with $N_2 = \lambda\bar{x}.Q_2$.
- let $\bar{M}_1 = (\lambda\bar{x}.\bar{P}_1)\bar{Q}_1$ such that $\bar{P}_1, \bar{Q}_1 \in \Lambda_c^\beta$ and $\bar{x} \in \mathsf{Var}_c$ then by Lemma 2.9.vii, $M_2 = (\lambda\bar{x}.P_2)Q_2$ such that $\bar{P}_1 \to_c^* P_2$ and $\bar{Q}_1 \to_c^* Q_2$. By compatibility:
  · Either $N_1 = (\lambda\bar{x}.P_1')\bar{Q}_1$ such that $\bar{P}_1 \to_\beta P_1'$. By IH, there exist P_2' such that $P_2 \to_\beta P_2'$ and $P_1' \to_c^* P_2'$. So it is done with $N_2 = (\lambda\bar{x}.P_2')Q_2$.
  · Or $N_1 = (\lambda\bar{x}.\bar{P}_1)Q_1'$ such that $\bar{Q}_1 \to_\beta Q_1'$. By IH, there exists Q_2' such that

$Q_2 \to_\beta Q'_2$ and $Q'_1 \to^*_c Q'_2$. So it is done with $N_2 = (\lambda\bar{x}.P_2)Q'_2$.
· Or $N_1 = \bar{P}_1[\bar{x} := \bar{Q}_1]$. By Lemma 2.9.viii, it is done with $N_2 = P_2[\bar{x} := Q_2]$.

- Let $\bar{M}_1 = c\bar{P}_1\bar{Q}_1$ such that $\bar{P}_1, \bar{Q}_1 \in \Lambda^\beta_c$. By Lemma 2.9.vii and Lemma 2.9.iv:
  · Either $M_2 = cP_2Q_2$ such that $\bar{P}_1 \to^*_c P_2$ and $\bar{Q}_1 \to^*_c Q_2$. By compatibility:
  Either $N_1 = cP'_1\bar{Q}_1$ such that $\bar{P}_1 \to_\beta P'_1$. By IH, there exists P'_2 such that $P_2 \to_\beta P'_2$ and $P'_1 \to^*_c P'_2$. So it is done with $N_2 = cP'_2Q_2$.
  Or $N_1 = c\bar{P}_1Q'_1$ such that $\bar{Q}_1 \to_\beta Q'_1$. By IH, there exists Q'_2 such that $Q_2 \to_\beta Q'_2$ and $Q'_1 \to^*_c Q'_2$. So it is done with $N_2 = cP_2Q'_2$.
  · Or $M_2 = P_2Q_2$ such that $\bar{P}_1 \to^*_c P_2$ and $\bar{Q}_1 \to^*_c Q_2$. By compatibility:
  Either $N_1 = cP'_1\bar{Q}_1$ such that $\bar{P}_1 \to_\beta P'_1$. By IH, there exists P'_2 such that $P_2 \to_\beta P'_2$ and $P'_1 \to^*_c P'_2$. So it is done with $N_2 = P'_2Q_2$.
  Or $N_1 = c\bar{P}_1Q'_1$ such that $\bar{Q}_1 \to_\beta Q'_1$. By IH, there exists Q'_2 such that $Q_2 \to_\beta Q'_2$ and $Q'_1 \to^*_c Q'_2$. So it is done with $N_2 = P_2Q'_2$.

ii By induction on the length of the reduction $M_1 \to^*_\beta N_1$ using Lemma 3.5.i.

□

Proof. [of Lemma 3.6]

⇒) Let $M \to^*_\beta N$. We prove that $M \to^*_1 N$ by induction on the size of the reduction $M \to^*_\beta N$.

- If $M = N$, then it is done since $M \to^*_1 N$.
- If $M \to^*_\beta M' \to_\beta N$. By Lemma 2.2.iii, $c \notin \mathrm{fv}(M')$. By IH, $M \to^*_1 M'$. We prove that $M' \to_1 N$ by induction on the structure of M'.
  · Let $M' \in \mathsf{Var}$ then it is done because M' does not reduce.
  · Let $M' = \lambda x.P$ such that $x \neq c$, then by compatibility $N = \lambda x.P'$ and $P \to_\beta P'$. By IH, $P \to_1 P'$. By definition, $\Psi_c(P) \to^*_\beta Q$ and $Q \to^*_c P'$. So $\Psi_c(\lambda x.P) = \lambda x.\Psi_c(P) \to^*_\beta \lambda x.Q$ and $\lambda x.Q \to^*_c \lambda x.P' = N$. Hence, $M' \to_1 N$.
  · Let $M' = PQ$.

  (a) If $P = \lambda x.P_1$ such that $x \neq c$ then by compatibility:
  Either $N = (\lambda x.P_2)Q$ such that $P_1 \to_\beta P_2$. By IH, $P_1 \to_1 P_2$. By definition, $\Psi_c(P_1) \to^*_\beta P'_1$ and $P'_1 \to^*_c P_2$. So, $\Psi_c(M') = (\lambda x.\Psi_c(P_1))\Psi_c(Q) \to^*_\beta (\lambda x.P'_1)\Psi_c(Q)$ and by lemma 2.9.i, $(\lambda x.P'_1)\Psi_c(Q) \to^*_c (\lambda x.P_2)Q = N$. Hence, $M' \to_1 N$.
  Or $N = (\lambda x.P_1)Q_1$ such that $Q \to_\beta Q_1$. By IH, $Q \to_1 Q_1$. By definition, $\Psi_c(Q) \to^*_\beta Q_2$ and $Q_2 \to^*_c Q_1$. So, $\Psi_c(M') = (\lambda x.\Psi_c(P_1))\Psi_c(Q) \to^*_\beta (\lambda x.\Psi_c(P_1))Q_2$ and by lemma 2.9.i, $(\lambda x.\Psi_c(P_1))Q_2 \to^*_c (\lambda x.P_1)Q_1 = N$. Hence, $M' \to_1 N$.
  Or $N = P_1[x := Q]$. So, $\Psi_c(M') = (\lambda x.\Psi_c(P_1))\Psi_c(Q) \to_\beta \Psi_c(P_1)[x := \Psi_c(Q)]$ and by lemma 2.9.i and lemma 2.9.viii $\Psi_c(P_1)[x := \Psi_c(Q)] \to^*_c P_1[x := Q]$. Hence, $M' \to_1 N$.

  (b) Else, by compatibility:
  Either $N = P'Q$ such that $P \to_\beta P'$. By IH, $P \to_1 P'$. By definition, $\Psi_c(P) \to^*_\beta P_1$ and $P_1 \to^*_c P'$. So, $\Psi_c(M') = c\Psi_c(P)\Psi_c(Q) \to^*_\beta cP_1\Psi_c(Q)$ and by lemma 2.9.i $cP_1\Psi_c(Q) \to^*_c cP'Q \to_c P'Q = N$. So $M' \to_1 N$.
  Or $N = PQ'$ such that $Q \to_\beta Q'$. By IH, $Q \to_1 Q'$. By definition, $\Psi_c(Q) \to^*_\beta Q_1$ and $Q_1 \to^*_c Q'$. So, $\Psi_c(M') = c\Psi_c(P)\Psi_c(Q) \to^*_\beta c\Psi_c(P)Q_1$

and by lemma 2.9.i $c\Psi_c(P)Q_1 \rightarrow_c^* cPQ' \rightarrow_c PQ' = N$. So $M' \rightarrow_1 N$.

$\Leftarrow$) Let $M \rightarrow_1^* N$. We prove that $M \rightarrow_\beta^* N$ by induction on the size of the derivation $M \rightarrow_1^* N$.

- Let $M = N$, then it is done since $M \rightarrow_\beta^* N$.
- Let $M \rightarrow_1^* M' \rightarrow_1 N$. By IH, $M \rightarrow_\beta^* M'$. Because $M' \rightarrow_1 N$ then by definition there exists P such that $\Psi_c(M') \rightarrow_\beta^* P$ and $P \rightarrow_c^* N$ and $c \notin \mathrm{fv}(M') \cup \mathrm{fv}(N)$. By Lemma 3.3, $\Psi_c(M') \in \Lambda_c^\beta$. By Lemma 2.9.i, $\Psi_c(M') \rightarrow_c^* M'$. By Lemma 3.5.ii, there exists Q such that $P \rightarrow_c^* Q$ and $M' \rightarrow_\beta^* Q$. By Lemma 2.2.iii, $c \notin \mathrm{fv}(Q)$. By Lemma 3.4.ii, $P \in \Lambda_c^\beta$. By lemma 2.9.x, $Q \rightarrow_c^* N$. By lemma 2.9.ix, $Q = N$. Hence $M' \rightarrow_\beta^* N$.

□

Proof. [of Lemma 3.7]

i By definition, there exist P_1, P_2 such that $\Psi_c(M) \rightarrow_\beta^* P_1$, $\Psi_c(M) \rightarrow_\beta^* P_2$, $P_1 \rightarrow_c^* M_1$, $P_2 \rightarrow_c^* M_2$ and $c \notin \mathrm{fv}(M) \cup \mathrm{fv}(M_1) \cup \mathrm{fv}(M_2)$. By Lemma 3.3, $\Psi_c(M) \in \Lambda_c^\beta$. So by Corollary 3.2, there exists P_3 such that $P_1 \rightarrow_\beta^* P_3$ and $P_2 \rightarrow_\beta^* P_3$. By Lemma 3.4.ii, $P_1, P_2, P_3 \in \Lambda_c^\beta$. By lemma 3.4.iv, there exists M_3 such that $P_3 \rightarrow_c^* M_3$ and $c \notin \mathrm{fv}(M_3)$. By Lemma 3.4.iii, $P_1 \rightarrow_c^* \Psi_c(M_1)$ and $P_2 \rightarrow_c^* \Psi_c(M_2)$. By Lemma 3.5.ii, there exist Q_1, Q_2 such that $P_3 \rightarrow_c^* Q_1$, $P_3 \rightarrow_c^* Q_2$, $\Psi_c(M_1) \rightarrow_\beta^* Q_1$ and $\Psi_c(M_2) \rightarrow_\beta^* Q_2$. By Lemma 2.9.x, $Q_1 \rightarrow_c^* M_3$ and $Q_2 \rightarrow_c^* M_3$. So $M_1 \rightarrow_1 M_3$ and $M_2 \rightarrow_1 M_3$.

ii By Lemma 3.7.i

□

Proof. [of Lemma 4.1] We prove the result by induction on the structure of $\bar{M}$:

- Let $\bar{M} = x \in \mathsf{Var}_c$ and $M \in s$ then $x[x := M] = M \in s$.
- Let $\bar{M} = \lambda\bar{x}.\bar{N}$. Let $\mathrm{fv}(\bar{N}) \setminus \{c, \bar{x}\} = \{x_1, \ldots, x_n\}$ and $M_1, \ldots, M_n \in s$. Let $\bar{x} \notin \mathrm{fv}(M_1) \cup \cdots \cup \mathrm{fv}(M_n)$. Because $s \in \mathsf{VAR}$ then $\bar{x} \in s$. By IH, $\bar{N}[x_1 := M_1, \ldots, x_n := M_n] \in s$. Because $s \in \mathsf{ABS}$ then $(\lambda\bar{x}.\bar{N})[x_1 := M_1, \ldots, x_n := M_n] \in s$.
- Let $\bar{M} = c\bar{P}\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c\} = \{x_1, \ldots, x_n, x'_1, \ldots, x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c\} = \{x_1, \ldots, x_n, x''_1, \ldots, x''_{n_2}\}$, $\{x'_1, \ldots, x'_{n_1}\} \cap \{x''_1, \ldots, x''_{n_2}\} = \varnothing$ and $M_1, \ldots, M_n, M'_1, \ldots, M'_{n_1}, M''_1, \ldots, M''_{n_2} \in s$. By IH, $\bar{P}[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}]$, $\bar{Q}[x_1 := M_1, \ldots, x_n := M_n, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}] \in s$. Because $s \in \mathsf{VAR}$ then $(c\bar{P}\bar{Q})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}] \in s$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{P})\bar{Q}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c, \bar{x}\} = \{x_1, \ldots, x_n, x'_1, \ldots, x'_{n_1}\}$, $\mathrm{fv}(\bar{Q}) \setminus \{c\} = \{x_1, \ldots, x_n, x''_1, \ldots, x''_{n_2}\}$ and $M_1, \ldots, M_n, M'_1, \ldots, M'_{n_1}, M''_1, \ldots, M''_{n_2} \in s$ and $\{x'_1, \ldots, x'_{n_1}\} \cap \{x''_1, \ldots, x''_{n_2}\} = \varnothing$. Let $\bar{x} \notin \mathrm{fv}(M_1) \cup \cdots \cup \mathrm{fv}(M_n) \cup \mathrm{fv}(M'_1) \cup \cdots \cup \mathrm{fv}(M'_{n_1}) \cup \mathrm{fv}(M''_1) \cup \cdots \cup \mathrm{fv}(M''_{n_2})$. By IH, $Q' = \bar{Q}[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}] \in s$. By IH, $\bar{P}[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}, \bar{x} := Q'] \in s$. Because $s \in \mathsf{SAT}$, $((\lambda\bar{x}.\bar{P})\bar{Q})[x_1 := M_1, \ldots, x_n := M_n, x'_1 := M'_1, \ldots, x'_{n_1} := M'_{n_1}, x''_1 := M''_1, \ldots, x''_{n_2} := M''_{n_2}] \in s$.
- Let $\bar{M} = c\bar{P}$. Let $\mathrm{fv}(\bar{P}) \setminus \{c\} = \{x_1, \ldots, x_n\}$ and $M_1, \ldots, M_n \in s$. By IH,

$\bar{P}[x_1 := M_1, \dots, x_n := M_n] \in s$. Because $s \in \mathsf{VAR}$ then $c(\bar{P}[x_1 := M_1, \dots, x_n := M_n]) = (c\bar{P})[x_1 := M_1, \dots, x_n := M_n] \in s$.

□

Proof. [of Lemma 4.4]

i By induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_c$. If $\bar{M} = \bar{x}$ then $\bar{M}[\bar{x} := \bar{N}] = \bar{N} \in \Lambda_c^{\beta\eta}$. Else $\bar{M}[\bar{x} := \bar{N}] = \bar{M} \in \Lambda_c^{\beta\eta}$.
- Let $\bar{M} = \lambda\bar{y}.\bar{P}$ such that $\bar{y} \in \mathsf{Var}_c$ and $\bar{P} \in \Lambda_c^{\beta\eta}$. Let $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$. By IH, $\bar{P}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = \lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$.
- Let $\bar{M} = (\lambda\bar{y}.\bar{P})\bar{Q}$ such that $\bar{y} \in \mathsf{Var}_c$ and $\bar{P}, \bar{Q} \in \Lambda_c^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = (\lambda\bar{y}.\bar{P}[\bar{x} := \bar{N}])\bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$, such that $\bar{y} \notin \mathrm{fv}(\bar{N}) \cup \{\bar{x}\}$.
- Let $\bar{M} = c\bar{P}\bar{Q}$ such that $\bar{P}, \bar{Q} \in \Lambda_c^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}], \bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = c\bar{P}[\bar{x} := \bar{N}]\bar{Q}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$.
- Let $\bar{M} = c\bar{P}$ such that $\bar{P} \in \Lambda_c^{\beta\eta}$. By IH, $\bar{P}[\bar{x} := \bar{N}] \in \Lambda_c^{\beta\eta}$. Then, $\bar{M}[\bar{x} := \bar{N}] = c(\bar{P}[\bar{x} := \bar{N}]) \in \Lambda_c^{\beta\eta}$.

ii We prove the lemma by induction on the length of the derivation $\bar{M} \to_{\beta\eta}^* N$.

- Let $\bar{M} = N$ then it is done.
- Let $\bar{M} \to_{\beta\eta}^* M' \to_{\beta\eta} N$. By IH, $M' \in \Lambda_c^{\beta\eta}$. We prove that $N \in \Lambda_c^{\beta\eta}$ by induction on the structure of M'.
  - · Let $M' \in \mathsf{Var}_c$ then it is done because M' does not reduce.
  - · Let $M' = \lambda x.P$ such that $x \in \mathsf{Var}_c$ and $P \in \Lambda_c^{\beta\eta}$. By compatibility:
    Either $N = \lambda x.P'$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda_c^{\beta\eta}$ so $N \in \Lambda_c^{\beta\eta}$.
    Or $P = Nx$ such that $x \notin \mathrm{fv}(N)$. Because $P \in \Lambda_c^{\beta\eta}$, by case on P, either $N = cN'$ such that $N' \in \Lambda_c^{\beta\eta}$, so $N = cN' \in \Lambda_c^{\beta\eta}$. Or $N = \lambda y.N'$ such that $y \in \mathsf{Var}_c$ and $N' \in \Lambda_c^{\beta\eta}$, so $N = \lambda y.N' \in \Lambda_c^{\beta\eta}$.
  - · Let $M' = (\lambda x.P)Q$ such that $x \in \mathsf{Var}_c$ and $P, Q \in \Lambda_c^{\beta\eta}$. By compatibility:
    Either $N = (\lambda x.P')Q$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda_c^{\beta\eta}$ so $N \in \Lambda_c^{\beta\eta}$.
    Or $N = P'Q$ and $P = P'x$ such that $x \notin \mathrm{fv}(P')$. Because $P \in \Lambda_c^{\beta\eta}$, either $P' = cP''$ such that $P'' \in \Lambda_c^{\beta\eta}$ so $N = cP''Q \in \Lambda_c^{\beta\eta}$. Or $P' = \lambda y.P''$ such that $P'' \in \Lambda_c^{\beta\eta}$ and $y \in \mathsf{Var}_c$ so $N = (\lambda y.P'')Q \in \Lambda_c^{\beta\eta}$.
    Or $N = (\lambda x.P)Q'$ such that $Q \to_{\beta\eta} Q'$. By IH, $Q' \in \Lambda_c^{\beta\eta}$ so $N \in \Lambda_c^{\beta\eta}$.
    Or $N = P[x := Q]$. So, by Lemma 4.4.i, $N \in \Lambda_c^{\beta\eta}$.
  - · Let $M' = cPQ$ such that $P, Q \in \Lambda_c^{\beta\eta}$. By compatibility:
    Either $N = cP'Q$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda_c^{\beta\eta}$ so $N \in \Lambda_c^{\beta\eta}$.
    Or $N = cPQ'$ such that $Q \to_{\beta\eta} Q'$. By IH, $Q' \in \Lambda_c^{\beta\eta}$ so $N \in \Lambda_c^{\beta\eta}$.
  - · Let $M' = cP$ such that $P \in \Lambda_c^{\beta\eta}$, so by compatibility $N = cP'$ such that $P \to_{\beta\eta} P'$. By IH, $P' \in \Lambda_c^{\beta\eta}$ so $N \in \Lambda_c^{\beta\eta}$.

iii We prove this lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_c$ then it is done because by lemma 2.9.vii, $N = \bar{M}$ and $\Psi_c(N) = \bar{M}$.
- Let $\bar{M} = \lambda\bar{x}.\bar{M}'$. By lemma 2.9.vii, $N = \lambda\bar{x}.N'$ such that $\bar{M}' \to_c^* N'$. By IH, $\bar{M}' \to_c^* \Psi_c(N')$. Hence, $\bar{M} \to_c^* \lambda\bar{x}.\Psi_c(N') = N$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{M}_1)\bar{M}_2$. By lemma 2.9.vii, $N = (\lambda\bar{x}.N_1)N_2$ such that $\bar{M}_1 \to_c^*$

N_1 and $\bar{M}_2 \to_c^* N_2$. By IH, $\bar{M}_1 \to_c^* \Psi_c(N_1)$ and $\bar{M}_2\Psi_c(N_2)$, so $\bar{M} \to_c^* (\lambda\bar{x}.\Psi_c(N_1))\Psi_c(N_2) = \Psi_c(N)$.

- Let $\bar{M} = c\bar{M}_1\bar{M}_2$. By lemma 2.9.vii and lemma 2.9.iv:
  - Either $N = N_1N_2$ such that $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. By IH, $\bar{M}_1 \to_c^* \Psi_c(N_1)$ and $\bar{M}_2 \to_c^* \Psi_c(N_2)$. If N_1 is a λ-abstraction then $\bar{M} \to_c^* c\Psi_c(N_1)\Psi_c(N_2) \to_c \Psi_c(N_1)\Psi_c(N_2) = \Psi_c(N)$ else $\bar{M} \to_c^* c\Psi_c(N_1)\Psi_c(N_2) = \Psi_c(N)$.
  - Or $N = cN_1N_2$ such that $\bar{M}_2 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. We obtain a contradiction because by IH, $c \notin \text{fv}(N)$.
- Let $\bar{M} = c\bar{M}'$. By lemma 2.9.vii:
  - Either $\bar{M}' \to_c^* N$. By IH, $\bar{M}' \to_c^* \Psi_c(N)$, so $\bar{M} \to_c \bar{M}' \to_c^* \Psi_c(N)$.
  - Or $N = cN'$ and $\bar{M}' \to_c^* N'$. We obtain a contradiction because by IH, $c \notin \text{fv}(N)$.

iv We prove this lemma by induction on the structure of $\bar{M}$.

- Let $\bar{M} \in \mathsf{Var}_c$ then it is done with $N = \bar{M}$.
- Let $\bar{M} = \lambda\bar{x}.\bar{M}'$. By IH there exists N' such that $c \notin \text{fv}(N')$ and $\bar{M}' \to_c^* N'$. So, $\bar{M} \to_c^* \lambda\bar{x}.N' = N$ and $c \notin \text{fv}(N)$.
- Let $\bar{M} = (\lambda\bar{x}.\bar{M}_1)\bar{M}_2$. By IH, there exists N_1, N_2 such that $c \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. So, $\bar{M} \to_c^* (\lambda\bar{x}.N_1)N_2 = N$ and $c \notin \text{fv}(N)$.
- Let $\bar{M} = c\bar{M}_1\bar{M}_2$. By IH, there exists N_1, N_2 such that $c \notin \text{fv}(N_1) \cup \text{fv}(N_2)$, $\bar{M}_1 \to_c^* N_1$ and $\bar{M}_2 \to_c^* N_2$. So, $\bar{M} \to_c^* cN_1N_2 \to_c N_1N_2 = N$ and $c \notin \text{fv}(N)$.
- Let $\bar{M} = c\bar{M}'$. By IH, there exists N such that $c \notin \text{fv}(N)$ and $\bar{M}' \to_c^* N$. So, $\bar{M} \to_c \bar{M}' \to_c^* N$.

□

Proof. [of Lemma 4.5]

i We prove this lemma by induction on the structure of M_1.

- Let $\bar{M}_1 \in \mathsf{Var}_c$, then it is done because $\bar{M}_1$ does not reduce.
- Let $\bar{M}_1 = \lambda\bar{x}.\bar{P}_1$ such that $\bar{x} \in \mathsf{Var}_c$ and $\bar{P}_1 \in \Lambda_c^{\beta\eta}$. Then, by Lemma 2.9.vii, $M_2 = \lambda\bar{x}.P_2$ such that $\bar{P}_1 \to_c^* P_2$. By compatibility:
  - Either $N_1 = \lambda\bar{x}.P_1'$ such that $\bar{P}_1 \to_{\beta\eta} P_1'$. By IH, there exits P_2' such that $P_2 \to_{\beta\eta} P_2'$ and $P_1' \to_c^* P_2'$. So it is done with $N_2 = \lambda\bar{x}.P_2'$.
  - Or $\bar{P}_1 = N_1\bar{x}$ such that $\bar{x} \notin \text{fv}(N_1)$. Because $\bar{P}_1 \in \Lambda_c^{\beta\eta}$ then by case on $\bar{P}_1$, $N_1 \in \Lambda_c^{\beta\eta}$. By Lemma 2.9.vii and Lemma 2.9.iv, $P_2 = N_1'\bar{x}$ and $N_1 \to_c^* N_1'$. By Lemma 2.9.ii, $\bar{x} \notin \text{fv}(N_1')$. So $M_2 = \lambda\bar{x}.N_1'\bar{x} \to_\eta N_1' = N_2$.
- Let $\bar{M}_1 = (\lambda\bar{x}.\bar{P}_1)\bar{Q}_1$ such that $\bar{x} \in \mathsf{Var}_c$ and $\bar{P}_1, \bar{Q}_1 \in \Lambda_c^{\beta\eta}$. By Lemma 2.9.vii, $M_2 = (\lambda\bar{x}.P_2)Q_2$ such that $\bar{P}_1 \to_c^* P_2$ and $\bar{Q}_1 \to_c^* Q_2$. By compatibility:
  - Either, $N_1 = \bar{P}_1[\bar{x} := \bar{Q}_1]$. We have, $M_2 \to_\beta P_2[\bar{x} := Q_2] = N_2$ and by Lemma 2.9.viii, $N_1 \to_c^* N_2$.
  - Or, $N_1 = (\lambda\bar{x}.P_1')\bar{Q}_1$ such that $\bar{P}_1 \to_{\beta\eta} P_1'$. By IH, there exists P_2' such that $P_2 \to_{\beta\eta} P_2'$ and $P_1' \to_c^* P_2'$. So, $M_2 = (\lambda\bar{x}.P_2)Q_2 \to_{\beta\eta} (\lambda\bar{x}.P_2')Q_2 = N_2$ and $N_1 \to_c^* N_2$.
  - Or $\bar{P}_1 = R_1\bar{x}$ such that $\bar{x} \notin \text{fv}(R_1)$ and $N_1 = R_1\bar{Q}_1$. Because $\bar{P}_1 \in \Lambda_c^{\beta\eta}$ then by case on $\bar{P}_1$, $R_1 \in \Lambda_c^{\beta\eta}$. By Lemma 2.9.vii and Lemma 2.9.iv, $P_2 = R_1'\bar{x}$ and $R_1 \to_c^* R_1'$. By Lemma 2.9.ii, $\bar{x} \notin \text{fv}(R_1')$. So $M_2 = (\lambda\bar{x}.R_1'\bar{x})Q_2 \to_\eta R_1'Q_2 = N_2$ and $N_1 = R_1\bar{Q}_1 \to_c^* N_2$.

- Or, $N_1 = (\lambda\bar{x}.\bar{P}_1)Q'_1$ such that $\bar{Q}_1 \rightarrow_{\beta\eta} Q'_1$. By IH, there exist Q'_2 such that $Q_2 \rightarrow_{\beta\eta} Q'_2$ and $Q'_1 \rightarrow^*_c Q'_2$. So, $M_2 = (\lambda\bar{x}.P_2)Q_2 \rightarrow_{\beta\eta} (\lambda\bar{x}.P_2)Q'_2 = N_2$ and $N_1 \rightarrow^*_c N_2$.

- Let $\bar{M}_1 = c\bar{P}_1\bar{Q}_1$ such that $\bar{P}_1, \bar{Q}_1 \in \Lambda^{\beta\eta}_c$. By compatibility:
  - Either $N_1 = cP'_1\bar{Q}_1$ such that $\bar{P}_1 \rightarrow_{\beta\eta} P'_1$. By Lemma 2.9.vii and Lemma 2.9.iv:
    Either $M_2 = P_2Q_2$ such $\bar{P}_1 \rightarrow^*_c P_2$ and $\bar{Q}_1 \rightarrow^*_c Q_2$. By IH, there exists P'_2 such that $P'_1 \rightarrow^*_c P'_2$ and $P_2 \rightarrow_{\beta\eta} P'_2$. So it is done with $N_2 = P'_2Q_2$.
    Or $M_2 = cP_2Q_2$ such that $\bar{P}_1 \rightarrow^*_c P_2$ and $\bar{Q}_1 \rightarrow^*_c Q_2$. By IH, there exists P'_2 such that $P'_1 \rightarrow^*_c P'_2$ and $P_2 \rightarrow_{\beta\eta} P'_2$. So it is done with $N_2 = cP'_2Q_2$.
  - Or $N_1 = c\bar{P}_1Q'_1$ such that $\bar{Q}_1 \rightarrow_{\beta\eta} Q'_1$. By Lemma 2.9.vii and Lemma 2.9.iv:
    Either $M_2 = P_2Q_2$ such $\bar{P}_1 \rightarrow^*_c P_2$ and $\bar{Q}_1 \rightarrow^*_c Q_2$. By IH, there exists Q'_2 such that $Q'_1 \rightarrow^*_c Q'_2$ and $Q_2 \rightarrow_{\beta\eta} Q'_2$. So it is done with $N_2 = P_2Q'_2$.
    Or $M_2 = cP_2Q_2$ such that $\bar{P}_1 \rightarrow^*_c P_2$ and $\bar{Q}_1 \rightarrow^*_c Q_2$. By IH, there exists Q'_2 such that $Q'_1 \rightarrow^*_c Q'_2$ and $Q_2 \rightarrow_{\beta\eta} Q'_2$. So it is done with $N_2 = cP_2Q'_2$.
- Let $\bar{M}_1 = c\bar{P}_1$ such that $\bar{P}_1 \in \Lambda^{\beta\eta}_c$. Then by compatibility $N_1 = cP'_1$ such that $\bar{P}_1 \rightarrow_{\beta\eta} P'_1$. By Lemma 2.9.vii:
  - Either $M_2 = P_2$ and $\bar{P}_1 \rightarrow^*_c P_2$. By IH, there exists P'_2 such that $P_2 \rightarrow_{\beta\eta} P'_2$ and $P'_1 \rightarrow^*_c P'_2$. So it is done with $N_2 = P'_2$.
  - Or $M_2 = cP_2$ and $\bar{P}_1 \rightarrow^*_c P_2$. By IH, there exists P'_2 such that $P_2 \rightarrow_{\beta\eta} P'_2$ and $P'_1 \rightarrow^*_c P'_2$. So it is done with $N_2 = cP'_2$.

ii Easy by Lemma 4.5.i.

□

Proof. [of Lemma 4.6]

⇒) Let $M \rightarrow^*_{\beta\eta} N$. We prove that $M \rightarrow^*_2 N$ by induction on the size of the reduction $M \rightarrow^*_{\beta\eta} N$.
- ▼ If $M = N$, then it is done since $M \rightarrow^*_2 N$.
- ▼ If $M \rightarrow^*_{\beta\eta} M' \rightarrow_{\beta\eta} N$. By Lemma 2.2.iii, $c \notin \mathrm{fv}(M')$. By IH, $M \rightarrow^*_2 M'$. We prove that $M' \rightarrow_2 N$ by induction on the structure of M'.
  - Let $M' \in \mathsf{Var}$. It is done because M' does not reduce.
  - Let $M' = \lambda x.P$ such that $x \neq c$. By compatibility:
    - Either $N = \lambda x.P'$ such that $P \rightarrow_{\beta\eta} P'$. By IH, $P \rightarrow_2 P'$. By definition there exists Q such that $\Psi_c(P) \rightarrow^*_{\beta\eta} Q$ and $Q \rightarrow^*_c P'$. Then $\Psi_c(M') = \lambda x.\Psi_c(P) \rightarrow^*_{\beta\eta} \lambda x.Q$ and $\lambda x.Q \rightarrow^*_c \lambda x.P'$. Hence, $M' \rightarrow_2 N$.
    - Or $P = Nx$ such that $x \notin \mathrm{fv}(N)$. By Lemma 2.9.iii, $x \notin \mathrm{fv}(\Psi_c(N))$.
      If N is a λ-abstraction then $\Psi_c(M') = \lambda x.\Psi_c(P) = \lambda x.\Psi_c(N)x \rightarrow_\eta \Psi_c(N)$ and by lemma 2.9.i, $\Psi_c(N) \rightarrow^*_c N$. Hence, $M' \rightarrow_2 N$.
      Else, $\Psi_c(M') = \lambda x.\Psi_c(P) = \lambda x.c\Psi_c(N)x \rightarrow_\eta c\Psi_c(N)$ and by lemma 2.9.i, $c\Psi_c(N) \rightarrow_c \Psi_c(N) \rightarrow^*_c N$. Hence, $M' \rightarrow_2 N$.
  - Let $M' = PQ$.
    - If $P = \lambda x.P_1$, such that $x \neq c$ then $M' = (\lambda x.P_1)Q$ and by compatibility:
      Either $N = (\lambda x.P_2)Q$ and $P_1 \rightarrow_{\beta\eta} P_2$. By IH, $P_1 \rightarrow_2 P_2$. By definition there exists P'_1 such that $\Psi_c(P_1) \rightarrow^*_{\beta\eta} P'_1$ and $P'_1 \rightarrow^*_c P_2$. So, $\Psi_c(M') = (\lambda x.\Psi_c(P_1))\Psi_c(Q) \rightarrow^*_{\beta\eta} (\lambda x.P'_1)\Psi_c(Q)$ and by lemma 2.9.i, $(\lambda x.P'_1)\Psi_c(Q) \rightarrow^*_c (\lambda x.P_2)Q = N$. Hence, $M' \rightarrow_2 N$.
      Or, $N = P_0Q$ and $P_1 = P_0x$ such that $x \notin \mathrm{fv}(P_0)$. By Lemma 2.9.iii,

$x \notin \text{fv}(\Psi_c(P_0))$. If P_0 is a λ-abstraction then $\Psi_c(M') = (\lambda x.\Psi_c(P_0)x)\Psi_c(Q) \to_\eta \Psi_c(P_0)\Psi_c(Q) = \Psi_c(N)$. Else, $\Psi_c(M') = (\lambda x.c\Psi_c(P_0)x)\Psi_c(Q) \to_\eta c\Psi_c(P_0)\Psi_c(Q) = \Psi_c(N)$. In both cases by lemma 2.9.i, $\Psi_c(N) \to_c^* N$, and so, $M' \to_2 N$.
Or $N = (\lambda x.P_1)Q_1$ such that $Q \to_{\beta\eta} Q_1$. By IH, $Q \to_2 Q_1$. By definition there exists Q_2 such that $\Psi_c(Q) \to_{\beta\eta}^* Q_2$ and $Q_2 \to_c^* Q_1$. So, $\Psi_c(M') = (\lambda x.\Psi_c(P_1))\Psi_c(Q) \to_{\beta\eta}^* (\lambda x.\Psi_c(P_1))Q_2$ and by lemma 2.9.i, $(\lambda x.\Psi_c(P_1))Q_2 \to_c^* (\lambda x.P_1)Q_1 = N$. Hence, $M' \to_2 N$.
Or $N = P_1[x := Q]$. So, $\Psi_c(M') = (\lambda x.\Psi_c(P_1))\Psi_c(Q) \to_\beta \Psi_c(P_1)[x := \Psi_c(Q)]$ and by lemma 2.9.i and lemma 2.9.viii, $\Psi_c(P_1)[x := \Psi_c(Q)] \to_c^* P_1[x := Q]$. Hence, $M' \to_1 N$.

· Else,
Either $N = P'Q$ such that $P \to_{\beta\eta} P'$. By IH, $P \to_2 P'$. By definition, there exists P_1 such that $\Psi_c(P) \to_{\beta\eta}^* P_1$ and $P_1 \to_c^* P'$. So, $\Psi_c(M') = c\Psi_c(P)\Psi_c(Q) \to_{\beta\eta}^* cP_1\Psi_c(Q)$ and by lemma 2.9.i, $cP_1\Psi_c(Q) \to_c^* cP'Q \to_c N$. So $M' \to_2 N$.
Or $N = PQ'$ such that $Q \to_{\beta\eta} Q'$. By IH, $Q \to_2 Q'$. By definition, there exists Q_1 such that $\Psi_c(Q) \to_{\beta\eta}^* Q_1$ and $Q_1\ ffoQ'$. So, $\Psi_c(M') = c\Psi_c(P)\Psi_c(Q) \to_\beta^* c\Psi_c(P)Q_1$ and by lemma 2.9.i, $c\Psi_c(P)Q_1 \to_c^* cPQ' \to_c N$. So $M' \to_2 N$.

$\Leftarrow$) Let $M \to_2^* N$. We prove that $M \to_{\beta\eta}^* N$ by induction on the size of the derivation $M \to_2^* N$.

- Let $M = N$, then it is done because $M \to_{\beta\eta}^* N$.
- Let $M \to_2^* M' \to_2 N$. By IH, $M \to_{\beta\eta}^* M'$. Because $M' \to_2 N$ then by definition there exists P such that $\Psi_c(M') \to_{\beta\eta}^* P$ and $P \to_c^* N$ and $c \notin \text{fv}(M') \cup \text{fv}(N)$. By Lemma 4.3, $\Psi_c(M') \in \Lambda_c^{\beta\eta}$. By Lemma 2.9.i, $\Psi_c(M') \to_c^* M'$. By Lemma 4.5.ii, there exists Q such that $P \to_c^* Q$ and $M' \to_{\beta\eta}^* Q$. By Lemma 2.2.iii, $c \notin \text{fv}(Q)$. By Lemma 4.4.ii, $P \in \Lambda_c^{\beta\eta}$. By lemma 2.9.x, $Q \to_c^* N$. By lemma 2.9.ix, $Q = N$. Hence $M' \to_{\beta\eta}^* N$.

□

Proof. [of Lemma 4.7]

i By definition, there exist P_1, P_2 such that $\Psi_c(M) \to_{\beta\eta}^* P_1$, $\Psi_c(M) \to_{\beta\eta}^* P_2$, $P_1 \to_c^* M_1$, $P_2 \to_c^* M_2$ and $c \notin \text{fv}(M) \cup \text{fv}(M_1) \cup \text{fv}(M_2)$. By Lemma 4.3, $\Psi_c(M) \in \Lambda_c^{\beta\eta}$. So by Corollary 4.2, there exists P_3 such that $P_1 \to_{\beta\eta}^* P_3$ and $P_2 \to_{\beta\eta}^* P_3$. By Lemma 4.4.ii, $P_1, P_2, P_3 \in \Lambda_c^{\beta\eta}$. By lemma 4.4.iv, there exists M_3 such that $P_3 \to_c^* M_3$ and $c \notin \text{fv}(M_3)$. By Lemma 4.4.iii, $P_1 \to_c^* \Psi_c(M_1)$ and $P_2 \to_c^* \Psi_c(M_2)$. By Lemma 4.5.ii, there exist Q_1, Q_2 such that $P_3 \to_c^* Q_1$, $P_3 \to_c^* Q_2$, $\Psi_c(M_1) \to_{\beta\eta}^* Q_1$ and $\Psi_c(M_2) \to_{\beta\eta}^* Q_2$. By Lemma 2.9.x, $Q_1 \to_c^* M_3$ and $Q_2 \to_c^* M_3$. So $M_1 \to_2 M_3$ and $M_2 \to_2 M_3$.

ii Easy by Lemma 4.7.i.

□

Proof. [of lemma 5.2]

ii Let $M \Rightarrow_\beta N$. We prove that $M \to_1 N$ by induction on the size of the derivation of $M \Rightarrow_\beta N$ and then by case on the last rule of the derivation.

- Let $M \Rightarrow_\beta M = N$ then it is done because by lemma 2.9.i, $\Psi_c(M) \rightarrow_c^* M$.
- Let $M = \lambda x.P \Rightarrow_\beta \lambda x.P' = N$ such that $P \Rightarrow_\beta P'$. Let $x \neq c$. Then $c \notin \text{fv}(P) \cup \text{fv}(P')$. By IH, $P \rightarrow_1 P'$. By definition, there exists Q where $\Psi_c(P) \rightarrow_\beta^* Q \rightarrow_c^* P'$. So $\Psi_c(M) = \lambda x.\Psi_c(P) \rightarrow_\beta^* \lambda x.Q \rightarrow_c^* \lambda x.P' = N$. Hence $M \rightarrow_1 N$.
- Let $M = PQ \Rightarrow_\beta P'Q' = N$ such that $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. Then $c \notin \text{fv}(P) \cup \text{fv}(P') \cup \text{fv}(Q) \cup \text{fv}(Q')$. By IH, $P \rightarrow_1 P'$ and $Q \rightarrow_1 Q'$. By definition, where P'' and Q'' such that $\Psi_c(P) \rightarrow_\beta^* P'' \rightarrow_c^* P'$ and $\Psi_c(Q) \rightarrow_\beta^* Q'' \rightarrow_c^* Q'$.
  · If P is a λ-abstraction then $\Psi_c(M) = \Psi_c(P)\Psi_c(Q) \rightarrow_\beta^* P''Q'' \rightarrow_c^* P'Q' = N$. So $M \rightarrow_1 N$.
  · Else $\Psi_c(M) = c\Psi_c(P)\Psi_c(Q) \rightarrow_\beta^* cP''Q'' \rightarrow_c^* P'Q' = N$. So $M \rightarrow_1 N$.
- Let $M = (\lambda x.P)Q \Rightarrow_\beta P'[x := Q'] = N$ such that $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. Let $x \neq c$. Then $c \notin \text{fv}(P) \cup \text{fv}(Q)$. By lemma 5.2.i, $c \notin \text{fv}(P') \cup \text{fv}(Q')$. By IH, $P \rightarrow_1 P'$ and $Q \rightarrow_1 Q'$. By definition, there exist P'' and Q'' such that $\Psi_c(P) \rightarrow_\beta^* P'' \rightarrow_c^* P'$ and $\Psi_c(Q) \rightarrow_\beta^* Q'' \rightarrow_c^* Q'$. So $\Psi_c(M) = (\lambda x.\Psi_c(P))\Psi_c(Q) \rightarrow_\beta^* (\lambda x.P'')Q'' \rightarrow_\beta P''[x := Q'']$ and by lemma 2.9.viii $P''[x := Q''] \rightarrow_c^* P'[x := Q'] = N$. So $M \rightarrow_1 N$.

iii. Let $M \Rightarrow_{\beta\eta} N$. We prove that $M \rightarrow_2 N$ by induction on the size of the derivation of $M \Rightarrow_{\beta\eta} N$ and then by case on the last rule of the derivation.

- Let $M \Rightarrow_{\beta\eta} M = N$ then it is done because by lemma 2.9.i, $\Psi_c(M) \rightarrow_c^* M$.
- Let $M = \lambda x.P \Rightarrow_{\beta\eta} \lambda x.P' = N$ such that $P \Rightarrow_{\beta\eta} P'$. Let $x \neq c$. Then $c \notin \text{fv}(P) \cup \text{fv}(P')$. By IH, $P \rightarrow_2 P'$. By definition, there exists Q such that $\Psi_c(P) \rightarrow_{\beta\eta}^* Q$ and $Q \rightarrow_c^* P'$. So $\Psi_c(M) = \lambda x.\Psi_c(P) \rightarrow_{\beta\eta}^* \lambda x.Q$ and $\lambda x.Q \rightarrow_c^* \lambda x.P' = N$. So $M \rightarrow_2 N$.
- Let $M = PQ \Rightarrow_{\beta\eta} P'Q' = N$ such that $P \Rightarrow_{\beta\eta} P'$ and $Q \Rightarrow_{\beta\eta} Q'$. Then $c \notin \text{fv}(P) \cup \text{fv}(P') \cup \text{fv}(Q) \cup \text{fv}(Q')$. By IH, $P \rightarrow_2 P'$ and $Q \rightarrow_2 Q'$. By definition, there exist P'' and Q'' such that $\Psi_c(P) \rightarrow_{\beta\eta}^* P''$, $\Psi_c(Q) \rightarrow_{\beta\eta}^* Q''$, $P'' \rightarrow_c^* P'$ and $Q'' \rightarrow_c^* Q'$.
  · If P is a λ-abstraction then $\Psi_c(M) = \Psi_c(P)\Psi_c(Q) \rightarrow_{\beta\eta}^* P''Q''$ and $P''Q'' \rightarrow_c^* P'Q' = N$. So $M \rightarrow_2 N$.
  · Else $\Psi_c(M) = c\Psi_c(P)\Psi_c(Q) \rightarrow_{\beta\eta}^* cP''Q''$ and $cP''Q'' \rightarrow_c P''Q'' \rightarrow_c^* P'Q' = N$. So $M \rightarrow_2 N$.
- Let $M = (\lambda x.P)Q \Rightarrow_{\beta\eta} P'[x := Q'] = N$ such that $P \Rightarrow_{\beta\eta} P'$ and $Q \Rightarrow_{\beta\eta} Q'$. Let $x \neq c$. Then $c \notin \text{fv}(P) \cup \text{fv}(Q)$. By lemma 5.2.i, $c \notin \text{fv}(P') \cup \text{fv}(Q')$. By IH, $P \rightarrow_2 P'$ and $Q \rightarrow_2 Q'$. By definition, there exist P'' and Q'' such that $\Psi_c(P) \rightarrow_{\beta\eta}^* P''$, $\Psi_c(Q) \rightarrow_{\beta\eta}^* Q''$, $P'' \rightarrow_c^* P'$ and $Q'' \rightarrow_c^* Q'$. So $\Psi_c(M) = (\lambda x.\Psi_c(P))\Psi_c(Q) \rightarrow_{\beta\eta}^* (\lambda x.P'')Q'' \rightarrow_\beta P''[x := Q'']$ and by lemma 2.9.viii $P''[x := Q''] \rightarrow_c^* P'[x := Q'] = N$. So $M \rightarrow_2 N$.
- Let $M = \lambda x.Px \Rightarrow_{\beta\eta} N$ such that $P \Rightarrow_{\beta\eta} N$ and $x \notin \text{fv}(P)$. Then $c \notin \text{fv}(P)$. Let $x \neq c$. By IH, $P \rightarrow_2 N$. By definition, there exists Q such that $\Psi_c(P) \rightarrow_{\beta\eta}^* Q$ and $Q \rightarrow_c^* N$. By lemma 2.9.iii, $x \notin \text{fv}(\Psi_c(P))$.
  · If P is a λ-abstraction then $\Psi_c(M) = \lambda x.\Psi_c(P)x \rightarrow_\eta \Psi_c(P) \rightarrow_{\beta\eta}^* Q$ and $Q \rightarrow_c^* N$. So $M \rightarrow_2 N$.
  · Else $\Psi_c(M) = \lambda x.c\Psi_c(P)x \rightarrow_\eta c\Psi_c(P) \rightarrow_{\beta\eta}^* cQ$ and $cQ \rightarrow_c Q \rightarrow_c^* N$. So $M \rightarrow_2 N$. □