Reducibility proofs in the λ-calculus

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Abstract

Reducibility has been used to prove a number of properties in the λ-calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we look at two related but different results in the literature. We show that one such result (which aims at giving reducibility proofs of Church-Rosser, standardization and weak normalisation) faces serious problems which break the reducibility method and then we provide a proposal to partially repair the method. Then, we consider a second result whose purpose is to use reducibility to show Church-Rosser of β-developments (without needing to use strong normalisation). We extend the second result to encompass both βI- and $\beta\eta$-reduction rather than simply β-reduction.

1 Introduction

Reducibility is a method based on realizability semantics [Kle45], developed by Tait [Tai67] in order to prove normalization of some functional theories. The idea is to interpret types by sets of λ-terms closed under some properties. Since, this method has been improved and generalized. Krivine uses it in [Kri90] to prove the strong normalization of system D [CDCV80]. Koletsos proves in [Kol85] that the set of simply typed λ-terms holds the Church-Rosser property. Some aspects of his method have been reused by Gallier in [Gal97, Gal03] to prove some results such as the strong normalization of λ-terms that are typable in systems like D or D^{Ω}. In his work, Gallier states some conditions a property needs to satisfy in order to be enjoyed by some typable terms under some restrictions. Similarly, Ghilezan and Likavec [GL02] state some conditions a property on λ-terms has to satisfy to be held by some λ-terms typable in a system close to system D^{Ω}. In addition, the authors state a condition that a property needs to satisfy in order to step from "a λ-term typable, under some restrictions on types holds the property" to "a λ-term of the untyped lambda-calculus holds the property". If it works, [GL02] would provide an attractive method to establishing properties like Church-Rosser for all the untyped λ-terms, simply by showing easier conditions on typed terms. However, we will see in this paper that both the method fails for the typed terms, and that the step of passing from typed to untyped terms fails. We will provide a solution to repair the first result, however, the second result seems unrepairable.

Step of establishing properties like Church-Rosser (or confluence) for typed λ-terms and concluding the properties for all the untyped λ-terms have been successfully exploited in the literature. Koletsos and Stravinos [KS08] use a reducibility method to state that λ-terms that are typable in system D hold the Church-Rosser

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property. Then, using this result together with a method based on β-developments [Klo80, Kri90], they show that β-developments are Church-Rosser and this in turn will imply the confluence of the untyped λ-calculus. Although Klop proves the confluence of β-developments [BBKV76], his proof is based on strong normalisation whereas [KS08] only uses an embedding of β-developments in the reduction of typable λ-terms. In this paper, we apply the method of [KS08] to βI-reduction and then generalise the method to $\beta\eta$-reduction.

In section 2 we introduce the formal machinery and establish the basic needed lemmas. In section 3 we present the reducibility method of [GL02] and show that it fails at a number of important propositions which makes it inapplicable. In particular, we give counterexamples which show that all the conditions stated in [GL02] are satisfied, yet the the claimed property does not hold. In section 4 we provide subsets of types which we use to partially salvage the reducibility method of [GL02] and we show that this can now be correctly used to establish confluence, standardization and weak head normal forms but only for restricted sets of lambda terms and types. In section 5 we adapt the Church-Rosser proof of [KS08] to βI-reduction. In section 6 we generalise the method of [KS08] to handle $\beta\eta$-reduction. We conclude in section 7.

2 The Formal Machinery

In this section we provide some known formal machinery and introduce new definitions and lemmas that are necessary for the paper. We take as convention that if a metavariable v ranges over a set S then the metavariables v_i such that $i \geq 0$ and the metavariables $v', v'', etc.$ also range over S.

2.1 Familiar background on λ-calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the λ-calculus and one lemma which deals with the shape of reductions.

Definition 2.1.

1. The set of terms of the λ-calculus is defined as follows:

   $$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$$

   We let $x, y, z, etc.$ range over $\mathcal{V}$, a denumerably infinite set of λ-term variables, and $M, N, P, Q, etc.$ range over Λ. We assume the usual definition of subterms: we write $N \subset M$ if N is a subterm of M. We also assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $M\ N_1 ... N_n$ instead of $(...(M\ N_1)\ N_2 ... N_{n-1})\ N_n$.

   We take terms modulo α-conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $FV(M)$ for the set of the free variables of term M.

2. Let $n \geq 0$. We define $M^n(N)$, by induction on n, as follows: $M^0(N) = N$ and $M^{n+1}(N) = M(M^n(N))$.

3. The set of term contexts is defined as follows:

   $$C \in \mathcal{C} ::= \Box \mid \lambda x.C \mid CM \mid MC$$

   We define $C[M]$, as the filling up of the context C with the term M, by induction on the structure of C: $\Box[M] = M$, $(\lambda x.C)[M] = \lambda x.C[M]$, $(NC)[M] = NC[M]$ and $(CN)[M] = C[M]N$.

4. The set $\Lambda I \subset \Lambda$, of terms of the λI-calculus is defined by the grammar:
   (a) If $x \in \mathcal{V}$ then $x \in \Lambda I$.
   (b) If $x \in FV(M)$ and $M \in \Lambda I$ then $\lambda x.M \in \Lambda I$.
   (c) If $M, N \in \Lambda I$ then $MN \in \Lambda I$.

5. We define as usual the substitution $M[x := N]$ of N for all free occurrences of x in M. We define the substitution $C[x := M]$ of N for all free occurrences of x in context C by: $\Box[x := N] = \Box$, $(\lambda y.C)[x := N] = \lambda y.C[x := N]$ ($x \neq y$ by (BC)), $(MC)[x := N] = M[x := N]C[x := N]$ and $(CM)[x := N] = C[x := N]M[x := N]$. We let $M[(x_i := N_i)_1^n]$ be the simultaneous substitution of N_i for all free occurrences of x_i in M for $1 \leq i \leq n$.

6. We assume the usual definition of compatibility. For $r \in \{\beta, \beta I, \beta\eta\}$, we define the reduction relation $\rightarrow_r$ on Λ as the least compatible relation closed under rule $(r) : L \rightarrow_r R$ below, and we call L an r-redex and R the contractum of L (or the L contractum). We define $\mathcal{R}^r$ to be the set of r-redexes.
   - (β): $(\lambda x.M)N \rightarrow_\beta M[x := N]$.
   - (βI): $(\lambda x.M)N \rightarrow_{\beta I} M[x := N]$ when $x \in FV(M)$.
   - (η): $\lambda x.Mx \rightarrow_\eta M$ when $x \notin FV(M)$.

   We define $\mathcal{R}^{\beta\eta} = \mathcal{R}^\beta \cup \mathcal{R}^\eta$ and $\rightarrow_{\beta\eta} = \rightarrow_\beta \cup \rightarrow_\eta$.

7. Let $r \in \{\beta, \beta I, \beta\eta\}$. We define $\mathcal{R}^r_M = \{C \mid C \in \mathcal{C} \wedge \exists R \in \mathcal{R}^r, C[R] = M\}$. If $M \rightarrow_r N$ by contracting the r-redex R in $M = C[R]$ then $C \in \mathcal{R}^r_M$ by definition, $N = C[R']$ where R' is the contractum of R and we write $M \stackrel{C}{\rightarrow}_r N$.

8. Let $M \in \Lambda$ and $\mathcal{F} \subseteq \Lambda$. $\mathcal{F} \upharpoonright M = \{N \mid N \in \mathcal{F} \wedge N \subset M\}$.

9. If $M = \lambda x_1. \ldots x_n.(\lambda x.M_0)M_1 \ldots M_m$ such that $n \geq 0$ and $m \geq 1$ then $(\lambda x.M_0)M_1$ is called the β-head redex of M.

10. If $M = (\lambda x.M_0 x)M_1 \ldots M_m$ such that $m \geq 1$ then $(\lambda x.M_0 x)$ is called the η-head redex of M.

11. Let $r \in \{\beta, \eta\}$. We write $M \rightarrow_{hr} M'$ (resp. $M \rightarrow_{ir} M'$) if M' is obtained by reducing the r-head (resp. a non r-head) redex of M.

12. We define: $\rightarrow_{\beta i\eta} = \rightarrow_\beta \cup \rightarrow_{i\eta}$

13. Let $r \in \{\rightarrow_\beta, \rightarrow_\eta, \rightarrow_{\beta\eta}, \rightarrow_{\beta I}, \rightarrow_{h\beta}, \rightarrow_{h\eta}, \rightarrow_{i\beta}, \rightarrow_{i\eta}, \rightarrow_{\beta i\eta}\}$. We use $\rightarrow^*_r$ to denote the reflexive transitive closure of $\rightarrow_r$. We let $\simeq_r$ denote the equivalence relation induced by $\rightarrow_r$.
    If the r-reduction from M to N is in k steps, we write $M \rightarrow^k_r N$.

14. Let $r \in \{\beta I, \beta\eta\}$, M not an application and $n \geq 0$. A term $M'N'_0N'_1 \ldots N'_n$ is a direct r-reduct of $MN_0N_1 \ldots N_n$ iff $\forall i \in \{0, \ldots, n\}, N_i \rightarrow^*_r N'_i$ and
    - if $r = \beta I$ then $M \rightarrow^*_{\beta I} M'$.
    - if $r = \beta\eta$ then $M \rightarrow^*_{\beta i\eta} M'$.

15. $\mathsf{NF}_\beta = \{\lambda x_1. \ldots \lambda x_n.x_0 N_1 \ldots N_m \mid n, m \geq 0, N_1, \ldots, N_m \in \mathsf{NF}_\beta\}$.

16. $\mathsf{WN}_\beta = \{M \in \Lambda \mid \exists N \in \mathsf{NF}_\beta, M \rightarrow^*_\beta N\}$.

17. Let $r \in \{\beta, \beta I, \beta\eta\}$.

- We say that M has the Church-Rosser property for r (has r-CR) if whenever $M \to_r^* M_1$ and $M \to_r^* M_2$ then there is an M_3 such that $M_1 \to_r^* M_3$ and $M_2 \to_r^* M_3$.
- $\mathsf{CR}^r = \{M \mid M \text{ has } r\text{-}\mathsf{CR}\}$.
- $\mathsf{CR}_0^r = \{xM_1 \ldots M_n \mid n \geq 0 \wedge x \in \mathcal{V} \wedge (\forall i \in \{1, \ldots, n\}, M_i \in \mathsf{CR}^r)\}$.
- We use CR to denote CR^β and CR_0 to denote CR_0^β.
- A term is a weak head normal form if it is an abstraction or if it starts with a variable. A term is weakly head normalizing if it reduces to a weak head normal form. Let $\mathsf{W}^r = \{M \in \Lambda \mid \exists n \geq 0, \exists x \in \mathcal{V}, \exists P, P_1, \ldots, P_n \in \Lambda, M \to_r^* \lambda x.P \text{ or } M \to_r^* xP_1 \ldots P_n\}$. We use W to denote W^β.

18. We say that M has the standardization property if whenever $M \to_\beta^* N$ then there is an M' such that $M \to_h^* M'$ and $M' \to_i^* N$. Let $\mathsf{S} = \{M \in \Lambda \mid M$ has the standardization property$\}$. □

The next lemma deals with the shape of reductions.

Lemma 2.2.

1. *If $M \to_{\beta\eta}^* M'$ then $FV(M') \subseteq FV(M)$.*
2. *If $M \to_{\beta I}^* M'$ then $FV(M) = FV(M')$ and if $M \in \Lambda I$ then $M' \in \Lambda I$.*
3. *$\lambda x.M \to_{\beta\eta} P$ iff either ($P = \lambda x.M'$ and $M \to_{\beta\eta} M'$) or ($M = Px$ and $x \notin FV(P)$).*
4. *$\lambda x.M \to_{\beta i\eta} P$ iff ($P = \lambda x.M'$ and $M \to_{\beta\eta} M'$).*
5. *Let $n \geq 0$. A direct $\beta\eta$-reduct of $(\lambda x.M)N_0N_1 \ldots N_n$, is a term $(\lambda x.M')N_0'N_1' \ldots N_n'$ such that $M \to_{\beta\eta}^* M'$ and $\forall i \in \{0, \ldots, n\}$, $N_i \to_{\beta\eta}^* N_i'$.*
6. *Let $r \in \{\beta I, \beta\eta\}$, M not an application, $n \geq 0$, P is not a direct r-reduct of $MN_0 \ldots N_n$ and $MN_0 \ldots N_n \to_r^k P$. Then the following holds:*
   (a) *$M = \lambda x.M'$, $k \geq 1$, and if $k = 1$ then $P = M'[x := N_0]N_1 \ldots N_n$.*
   (b) *There exists a direct r-reduct $(\lambda x.M'')N_0'N_1' \ldots N_n'$ of $MN_0 \ldots N_n$ such that $M''[x := N_0']N_1' \ldots N_n' \to_r^* P$.*
7. *Let $r \in \{\beta I, \beta\eta\}$, $n \geq 0$ and $(\lambda x.M)N_0N_1 \ldots N_n \to_r^* P$. There exist P' such that $P \to_r^* P'$ and*
   (a) *If $r = \beta I$ and $x \in FV(M)$ then $M[x := N_0]N_1 \ldots N_n \to_r^* P'$.*
   (b) *If $r = \beta\eta$ then $M[x := N_0]N_1 \ldots N_n \to_r^* P'$.*

□

2.2 Formalising the background on developments

In this section we go through some needed background from [Kri90] on developments and we precisely formalise and establish all the necessary properties. In order not to clutter the paper, we have put all the proofs of this section in an appendix. Throughout the paper, we take c to be a metavariable ranging over $\mathcal{V}$. As far as we know, this is the first precise formalisation of developments.

The next definition adapts Λ_c of [Kri90] to deal with βI- and $\beta\eta$-reduction. Basically, ΛI_c is Λ_c where in the abstraction construction rule (R1).2, we restrict abstraction to ΛI. In $\Lambda\eta_c$ we introduce the new rule (R4) and replace the abstraction rule of Λ_c by (R1).3 and (R1).4.

Definition 2.3 ($\Lambda\eta_c$, ΛI_c)**.**

1. We let $\mathcal{M}_c$ range over $\Lambda\eta_c, \Lambda I_c$ defined as follows (note that $\Lambda I_c \subset \Lambda I$):
   - (R1) If x is a variable distinct form c then
     1. $x \in \mathcal{M}_c$.
     2. If $M \in \Lambda I_c$ and $x \in FV(M)$ then $\lambda x.M \in \Lambda I_c$.
     3. If $M \in \Lambda\eta_c$ then $\lambda x.M[x := c(cx)] \in \Lambda\eta_c$.
     4. If $Nx \in \Lambda\eta_c$ such that $x \notin FV(N)$ and $N \neq c$ then $\lambda x.Nx \in \Lambda\eta_c$.
   - (R2) If $M, N \in \mathcal{M}_c$ then $cMN \in \mathcal{M}_c$.
   - (R3) If $M, N \in \mathcal{M}_c$ and M is a λ-abstraction then $MN \in \mathcal{M}_c$.
   - (R4) If $M \in \Lambda\eta_c$ then $cM \in \Lambda\eta_c$.
2. Let $C \in \mathcal{C}$ and $M \in \mathcal{M}_c$. If $\exists R \in \Lambda$ such that $C[R] = M$ then we call C a $\mathcal{M}_c$-context. □

Here is a lemma related to terms of $\mathcal{M}_c$.

Lemma 2.4 (Generation)**.**

1. *$M[x := c(cx)] \neq x$ and for any N, $M[x := c(cx)] \neq Nx$.*
2. *Let $x \notin FV(M)$. Then, $M[y := c(cx)] \neq x$ and for any N, $M[y := c(cx)] \neq Nx$.*
3. *If $M \in \mathcal{M}_c$ then $M \neq c$.*
4. *If $M, N \in \mathcal{M}_c$ then $M[x := N] \neq c$.*
5. *Let $MN \in \mathcal{M}_c$. Then $N \in \mathcal{M}_c$ and either*
   - *$M = cM'$ where $M' \in \mathcal{M}_c$ or*
   - *$M = c$ and $\mathcal{M}_c = \Lambda\eta_c$ or*
   - *$M = \lambda x.P$ is in $\mathcal{M}_c$*
6. *If $\lambda x.P \in \Lambda\eta_c$ then either*
   - *$P = Nx$ where $N, Nx \in \Lambda\eta_c$ where $x \notin FV(N)$ and $N \neq c$ or*
   - *$P = N[x := c(cx))]$ where $N \in \Lambda\eta_c$*
7. *If $\lambda x.P \in \Lambda I_c$ then $x \in FV(P)$ and $P \in \Lambda I_c$.*
8. *If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $M[x := N] \in \mathcal{M}_c$.*
9. *Let $M \in \Lambda\eta_c$.*
   - *(a) If $M = \lambda x.P$ then $P \in \Lambda\eta_c$.*
   - *(b) If $M = \lambda x.Px$ then $Px, P \in \Lambda\eta_c$, $x \notin FV(P)$ and $P \neq c$.*
   - *(c) Let $x \neq c$. If $M[x := c(cx)] \to_{\beta\eta} M'$ then $M' = N[x := c(cx)]$ and $M \to_{\beta\eta} N$.*
   - *(d) Let $n \geq 0$. If $c^n(M) \to_{\beta\eta} M'$ then $\exists N \in \Lambda\eta_c, M' = c^n(N)$ and $M \to_{\beta\eta} N$.*

□

Here is a lemma about the contexts surrounding the set of redexes in a term:

Lemma 2.5. *Let $r \in \{\beta I, \beta\eta\}$.*

- *If $M \in \mathcal{V}$ then $\mathcal{R}^r_M = \varnothing$.*
- *If $M = \lambda x.N$ then:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$.*
  - *else, $\mathcal{R}^r_M = \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$.*
- *If $M = PQ$ then:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{\Box\} \cup \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$.*
  - *else, $\mathcal{R}^r_M = \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$.*

$\Box$

Here is a lemma about the set of redexes in a term:

Lemma 2.6. *Let $r \in \{\beta I, \beta\eta\}$ and $\mathcal{F} \subseteq \mathcal{R}^r_M$.*

- *If $M \in \mathcal{V}$ then $\mathcal{F} = \varnothing$.*
- *If $M = \lambda x.N$ then $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^r_N$ and:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{F} \setminus \{\Box\} = \{\lambda x.C \mid C \in \mathcal{F}'\}$.*
  - *else, $\mathcal{F} = \{\lambda x.C \mid C \in \mathcal{F}'\}$.*
- *If $M = PQ$ then $\mathcal{F}_1 = \{C \mid CQ \in \mathcal{F}\} \subseteq \mathcal{R}^r_P$, $\mathcal{F}_2 = \{C \mid PC \in \mathcal{F}\} \subseteq \mathcal{R}^r_Q$ and:*
  - *if $M \in \mathcal{R}^r$ then $\mathcal{F} \setminus \{\Box\} = \{CQ \mid C \in \mathcal{F}_1\} \cup \{PC \mid C \in \mathcal{F}_2\}$.*
  - *else, $\mathcal{F} = \{CQ \mid C \in \mathcal{F}_1\} \cup \{PC \mid C \in \mathcal{F}_2\}$.*

$\Box$

Now we show that substitutions propagate inside contexts and redexes.

Lemma 2.7. *Let $r \in \{\beta I, \beta\eta\}$ and $C \in \mathcal{R}^r_M$. We have:*
$M[x := N] = C[x := N][R]$ iff $R = R'[x := N]$ and $M = C[R']$. $\Box$

Obviously, substitution dismisses non free variables:

Lemma 2.8. *If $x \notin FV(R)$ then $C[x := N][R] = C[R][x := N]$.* $\Box$

The next lemma shows the role on redexes of substitutions involving c.

Lemma 2.9. *Let $r \in \{\beta\eta, \beta I\}$. and $x \neq c$.*

1. *Let $x \neq y$. Then:*
   - *if $M[x := c(cx)] = y$ then $M = y$,*
   - *if $M[x := c(cx)] = Py$ then $M = Ny$ and $P = N[x := c(cx)]$ and*
   - *if $M[x := c(cx)] = \lambda y.P$ then $M = \lambda y.N$ and $P = N[x := c(cx)]$.*
2. *$M \in \mathcal{R}^{\beta\eta}$ iff $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.*
3. *$C \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$ iff $C = \lambda x.C'$ and $C' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.*
4. *$C \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$ iff $C = C'[x := c(cx)]$ and $C' \in \mathcal{R}^{\beta\eta}_M$.*
5. *Let $n \geq 0$ then $\mathcal{R}^{\beta\eta}_{c^n(M)} = \{c^n(C) \mid C \in \mathcal{R}^{\beta\eta}_M\}$.* $\Box$

The next lemma shows that any element $(\lambda x.P)Q$ of $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) is a βI- (resp. $\beta\eta$-) redex.

Lemma 2.10. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$. If $M = (\lambda x.P)Q$ then $M \in \mathcal{R}^r$.* □

The next lemma shows that $\Lambda\mathrm{I}_c$ (resp. $\Lambda\eta_c$) contains all the βI-redexes (resp. $\beta\eta$-redexes) of all its terms.

Lemma 2.11. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$. If $C \in \mathcal{R}^r_M$ and $M = C[R]$ then $R \in \mathcal{M}_c$.* □

In order to deal with βI- and $\beta\eta$-reduction, the next lemma generalises a lemma given in [Kri90] (and used in [KS08]). It states that $\Lambda\eta_c$ and $\Lambda\mathrm{I}_c$ are closed under $\to_{\beta\eta}$- resp. $\to_{\beta I}$-reduction.

Lemma 2.12.

1. *If $M \in \Lambda\eta_c$ and $M \to_{\beta\eta} M'$ then $M' \in \Lambda\eta_c$.*
2. *If $M \in \Lambda I_c$ and $M \to_{\beta I} M'$ then $M' \in \Lambda I_c$.* □

The next definition again taken from [Kri90], erases all the c's from a $\mathcal{M}_c$-term.

Definition 2.13 ($|-|^c$)**.** Let $M \in \Lambda$. We define $|M|^c$ inductively as follows:
- $|x|^c = x$
- $|\lambda x.N|^c = \lambda x.|N|^c$
- $|cP|^c = |P|^c$
- $|NP|^c = |N|^c|P|^c$ if $N \neq c$. □

The next definition erases all the c's from a $\mathcal{M}_c$-context.

Definition 2.14 ($|-|^c_{\mathcal{C}}$)**.** Let $C \in \mathcal{C}$. We define $|C|^c_{\mathcal{C}}$ inductively as follows:
- $|\Box|^c_{\mathcal{C}} = \Box$
- $|\lambda x.N|^c_{\mathcal{C}} = \lambda x.|C|^c_{\mathcal{C}}$
- $|C'N|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}|N|^c$
- $|cC'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$
- $|NC'|^c_{\mathcal{C}} = |N|^c|C'|^c_{\mathcal{C}}$ if $N \neq c$

Let $\mathcal{F} \subseteq \mathcal{C}$ then we define $|\mathcal{F}|^c_{\mathcal{C}} = \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{F}\}$. □

Now, c^n is indeed erased from $|c^n(M)|^c$.

Lemma 2.15. *Let $n \geq 0$ then $|c^n(M)|^c = |M|^c$.* □

Also, c^n is erased from $|c^n(N)|^c$ for any $c^n(N)$ subterm of M.

Lemma 2.16. *Let $|M|^c = P$.*

- *If $P \in \mathcal{V}$ then $\exists n \geq 0$ such that $M = c^n(P)$.*
- *If $P = \lambda x.Q$ then $\exists n \geq 0$ such that $M = c^n(\lambda x.N)$ and $|N|^c = Q$.*
- *If $P = P_1P_2$ then $\exists n \geq 0$ such that $M = c^n(M_1M_2)$, $|M_1|^c = P_1$ and $|M_2|^c = P_2$.* □

If the c-ersure of two reduction contexts of M are equal, then these contexts are also equal:

Lemma 2.17. *Let $r \in \{\beta I, \beta\eta\}$. If $C, C' \in \mathcal{R}^r_M$ and $|C|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$ then $C = C'$.* □

Inside a term, substituting x by $c(cx)$ is undone by c-erasure.

Lemma 2.18. *Let $x \neq c$. $|M[x := c(cx)]|^c = |M|^c$.* □

Inside a context, substituting x by $c(cx)$ is undone by c-erasure.

Lemma 2.19. *Let $x \neq c$. $|C[x := c(cx)]|^c_{\mathcal{C}} = |C|^c_{\mathcal{C}}$.* □

Erasure propagates through substitutions.

Lemma 2.20. *If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $|M[x := N]|^c = |M|^c[x := |N|^c]$.* □

The next lemma shows that c is definitely erased from the free variables of $|M|^c$.

Lemma 2.21. *If $M \in \mathcal{M}_c$ then $FV(M) \setminus \{c\} = FV(|M|^c)$.* □

Now, c-erasing an ΛI_c-term returns an ΛI-term.

Lemma 2.22. *If $M \in \Lambda I_c$ then $|M|^c \in \Lambda I$.* □

The next six lemmas show that c-erasure preserves redexes, their contractum and their contexts.

Lemma 2.23. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $R \in \mathcal{R}^r$. If $R \in \mathcal{M}_c$ then $|R|^c \in \mathcal{R}^r$ and if R' is the contractum of $|R|^c$ then $R' = |R''|^c$ and R'' is the contractum of R.* □

Lemma 2.24. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$. If $C \in \mathcal{R}^r_M$ and $M = C[R]$ then $|M|^c = |C|^c_{\mathcal{C}}[|R|^c]$.* □

Lemma 2.25. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$, $M \in \mathcal{M}_c$ and $C \in \mathcal{R}^r_M$. Then, $M = C[R]$ and $|C[R]|^c \xrightarrow{|C|^c_{\mathcal{C}}}_r |C[R']|^c$ such that R' is the contractum of R.* □

Lemma 2.26. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$ and $M \in \mathcal{M}_c$. If $C \in \mathcal{R}^r_M$ and $M \xrightarrow{C}_r M'$ then $|M|^c \xrightarrow{|C|^c_{\mathcal{C}}}_r |M'|^c$.* □

Lemma 2.27. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$, $(\lambda x.M_1)N_1, (\lambda x.M_2)N_2 \in \mathcal{M}_c$ such that $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, $|\mathcal{R}^r_{N_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{N_2}|^c_{\mathcal{C}}$, $|M_1|^c = |M_2|^c$ and $|N_1|^c = |N_2|^c$. We have $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.* □

Lemma 2.28. *Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda\eta_c, \beta\eta)\}$, $M_1, M_2 \in \mathcal{M}_c$ such that $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}}$
$\subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$ and $|M_1|^c = |M_2|^c$. If $M_1 \xrightarrow{C_1}_r M'_1$, $M_2 \xrightarrow{C_2}_r M'_2$ such that $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$ then $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.* □

2.3 Background on Types and Type Systems

In this section we give the background necessary for the type systems used in this paper.

Definition 2.29. Let $i \in \{1, 2\}$.

1. Let $\mathcal{A}$ be a denumerably infinite set of type variables and let $\Omega \notin \mathcal{A}$ be a constant type. The sets of types $\mathsf{Type}^1 \subset \mathsf{Type}^2$ are defined as follows:

$$\sigma^1 \in \mathsf{Type}^1 ::= \alpha \mid \sigma^1_1 \to \sigma^1_2 \mid \sigma^1_1 \cap \sigma^1_2$$

$$\sigma^2 \in \mathsf{Type}^2 ::= \alpha \mid \sigma^2_1 \to \sigma^2_2 \mid \sigma^2_1 \cap \sigma^2_2 \mid \Omega$$

   We let α range over $\mathcal{A}$; $\sigma^1, \tau^1, \rho^1, etc.$ range over Type^1; $\sigma^2, \tau^2, \rho^2, etc.$ range over Type^2 and $\sigma, \tau, \rho, etc.$ range over Type^i.

2. We let $\mathcal{B}^i = \{\Gamma = \{x : \sigma \mid x \in \mathcal{V}, \sigma \in \mathsf{Type}^i\} \mid \forall x : \sigma, y : \tau \in \Gamma$, if $\sigma \neq \tau$ then $x \neq y\}$. We let Γ, Δ range over $\mathcal{B}^i$. We define $\mathrm{dom}(\Gamma) = \{x \mid x : \sigma \in \Gamma\}$. When $x \notin \mathrm{dom}(\Gamma)$, we write $\Gamma, x : \sigma$ for $\Gamma \cup \{x : \sigma\}$. We denote $\Gamma = x_m : \sigma_m, \ldots, x_n : \sigma_n$ where $n \geq m \geq 0$, by $(x_i : \sigma_i)^m_n$. If $m = 1$, we simply denote Γ by $(x_i : \sigma_i)_n$.

(ref)	$\sigma \leq \sigma$	(Ω)	$\sigma \leq \Omega$
(tr)	$\sigma \leq \tau \wedge \tau \leq \rho \Rightarrow \sigma \leq \rho$	$(\Omega'\text{-}lazy)$	$\sigma \to \Omega \leq \Omega \to \Omega$
(in_L)	$\sigma \cap \tau \leq \sigma$	$(idem)$	$\sigma \leq \sigma \cap \sigma$
(in_R)	$\sigma \cap \tau \leq \tau$	$(\Omega\text{-}\eta)$	$\Omega \leq \Omega \to \Omega$
$(\to\text{-}\cap)$	$(\sigma \to \tau) \cap (\sigma \to \rho) \leq \sigma \to (\tau \cap \rho)$	$(\Omega\text{-}lazy)$	$\sigma \to \tau \leq \Omega \to \Omega$
(mon')	$\sigma \leq \tau \wedge \sigma \leq \rho \Rightarrow \sigma \leq \tau \cap \rho$		
(mon)	$\sigma \leq \sigma' \wedge \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$		
$(\to\text{-}\eta)$	$\sigma \leq \sigma' \wedge \tau' \leq \tau \Rightarrow \sigma' \to \tau' \leq \sigma \to \tau$		

Figure 1: Ordering axioms on types

If $\Gamma_1 = (x_i : \sigma_i)_n, (y_i : \tau_i)_p$ and $\Gamma_2 = (x_i : \sigma'_i)_n, (z_i : \rho_i)_q$ where $x_1, \dots, x_n$ are the only shared variables, then $\Gamma_1 \sqcap \Gamma_2 = (x_i : \sigma_i \cap \sigma'_i)_n, (y_i : \tau_i)_p, (z_i : \rho_i)_q$.
Let $X \subseteq \mathcal{V}$. We define $\Gamma \upharpoonright X = \Gamma' \subseteq \Gamma$ where $\mathrm{dom}(\Gamma') = \mathrm{dom}(\Gamma) \cap X$.
Let $\sqsubseteq$ be the reflexive transitive closure of the axioms $\sigma \cap \tau \sqsubseteq \sigma$ and $\sigma \cap \tau \sqsubseteq \tau$.
If $\Gamma = (x_i : \sigma_i)_n$ and $\Gamma' = (x_i : \sigma'_i)_n$ then $\Gamma \sqsubseteq \Gamma'$ iff $\forall i, \sigma_i \sqsubseteq \sigma'_i$.

3.
- – Let $\nabla_1 = \{(ref), (tr), (in_L), (in_R), (\to\text{-}\cap), (mon'), (mon), (\to\text{-}\eta)\}$.
  – Let $\nabla_2 = \nabla_1 \cup \{(\Omega), (\Omega' - lazy)\}$.
  – Let $\nabla_D = \{(in_L), (in_R)\}$.
  – Let $\nabla_{D_I} = \nabla_D \cup \{(idem)\}$
- – $\mathsf{Type}^{\nabla_1} = \mathsf{Type}^{\nabla_D} = \mathsf{Type}^{\nabla_{D_I}} = \mathsf{Type}^1$.
  – $\mathsf{Type}^{\nabla_2} = \mathsf{Type}^2$.
- – Let ∇ be a set of axioms from Figure 1. The relation $\leq^\nabla$ is defined on types Type^∇ and axioms ∇. We use $\leq^1$ instead of $\leq^{\nabla_1}$ and $\leq^2$ instead of $\leq^{\nabla_2}$.
  – The equivalence relation is defined by: $\sigma \sim^\nabla \tau \iff \sigma \leq^\nabla \tau \wedge \tau \leq^\nabla \sigma$. We use $\sim^1$ instead of $\sim^{\nabla_1}$ and $\sim^2$ instead of $\sim^{\nabla_2}$.
- – We define $\lambda\cap^1$ to be the type system $\langle \Lambda, \mathsf{Type}^1, \vdash^1 \rangle$ such that $\vdash^1$ is the type derivability relation on $\mathcal{B}^1$, Λ and Type^1 generated using the following typing rules of Figure 2: (ax), $(\to_E)$, $(\to_I)$, $(\cap_I)$ and $(\leq^1)$).
  – We define $\lambda\cap^2$ to be the type system $\langle \Lambda, \mathsf{Type}^2, \vdash^2 \rangle$ such that $\vdash^2$ is type derivability relation on $\mathcal{B}^2$, Λ and Type^2 generated using the following typing rules of Figure 2: (ax), $(\to_E)$, $(\to_I)$, $(\cap_I)$, $(\leq^2)$ and (Ω).
  – We define D to be the type system $\langle \Lambda, \mathsf{Type}^1, \vdash^{\beta\eta} \rangle$ where $\vdash^{\beta\eta}$ is the type derivability relation on $\mathcal{B}^1$, Λ and Type^1 generated using the following typing rules of Figure 2: (ax), $(\to_E)$, $(\to_I)$, $(\cap_I)$, $(\cap_{E1})$ and $(\cap_{E2})$.
  – We define D_I to be the type system $\langle \Lambda, \mathsf{Type}^1, \vdash^{\beta I} \rangle$ where $\vdash^{\beta I}$ is the type derivability relation on $\mathcal{B}^1$, Λ and Type^1 generated using the following typing rule of Figure 2: (ax^I), $(\to_{E^I})$, $(\to_I)$, $(\cap_I)$, $(\cap_{E1})$ and $(\cap_{E2})$. Moreover, in this type system, we assume that $\sigma \cap \sigma = \sigma$.

□

3 Problems of the reducibility method of [GL02]

In this section we introduce the reducibility method of [GL02] and show where exactly it fails.

$$\frac{}{\Gamma, x:\sigma \vdash x:\sigma}\ (ax) \qquad \frac{}{x:\sigma \vdash x:\sigma}\ (ax^I)$$

$$\frac{\Gamma \vdash M:\sigma \to \tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}\ (\to_E) \qquad \frac{\Gamma_1 \vdash M:\sigma \to \tau \quad \Gamma_2 \vdash N:\sigma}{\Gamma_1 \sqcap \Gamma_2 \vdash MN:\tau}\ (\to_{E^I})$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma \to \tau}\ (\to_I) \qquad \frac{\Gamma \vdash M:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M:\sigma \cap \tau}\ (\cap_I)$$

$$\frac{\Gamma \vdash M:\sigma \cap \tau}{\Gamma \vdash M:\sigma}\ (\cap_{E1}) \qquad \frac{\Gamma \vdash M:\sigma \cap \tau}{\Gamma \vdash M:\tau}\ (\cap_{E2})$$

$$\frac{\Gamma \vdash M:\sigma \quad \sigma \leq^{\nabla} \tau}{\Gamma \vdash M:\tau}\ (\leq^{\nabla}) \qquad \frac{}{\Gamma \vdash M:\Omega}\ (\Omega)$$

Figure 2: Typing rules

Definition 3.1 (Type systems and reducibility of [GL02])**.** Let $i \in \{1, 2\}$.

1. Let $\mathcal{P} \subseteq \Lambda$. The type interpretation $[\![-]\!]^i : \mathsf{Type}^i \to 2^{\Lambda}$ is defined by:
   - $[\![\alpha]\!]^i = \mathcal{P}$, where $\alpha \in \mathcal{A}$.
   - $[\![\sigma \cap \tau]\!]^i = [\![\sigma]\!]^i \cap [\![\tau]\!]^i$.
   - $[\![\Omega]\!]^2 = \Lambda$.
   - $[\![\sigma \to \tau]\!]^1 = [\![\sigma]\!]^1 \Rightarrow [\![\tau]\!]^1 = \{M \in \Lambda | \forall N \in [\![\sigma]\!]^1, MN \in [\![\tau]\!]^1\}$.
   - $[\![\sigma \to \tau]\!]^2 = ([\![\sigma]\!]^2 \Rightarrow [\![\tau]\!]^2) \cap \mathcal{P} = \{M \in \mathcal{P} | \forall N \in [\![\sigma]\!]^2, MN \in [\![\tau]\!]^2\}$.
2. A valuation of term variables in Λ is a function $\nu : \mathcal{V} \to \Lambda$. We write $v(x := M)$ for the function v' where $v'(x) = M$ and $v'(y) = v(y)$ if $y \neq x$.
3. let ν be a valuation of term variables in Λ. Then $[\![-]\!]_\nu : \Lambda \to \Lambda$ is defined by: $[\![M]\!]_\nu = M[x_1 := \nu(x_1), \ldots, x_n := \nu(x_n)]$, where $FV(M) = \{x_1, \ldots, x_n\}$.
4. - $\nu \models^i M : \sigma$ iff $[\![M]\!]_\nu \in [\![\sigma]\!]^i$
   - $\nu \models^i \Gamma$ iff $\forall (x : \sigma) \in \Gamma, \nu(x) \in [\![\sigma]\!]^i$
   - $\Gamma \models^i M : \sigma$ iff $\forall \nu \models^i \Gamma, \nu \models^i M : \sigma$
5. Let $\mathcal{X} \subseteq \Lambda$. We say that:
   - (VAR^i) $\mathcal{P}$ satisfies the variable property, denoted $VAR^i(\mathcal{P}, \mathcal{X})$, if

     $$\forall x, x \in \mathcal{X}$$

   - (SAT^1) $\mathcal{P}$ is 1-saturated, denoted $SAT^1(\mathcal{P}, \mathcal{X})$, if

     $$\forall M, \forall x, \forall N \in \mathcal{P}, M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X}$$

   - (SAT^2) $\mathcal{P}$ is 2-saturated, denoted $SAT^2(\mathcal{P}, \mathcal{X})$, if

     $$\forall M, \forall N, \forall x, M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X}$$

   - (CLO^1) $\mathcal{P}$ is closed by variable application, denoted $CLO^1(\mathcal{P}, \mathcal{X})$, if

     $$\forall M, \forall x, Mx \in \mathcal{X} \Rightarrow M \in \mathcal{P}$$

- (CLO^2) $\mathcal{P}$ is closed by abstraction, denoted $CLO^2(\mathcal{P}, \mathcal{X})$, if

$$\forall M, \forall x, M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P}$$

For $\mathcal{R} \in \{VAR^i, SAT^i, CLO^i\}$, let $\mathcal{R}(\mathcal{P}) \iff \forall \sigma \in \mathsf{Type}^i, \mathcal{R}(\mathcal{P}, [\![\sigma]\!]^i)$

6. Let $\mathcal{X} \subseteq \Lambda$. We say that:
   - ($\mathcal{P} - VAR$) $\mathcal{X}$ satisfies the $\mathcal{P}$-variable property, denoted $VAR(\mathcal{P}, \mathcal{X})$, if

   $$\forall x, \forall n \geq 0, \forall N_1, \ldots, N_n \in \mathcal{P}, x N_1 \ldots N_n \in \mathcal{X}$$

   - ($\mathcal{P} - SAT$) $\mathcal{X}$ is $\mathcal{P}$-saturated, denoted $SAT(\mathcal{P}, \mathcal{X})$, if

   $$\forall M, \forall N, \forall x, \forall n \geq 0, \forall N_1, \ldots, N_n \in \mathcal{P},$$
   $$M[x := N] N_1 \ldots N_n \in \mathcal{X} \Rightarrow (\lambda x.M) N N_1 \ldots N_n \in \mathcal{X}$$

   - ($\mathcal{P} - CLO$) $\mathcal{X}$ is $\mathcal{P}$-closed, denoted $CLO(\mathcal{P}, \mathcal{X})$, if

   $$\forall M, \forall x, M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P}$$

7. A set $\mathcal{P} \subseteq \Lambda$ is said to be invariant under abstraction if

$$\forall M, \forall x, M \in \mathcal{P} \iff \lambda x.M \in \mathcal{P}.$$

□

Lemma 3.2 (Basic lemmas proved in [GL02])**.**

1. *(a)* $[\![M]\!]_{\nu(x:=N)} \equiv [\![M]\!]_{\nu(x:=x)}[x := N]$
   (b) $[\![MN]\!]_\nu \equiv [\![M]\!]_\nu [\![N]\!]_\nu$
   (c) $[\![\lambda x.M]\!]_\nu \equiv \lambda x.[\![M]\!]_{\nu(x:=x)}$
2. *If $VAR^1(\mathcal{P})$ and $CLO^1(\mathcal{P})$ are satisfied then*
   (a) $\forall \sigma \in \mathsf{Type}^1, [\![\sigma]\!]^1 \subseteq \mathcal{P}$.
   (b) If $SAT^1(\mathcal{P})$ and $\Gamma \vdash^1 M : \sigma$ then we have $\Gamma \models^1 M : \sigma$ and $M \in \mathcal{P}$
3. *$\forall \sigma \in \mathsf{Type}^2$, if $\sigma \not\sim^2 \Omega$ then $[\![\sigma]\!]^2 \subseteq \mathcal{P}$*
4. *If $\sigma \leq^2 \tau$ then $[\![\sigma]\!]^2 \subseteq [\![\tau]\!]^2$.*
5. *If $VAR^2(\mathcal{P})$, $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ hold then $\Gamma \vdash^2 M : \sigma \Rightarrow \Gamma \models^2 M : \sigma$*
6. *If $VAR^2(\mathcal{P})$, $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ hold then $\forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}$*
7. *$CLO(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \Rightarrow CLO^2(\mathcal{P}, [\![\sigma]\!]^2)$.* □

Proof. We only prove 5. By induction on $\Gamma \vdash^2 M : \sigma$. (ax) and (Ω) are easy. $(\cap_I)$ (resp. $(\rightarrow_E)$ resp. $(\leq^2)$) is by IH (resp. IH and 1, resp. IH and 4).

$(\rightarrow_I)$ By IH, $\Gamma, x : \sigma \models^2 M : \tau$. Let $\nu \models^2 \Gamma$ and $N \in [\![\sigma]\!]^2$. Then $\nu(x := N) \models^2 \Gamma$ since $x \notin \mathrm{dom}(\Gamma)$ and $\nu(x := N) \models^2 x : \sigma$ since $N \in [\![\sigma]\!]^2$. Therefore $\nu(x := N) \models^2 M : \tau$, i.e. $[\![M]\!]_{\nu(x:=N)} \in [\![\tau]\!]^2$. Hence, by lemma 3.2.1, $[\![M]\!]_{\nu(x:=x)}[x := N] \in [\![\tau]\!]^2$. Hence by applying $SAT^2(\mathcal{P})$, we get $(\lambda x.[\![N]\!]_{\nu(x:=x)})N \in [\![\tau]\!]^2$. Again by lemma 3.2.1, $([\![\lambda x.M]\!]_\nu)N \in [\![\tau]\!]^2$. Hence $[\![\lambda x.M]\!]_\nu \in [\![\sigma]\!]^2 \Rightarrow [\![\tau]\!]^2$.
By $VAR^2(\mathcal{P})$, $x \in [\![\sigma]\!]^2$, hence by the same argument as above we obtain $[\![M]\!]_{\nu(x:=x)} \in [\![\tau]\!]^2$. So by $CLO^2(\mathcal{P})$, $\lambda x.[\![M]\!]_{\nu(x:=x)} \in \mathcal{P}$ and by lemma 3.2.1, $[\![\lambda x.M]\!]_\nu \in \mathcal{P}$. Hence, we conclude that $[\![\lambda x.M]\!]_\nu \in [\![\sigma \rightarrow \tau]\!]^2$. □

After giving the above definitions and lemmas, [GL02] states that since the properties (VAR^i), (SAT^i) and (CLO^i) for $1 \leq i \leq 2$ have been shown to be sufficient to develop the reducibility method, and since in order to prove these properties one needs stronger induction hypotheses which are easier to prove, the paper sets out to show that these stronger conditions when $i = 2$ are $(\mathcal{P} - VAR)$, $(\mathcal{P} - SAT)$ and $(\mathcal{P} - CLO)$. However, as we show below, this attempt fail.

Lemma 3.3 (Lemma 3.16 of [GL02] is false)**.** *The lemma of [GL02] stated below is false.*

$$VAR(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \to \tau \Rightarrow VAR(\mathcal{P}, [\![\sigma]\!]^2).$$

□

Proof. To show that the above statement is false, we give the following counterexample. Let σ be $\alpha \to \Omega \to \alpha \not\sim^2 \Omega \to \tau$, where $\alpha \in \mathcal{A}$. $VAR(\mathcal{P}, [\![\sigma]\!]^2)$ is true if $\forall x \in \mathcal{V}, \forall n \geq 0, \forall N_1, \ldots, N_n \in \mathcal{P}, xN_1 \ldots N_n \in [\![\sigma]\!]^2$, in particular if $x \in [\![\sigma]\!]^2$, where $x \in \mathcal{V}$. Let $\mathcal{P}$ be the set of strong normalizing terms. We have to notice that $VAR(\mathcal{P}, \mathcal{P})$ is true. Since $x \in \mathcal{P}$, $xx \in [\![\Omega \to \alpha]\!]^2$. Since $\circledast\circledast \in \Lambda = [\![\Omega]\!]^2$, where $\circledast = \lambda x.xx$, $xx(\circledast\circledast) \in [\![\alpha]\!]^2 = \mathcal{P}$. But $\circledast\circledast \notin \mathcal{P}$, hence $xx(\circledast\circledast) \notin \mathcal{P}$, so $VAR(\mathcal{P}, [\![\sigma]\!]^2)$ is false. □

Remark 3.4 (It is not clear that Lemma 3.18 of [GL02] holds).
It is not clear that the lemma of [GL02] stated below holds.

$$SAT(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \to \tau \Rightarrow SAT(\mathcal{P}, [\![\sigma]\!]^2).$$

□

Of remark 3.4. The proof given in [GL02] does not go through and we have neither been able to prove nor disprove this lemma. It remains that this lemma is not yet proved and hence cannot be used in further proofs. □

Then, [GL02] gives the following proposition which is the reducibility method for typable terms:
Proposition 3.21 of [GL02] Let $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ and $CLO(\mathcal{P}, \mathcal{P})$, then

$$\forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \sigma \not\sim^2 \Omega \to \tau \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}.$$

However, the proof of that proposition depends on two problematic lemmas (lemma 3.16 which we showed to fail in our lemma 3.3, and lemma 3.18 which we explained in remark 3.4 that it is not clear why it should hold). Below, we show that proposition 3.21 of [GL02] fails by giving a counterexample. First, here is a lemma:

Lemma 3.5. *$VAR(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$, $CLO(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $SAT(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ hold.* □

Proof.

- $VAR(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ is satisfied, since $\forall x \in \mathcal{V}$, $\forall n \geq 0$, $\forall N_1, \ldots, N_n \in \mathsf{WN}_\beta$, $xN_1 \ldots N_n \in \mathsf{WN}_\beta$.
- $CLO(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ is satisfied, since if $\exists n, m \geq 0$, $\exists x_0 \in \mathcal{V}$, $\exists N_1, \ldots, N_m \in \mathsf{NF}_\beta$ such that $M \to^*_\beta \lambda x_1. \ldots \lambda x_n.x_0 N_1 \ldots N_m$ then
$\forall y \in \mathcal{V}$, $\lambda y.M \to^*_\beta \lambda y.\lambda x_1. \ldots \lambda x_n.x_0 N_1 \ldots N_m$ and $\lambda y.M \in \mathsf{WN}_\beta$.
- $SAT(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ is satisfied, since if $M[x := N]N_1 \ldots N_n \in \mathsf{WN}_\beta$ where $n \geq 0$ and $N_1, \ldots, N_n \in \mathsf{WN}_\beta$ then $\exists P \in \mathsf{NF}_\beta$ such that
$M[x := N]N_1 \ldots N_n \to^*_\beta P$.
Hence, $(\lambda x.M)NN_1 \ldots N_n \to_\beta M[x := N]N_1 \ldots N_n \to^*_\beta P$. □

Lemma 3.6 (Proposition 3.21 of [GL02] fails)**.**
Let $VAR(\mathcal{P},\mathcal{P})$, $SAT(\mathcal{P},\mathcal{P})$ and $CLO(\mathcal{P},\mathcal{P})$, then it is **not** *the case that*
$\forall\sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \wedge \sigma \not\sim^2 \Omega \to \tau \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}$. □

Proof. Let $\mathcal{P}$ be WN_β of Definition 2 and recall that $\circledast = \lambda x.xx$. Note that $\lambda y.\circledast\circledast \notin \mathsf{WN}_\beta$. Moreover, $\forall\rho \in \mathsf{Type}^2$, we can construct the typing judgment $\vdash^2 \lambda y.\circledast\circledast : \rho \to \Omega$. Let σ be $\rho \to \Omega$. Obviously, $\sigma \not\sim^2 \Omega$. Let $\tau \in \mathsf{Type}^2$.
If $\tau \not\sim^2 \Omega$ then obviously $\sigma = \rho \to \Omega \not\sim^2 \Omega \to \tau$.
If $\tau \sim^2 \Omega$ then let $\rho \not\sim^2 \Omega$. Obviously $\sigma = \rho \to \Omega \not\sim^2 \Omega \to \tau$.
Lemma 3.5 and the above, give a counterexample for Proposition 3.21 of [GL02]. □

Finally, also the proof method for untyped terms given in [GL02] fails.

Lemma 3.7 (Proposition 3.23 of [GL02] fails)**.**
Proposition 3.23 of [GL02] which states that "If $\mathcal{P} \subseteq \Lambda$ is invariant under abstraction, $VAR(\mathcal{P},\mathcal{P})$ and $SAT(\mathcal{P},\mathcal{P})$ then $\mathcal{P} = \Lambda$" fails. □

Proof. The proof given in [GL02] depends on Proposition 3.21 which we have shown to fail. Furthermore, since WN_β is invariant under abstraction and by lemma 3.5, $VAR(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ and $SAT(\mathsf{WN}_\beta, \mathsf{WN}_\beta)$ hold, we have a counterexample for Proposition 3.23. □

4 Salvaging the reducibility method of [GL02]

In this section we provide subsets of types which we use to partially salvage the reducibility method of [GL02] and we show that this can now be correctly used to establish confluence, standardization and weak head normal forms but only for restricted sets of lambda terms and types.

REMARK 4.1. Note that in the proof of proposition 3.2.5, the properties $VAR^2(\mathcal{P})$, $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ are not needed for all types in Type^2. If $\Gamma \vdash^2 M : \sigma \to \tau$, we only need to have $VAR^2(\mathcal{P})$ for σ and $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ for τ. □

Lemma 4.2. *If $\Gamma \vdash^2 M : \rho$ and (if $\rho = \sigma \to \tau$ then $VAR^2(\mathcal{P}, [\![\sigma]\!]^2)$, $SAT^2(\mathcal{P}, [\![\tau]\!]^2)$ and $CLO^2(\mathcal{P}, [\![\tau]\!]^2)$) then $\Gamma \models^2 M : \rho$* □

Proof. By induction on $\Gamma \vdash^2 M : \rho$. The proof is exactly the same as that of the proof of proposition 3.2.5, except with the replacement of $VAR^2(\mathcal{P})$ $SAT^2(\mathcal{P})$ and $CLO^2(\mathcal{P})$ by $VAR^2(\mathcal{P}, [\![\sigma]\!]^2)$, $SAT^2(\mathcal{P}, [\![\tau]\!]^2)$, and $CLO^2(\mathcal{P}, [\![\tau]\!]^2)$ resp. □

In order to salvage the reducibility method of [GL02], we introduce the following:

Definition 4.3.

- $\sigma^{2+} \in \mathsf{Type}^{2+} = \{\sigma \in \mathsf{Type}^2 \mid \sigma \sim^2 \Omega\}$.
- $\sigma^{2-} \in \mathsf{Type}^{2-} = \{\sigma \in \mathsf{Type}^2 \mid \sigma \not\sim^2 \Omega\}$.
- $\sigma^{S_1} \in S_1 ::= \alpha \mid \sigma_1^{2+} \to \sigma_2^{2+} \mid \sigma^{2-} \to \sigma^{S_1} \mid \sigma^{S_1} \cap \sigma^{S_1}$.
- $\sigma^{S_2} \in S_2 ::= \Omega \to \Omega \mid \sigma^1$.

We let $\sigma, \tau, \rho, \sigma_1, \sigma_2, \ldots$ range over Type^1, Type^2, Type^{2+}, Type^{2-}, S_1 or S_2. □

Lemma 4.4.

1. *$S_2 \subseteq S_1$.*
2. *Let $\sigma \in S_1$. If $\sigma = \tau \to \rho \wedge \tau \not\sim^2 \Omega$ then $\rho \in S_1$. If $\sigma = \tau \cap \rho$, then $\tau, \rho \in S_1$.* □

Proof.

1. Let $\sigma \in S_2$. We prove this lemma by case on S_2. Either $\sigma = \Omega \to \Omega$ then $\sigma \in S_1$ since $\Omega \in \mathsf{Type}^{2+}$. Or $\sigma \in \mathsf{Type}^1$. Note that $\mathsf{Type}^1 \subset \mathsf{Type}^{2-}$ and $\mathcal{A} \subset S_1$. We prove the statement by induction on $\sigma \in \mathsf{Type}^1$.
   - If $\sigma = \tau \to \rho$ where $\tau, \rho \in \mathsf{Type}^1 \subset \mathsf{Type}^{2-}$ then by IH, $\rho \in S_1$. Hence $\sigma \in S_1$.
   - If $\sigma = \tau \cap \rho$ such that $\tau, \rho \in \mathsf{Type}^1$ then by IH, $\tau, \rho \in S_1$ and so, $\sigma \in S_1$.
2. Easy. □

Using S_1, we can establish a revised version of Lemmas 3.16 and 3.18 of [GL02].

Lemma 4.5.

1. $VAR(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in S_1, VAR(\mathcal{P}, [\![\sigma]\!]^2)$.
2. $SAT(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in S_1, SAT(\mathcal{P}, [\![\sigma]\!]^2)$. □

Proof. Let $\sigma \in S_1$ and $N_1, \dots, N_n \in \mathcal{P}$ such that $n \geq 0$.
1. By induction on σ. Assume $VAR(\mathcal{P}, \mathcal{P})$ and let $x \in \mathcal{V}$.

- $\sigma \in \mathcal{A}$. Then use $VAR(\mathcal{P}, \mathcal{P})$ and the definition of $[\![.]\!]^2$.
- $\sigma = \tau \to \rho$. By $VAR(\mathcal{P}, \mathcal{P})$, $xN_1 \dots N_n \in \mathcal{P}$. Let $N \in [\![\tau]\!]^2$ ($[\![\tau]\!]^2 = \varnothing$ is easy).
  - If $\tau \sim^2 \Omega$ then since $\sigma = \tau \to \rho \in S_1$, it should hold that $\rho \sim^2 \Omega$, so $xN_1 \dots N_n N \in \Lambda = [\![\rho]\!]^2$. Thus $xN_1 \dots N_n \in [\![\sigma]\!]^2$.
  - Else, $\tau \not\sim^2 \Omega$. Then by lemma 3.2.3, $N \in \mathcal{P}$. Moreover, by lemma 4.4.2, $\rho \in S_1$. Hence, by IH, $xN_1 \dots N_n N \in [\![\rho]\!]^2$. Thus $xN_1 \dots N_n \in [\![\sigma]\!]^2$.
- $\sigma = \tau \cap \rho$. By lemma 4.4.2, $\tau, \rho \in S_1$. By IH, $xN_1, \dots, N_n \in [\![\tau]\!]^2 \cap [\![\rho]\!]^2 = [\![\sigma]\!]^2$.
- $\sigma = \Omega$. Then $xN_1, \dots, N_n \in \Lambda = [\![\Omega]\!]^2$.

2. By induction on σ. Assume $SAT(\mathcal{P}, \mathcal{P})$ and $M[x := N]N_1 \dots N_n \in [\![\sigma]\!]^2$.

- $\sigma \in \mathcal{A}$. Then use $SAT(\mathcal{P}, \mathcal{P})$ and the definition of $[\![.]\!]^2$.
- $\sigma = \tau \to \rho$. By lemma 3.2.3, $M[x := N]N_1 \dots N_n \in \mathcal{P}$ and by $SAT(\mathcal{P}, \mathcal{P})$ $(\lambda x.M)NN_1 \dots N_n \in \mathcal{P}$. Let $P \in [\![\tau]\!]^2$ (case $[\![\tau]\!]^2 = \varnothing$ is immediate).
  - If $\tau \sim^2 \Omega$ then since $\sigma = \tau \to \rho \in S_1$, it should hold that $\rho \sim^2 \Omega$, so $(\lambda x.M)NN_1 \dots N_n P \in \Lambda = [\![\rho]\!]^2$. Thus $(\lambda x.M)NN_1 \dots N_n \in [\![\sigma]\!]^2$.
  - Else, $\tau \not\sim^2 \Omega$. Then by lemma 3.2.3, $P \in \mathcal{P}$. Moreover, by lemma 4.4.2, $\rho \in S_1$. Hence, since $M[x := N]N_1 \dots N_n P \in [\![\rho]\!]^2$, by IH, we get $(\lambda x.M)NN_1 \dots N_n P \in [\![\rho]\!]^2$. Thus $(\lambda x.M)NN_1 \dots N_n \in [\![\sigma]\!]^2$.
- $\sigma = \tau \cap \rho$. Then, $M[x := N]N_1 \dots N_n \in [\![\tau]\!]^2 \cap [\![\rho]\!]^2$ and by lemma 4.4.2, $\tau, \rho \in S_1$. By IH, $(\lambda x.M)NN_1, \dots, N_n \in [\![\tau]\!]^2 \cap [\![\rho]\!]^2 = [\![\sigma]\!]^2$.
- $\sigma = \Omega$. Then $(\lambda x.M)NN_1, \dots, N_n \in \Lambda = [\![\Omega]\!]^2$. □

Corollary 4.6.

1. $VAR(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in S_1, VAR^2(\mathcal{P}, [\![\sigma]\!]^2)$.
2. $SAT(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in S_1, SAT^2(\mathcal{P}, [\![\sigma]\!]^2)$. □

REMARK 4.7. $\sigma \not\sim^2 \Omega$ is not a sufficient hypothesis in Proposition 3.21. We saw in remark 4.1 that if $\sigma = \tau \to \rho$, we need to have $CLO^2(\mathcal{P})$ only for ρ (not for all types in Type^2). Hence, since $CLO(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \sigma \in \mathsf{Type}^2, \sigma \not\sim^2 \Omega \Rightarrow CLO^2(\mathcal{P}, [\![\sigma]\!]^2)$, at least, we need to have $\rho \not\sim^2 \Omega$. The same remark holds for the hypothesis $\sigma \not\sim^2 \Omega \to \tau$. Similarly, the same remark holds if we replace $\sigma \not\sim^2 \Omega \wedge \sigma \not\sim^2 \Omega \to \tau$ by $\sigma \not\sim^2 \Omega \wedge \sigma \in S_1$. □

Lemma 4.8 (Using S_1 in Proposition 3.21 of [GL02] does not help)**.**
If $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ and $CLO(\mathcal{P}, \mathcal{P})$, then it is **not** *the case that* $\forall \sigma \in \mathit{Type}^2, \sigma \not\sim^2 \Omega \wedge \sigma \in S_1 \wedge \Gamma \vdash^2 M : \sigma \Rightarrow M \in \mathcal{P}$. □

Proof. Take the same counterexample given in the proof of Lemma 3.6 and choose $\rho = \Omega$. Since σ belongs to S_2 so to S_1 by lemma 4.4.1. □

However, we can rescue the reducibility method for typable terms as follows:

Proposition 4.9. *Let $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ and $CLO(\mathcal{P}, \mathcal{P})$, then*

$$\forall \sigma \in \mathit{Type}^2, \sigma \not\sim^2 \Omega \wedge \Gamma \vdash^2 M : \sigma \wedge (\sigma = \tau \to \rho \Rightarrow \tau, \rho \in S_1 \wedge \rho \not\sim^2 \Omega) \Rightarrow M \in \mathcal{P}.$$

□

Proof. By proposition 3.2.6, corollaries 4.6.1 and 4.6.2, lemma 3.2.7 and lemma 4.2. □

[GL02] applied the method to confluence of β in Λ and standardisation in Λ by showing that the method of their Proposition 3.23 is applicable to the sets CR and S of Definition 2. It applied the method to the existence of weak head normal forms in $\lambda\cap^2$ (under some restrictions on types) by showing that the method of their Proposition 3.21 is applicable to the set W of Definition 2. However, since we showed in lemma 3.6 that proposition 3.21 fails, we need to review the applications and show where exactly they work. First, here is a lemma proven in [GL02].

Lemma 4.10. *Let $\mathcal{P} \in \{\mathit{CR}, \mathit{S}, \mathit{W}\}$. Then $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ and $CLO(\mathcal{P}, \mathcal{P})$.* □

However, we need to reformulate Propositions 4.5, 4.12 and 4.15 of [GL02], since the method of Proposition 3.21 does not work. We take into account the conditions given in proposition 4.9.

Proposition 4.11. *Let $M \in \Lambda$. If $\exists \Gamma, \sigma$ such that $\Gamma \vdash^2 M : \sigma$ and $(\sigma = \tau \to \rho \Rightarrow \tau, \rho \in S_1 \wedge \rho \not\sim^2 \Omega)$ then $M \in \mathit{CR}$, $M \in \mathit{S}$, and $M \in \mathit{W}$.* □

Proof. By lemma 4.10 and proposition 4.9. □

5 Adapting the CR proof of [KS08] to βI-reduction

[KS08] gave a proof of Church-Rosser for β-reduction for the intersection type system D of Definition 2.29 (studied in detail in [Kri90]) and showed that this can be used to establish confluence of β-developments without using strong normalisation. In this section, we adapt his proof to βI and at the same time, set the formal ground for generalising the method for $\beta\eta$ in the next section. First, we adapt and formalise a number of definitions and lemmas given in [Kri90] in order to make them applicable to βI-developments. Then, we define type interpretations for both βI and $\beta\eta$, establish the soundness and Church-Rosser of both systems D and D_I (for $\beta\eta$- resp. βI-reduction), and finally, adapt [KS08] to establish the confluence of βI-developments.

All proofs from this section are located in appendix B.

5.1 Formalising βI-developments

The next definition, taken from [Kri90] (and used in [KS08]) uses the variable c to destroy the βI-redexes of M which are not in the set $\mathcal{F}$ of βI-redex occurrences in M, and to neutralise applications so that they cannot be transformed into redexes after βI-reduction. For example, in $c(\lambda x.x)y$, c is used to destroy the βI-redex $(\lambda x.x)y$.

Definition 5.1 ($\Phi^{\beta I}(-,-)$)**.** Let $M \in \Lambda\mathrm{I}$, such that $c \notin FV(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$.

1. If $M = x$ then $\mathcal{F} = \varnothing$ and $\Phi^{\beta I}(x, \mathcal{F}) = x$
2. If $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ then $\Phi^{\beta I}(\lambda x.N, \mathcal{F}) = \lambda x.\Phi^{\beta I}(N, \mathcal{F}')$
3. If $M = NP$, $\mathcal{F}_1 = \{C \mid CP \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta I}$ then
$$\Phi^{\beta I}(NP, \mathcal{F}) = \begin{cases} c\Phi^{\beta I}(N, \mathcal{F}_1)\Phi^{\beta I}(P, \mathcal{F}_2) & \text{if } \Box \notin \mathcal{F} \\ \Phi^{\beta I}(N, \mathcal{F}_1)\Phi^{\beta I}(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

$\Box$

The next lemma is an adapted version of a lemma which appears in [KS08] and which in turns adapts a lemma from [Kri90].

Lemma 5.2.

1. *If $M \in \Lambda I$, $c \notin FV(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$ then*
   (a) $FV(M) = FV(\Phi^{\beta I}(M, \mathcal{F})) \setminus \{c\}$.
   (b) $\Phi^{\beta I}(M, \mathcal{F}) \in \Lambda I_c$.
   (c) $|\Phi^{\beta I}(M, \mathcal{F})|^c = M$.
   (d) $|\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F})}|^c_{\mathcal{C}} = \mathcal{F}$.
2. *Let $M \in \Lambda I_c$.*
   (a) $|\mathcal{R}_M^{\beta I}|^c_{\mathcal{C}} \subseteq \mathcal{R}^{\beta I}_{|M|^c}$ *and* $M = \Phi^{\beta I}(|M|^c, |\mathcal{R}_M^{\beta I}|^c_{\mathcal{C}})$.
   (b) $(|M|^{\beta I}, |\mathcal{R}_M^{\beta I}|^c_{\mathcal{C}})$ *is the one and only pair* $(N, \mathcal{F})$ *such that* $N \in \Lambda I$, $c \notin FV(N)$, $\mathcal{F} \subseteq \mathcal{R}_N^{\beta I}$ *and* $\Phi^{\beta I}(N, \mathcal{F}) = M$.

$\Box$

The next lemma is needed to define βI-developments.

Lemma 5.3. *Let $M \in \Lambda I$, such that $c \notin FV(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta I} M'$. Then, there is a unique set $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$, such that $\Phi^{\beta I}(M, \mathcal{F}) \xrightarrow{C'}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$ and $|C'|^c_{\mathcal{C}} = C$.* $\Box$

We follow [Kri90] and define the set of βI-residuals of a set of βI-redexes $\mathcal{F}$ relative to a sequence of βI-redexes. First, we give the definition relative to one redex.

Definition 5.4. Let $M \in \Lambda\mathrm{I}$, such that $c \notin FV(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta I} M'$. By lemma 5.3, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $\Phi^{\beta I}(M, \mathcal{F}) \xrightarrow{C'}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$ and $|C'|^{\beta I} = C$. We call $\mathcal{F}'$ the set of **βI-residuals of $\mathcal{F}$ in M' relative to C.** $\Box$

Definition 5.5 (βI-development)**.** Let $M \in \Lambda\mathrm{I}$, where $c \notin FV(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$. A one-step βI-development of $(M, \mathcal{F})$, denoted $(M, \mathcal{F}) \to_{\beta Id} (M', \mathcal{F}')$, is a βI-reduction $M \xrightarrow{C}_{\beta I} M'$ where $C \in \mathcal{F}$ and $\mathcal{F}'$ is the set of βI-residuals of $\mathcal{F}$ in M' relative to C. A βI**-development** is the transitive closure of a one-step βI-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta Id} M_n$ for the βI-development $(M, \mathcal{F}) \to^*_{\beta Id} (M_n, \mathcal{F}_n)$. □

The next two lemmas are informative about developments.

Lemma 5.6. *Let $M \in \Lambda I$, such that $c \notin FV(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$. $(M, \mathcal{F}) \to^*_{\beta Id} (M', \mathcal{F}') \iff \Phi^{\beta I}(M, \mathcal{F}) \to^*_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$.* □

Lemma 5.7. *Let $M \in \Lambda I$, such that $c \notin FV(M)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}_M^{\beta I}$. If $(M, \mathcal{F}_1) \to_{\beta Id} (M', \mathcal{F}_1')$ then $\exists \mathcal{F}_2' \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2'$ and $(M, \mathcal{F}_2) \to_{\beta Id} (M', \mathcal{F}_2')$.* □

5.2 Confluence of βI-developments, hence of βI-reduction

Definition 5.8.

1. Let $r \in \{\beta I, \beta\eta\}$. We define the type interpretation $[\![-]\!]^r : \mathsf{Type}^1 \to 2^\Lambda$ by:
   - $[\![\alpha]\!]^r = CR^r$, where $\alpha \in \mathcal{A}$.
   - $[\![\sigma \cap \tau]\!]^r = [\![\sigma]\!]^r \cap [\![\tau]\!]^r$.
   - $[\![\sigma \to \tau]\!]^r = ([\![\sigma]\!]^r \Rightarrow [\![\tau]\!]^r) \cap CR^r = \{t \in CR | \forall u \in [\![\sigma]\!]^r, tu \in [\![\tau]\!]^r\}$.
2. A set $\mathcal{X} \subseteq \Lambda$ is saturated if $\forall n \geq 0, \forall M, N, M_1, \ldots, M_n \in \Lambda, \forall x \in \mathcal{V}$,
$$M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$$
3. A set $\mathcal{X} \subseteq \Lambda\mathrm{I}$ is I-saturated if $\forall n \geq 0, \forall M, N, M_1, \ldots, M_n \in \Lambda, \forall x \in \mathcal{V}$,
$$x \in FV(M) \Rightarrow M[x := N]M_1 \ldots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$$

□

Here is a background lemma:

Lemma 5.9.

1. *If $\Gamma \vdash^{\beta I} M : \sigma$ then $M \in \Lambda I$ and $FV(M) = \mathrm{dom}(\Gamma)$.*
2. *Let $\Gamma \vdash^{\beta\eta} M : \sigma$. Then $FV(M) \subseteq \mathrm{dom}(\Gamma)$ and if $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash^{\beta\eta} M : \sigma$.*
3. *Let $r \in \{\beta I, \beta\eta\}$. If $\Gamma \vdash^r M : \sigma$, $\sigma \sqsubseteq \sigma'$ and $\Gamma' \sqsubseteq \Gamma$ then $\Gamma' \vdash^r M : \sigma'$.* □

The next lemma states that the interpretations of types are saturated and only contain terms that are Church-Rosser. In [Kri90] it was shown for $r = \beta$ and where CR_0^r and CR^r were replaced by the corresponding sets of strongly normalising terms. [KS08] adapted Krivine's lemma for β Church-Rosser instead of strong normalisation. Here, we prove it for βI and $\beta\eta$.

Lemma 5.10. *Let $r \in \{\beta I, \beta\eta\}$.*

1. *$\forall \sigma \in \mathsf{Type}^1$, $CR_0^r \subseteq [\![\sigma]\!]^r \subseteq CR^r$.*
2. *$CR^{\beta I}$ is I-saturated.*
3. *$CR^{\beta\eta}$ is saturated.*

4. $\forall\sigma \in \mathsf{Type}^1$, $[\![\sigma]\!]^{\beta I}$ *is I-saturated.*

5. $\forall\sigma \in \mathsf{Type}^1$, $[\![\sigma]\!]^{\beta\eta}$ *is saturated.* □

Next we adapt the soundness lemma of [Kri90] to both $\vdash^{\beta I}$ and $\vdash^{\beta\eta}$.

Lemma 5.11. *Let $r \in \{\beta I, \beta\eta\}$. If $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$ and $\forall i \in \{1, \ldots, n\}$, $N_i \in [\![\sigma_i]\!]^r$ then $M[(x_i := N_i)_1^n] \in [\![\sigma]\!]^r$.* □

Finally, we adapt a corollary from [KS08] to show that every term of Λ typable in system D has the $\beta\eta$ Church-Rosser property and every term of Λ typable in system D_I has the βI Church-Rosser property.

Corollary 5.12. *Let $r \in \{\beta I, \beta\eta\}$. If $\Gamma \vdash^r M : \sigma$ then $M \in \mathsf{CR}^r$.* □

Proof. Let $\Gamma = (x_i : \sigma_i)_n$. By lemma 5.10, $\forall i \in \{1, \ldots, n\}, x_i \in [\![\sigma_i]\!]^r$, so by lemma 5.11 and again by lemma 5.10, $M \in [\![\sigma]\!]^r \subseteq \mathsf{CR}^r$. □

In order to accommodate βI- and $\beta\eta$-reduction, the next lemma generalises a lemma given in [Kri90] (and used in [KS08]). Basically this lemma states that every term of $\Lambda\mathrm{I}_c$ is typable in system D and every term of $\Lambda\eta_c$ is typable in D_I.

Lemma 5.13. *Let $FV(M) \setminus \{c\} = \{x_1, \ldots, x_n\} \subseteq \mathrm{dom}(\Gamma)$ where $c \notin \mathrm{dom}(\Gamma)$.*

1. *If $M \in \Lambda I_c$ then for $\Gamma' = \Gamma \upharpoonright FV(M)$, $\exists\sigma, \tau \in \mathsf{Type}^1$ such that if $c \in FV(M)$ then $\Gamma', c : \sigma \vdash^{\beta I} M : \tau$, and if $c \notin FV(M)$ then $\Gamma' \vdash^{\beta I} M : \tau$.*
2. *If $M \in \Lambda\eta_c$ then $\exists\sigma, \tau \in \mathsf{Type}^1$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} M : \tau$.* □

The next lemma is an adaptation of the main theorem in [KS08] where as far as we know appears for the first time.

Lemma 5.14 (confluence of the βI-developments)**.** *Let $M \in \Lambda I$, such that $c \notin FV(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exist sets $\mathcal{F}'_1 \subseteq \mathcal{R}^{\beta I}_{M_1}$, $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M_2}$ and a term $M_3 \in \Lambda I$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta Id} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta Id} M_3$.* □

We follow [Bar84] and [KS08] and define one reduction as follows:

Notation 5.15. Let $M, M' \in \Lambda\mathrm{I}$, such that $c \notin FV(M)$. We define one reduction by: $M \to_{1I} M' \iff \exists\mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \to^*_{\beta Id} (M', \mathcal{F}')$. □

Lemma 5.16. *Let $c \notin FV(M)$. $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)} = \varnothing$.* □

Lemma 5.17. *Let $c \notin FV(MN)$. $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \varnothing$.* □

Lemma 5.18. *Let $c \notin FV(M)$. If $C \in \mathcal{R}^{\beta I}_M$ and $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} M'$ then $\mathcal{R}^{\beta I}_{M'} = \varnothing$.* □

Lemma 5.19. *Let $c \notin FV(M)$. If $C \in \mathcal{R}^{\beta I}_M$ and $M \xrightarrow{C}_{\beta I} M'$ then $(M, \{C\}) \to_{\beta Id} (M', \varnothing)$.* □

Lemma 5.20. $\to^*_{\beta I} = \to^*_{1I}$. □

Finally, we achieve what we started to do: the confluence of βI-reduction on $\Lambda\mathrm{I}$.

Lemma 5.21. *If $M \in \Lambda I$ such that $c \notin FV(M)$ then $M \in \mathsf{CR}^{\beta I}$.* □

6 Generalisation of the method to $\beta\eta$-reduction

In this section, we generalise the method of [KS08] to handle $\beta\eta$-reduction. This generalisation is not trivial since we needed to develop developments involving η-reduction and to establish the important result of the closure under η-reduction of a defined set of frozen terms. It is for reasons like this that we extended the various definitions related to developments. For example, clause (R4) of the definition of $\Lambda\eta_c$ in Definition 2.3 aims to ensure closure under η-reduction. The definition of Λ_c in [Kri90] exluded such a rule and hence we lose closure under η-reduction as can be seen in the following example: Let $M = \lambda x.cNx \in \Lambda_c$ where $x \notin FV(N)$ and $N \in \Lambda_c$, then $M \to_\eta cN \notin \Lambda_c$.

Again here, the proofs are moved to appendix C.

The next two definitions adapt definition 5.1 to deal with $\beta\eta$-reduction. The variable c enables to destroy the $\beta\eta$-redexes of M which are not in the set $\mathcal{F}$ of $\beta\eta$-redex occurrences in M; to neutralise applications so that they cannot be transformed into redexes after $\beta\eta$-reduction; and to neutralise bound variables so λ-abstraction cannot be transformed into redexes after $\beta\eta$-reduction. For example, in $\lambda x.y(c(cx))$ $(x \neq x)$, c is used to destroy the η-redex $\lambda x.yx$.

Definition 6.1 ($\Phi^{\beta\eta}(-,-), \Phi_0^{\beta\eta}(-,-)$)**.** Let $c \notin FV(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.

(P1) If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \varnothing$ and

$$\Phi^{\beta\eta}(M,\mathcal{F}) = \{c^n(M) \mid n > 0\}$$

$$\Phi_0^{\beta\eta}(M,\mathcal{F}) = \{M\}$$

(P2) If $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$:

$$\Phi^{\beta\eta}(M,\mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Phi^{\beta\eta}(N,\mathcal{F}')\} & \text{if } \Box \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(N,\mathcal{F}')\} & \text{otherwise} \end{cases}$$

$$\Phi_0^{\beta\eta}(M,\mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Phi^{\beta\eta}(N,\mathcal{F}')\} & \text{if } \Box \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Phi_0^{\beta\eta}(N,\mathcal{F}')\} & \text{otherwise} \end{cases}$$

(P3) If $M = NP$, $\mathcal{F}_1 = \{C \mid CP \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta\eta}$ then:

$$\Phi^{\beta\eta}(M,\mathcal{F}) = \begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(N,\mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(P,\mathcal{F}_2)\} & \text{if } \Box \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(N,\mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(P,\mathcal{F}_2)\} & \text{otherwise} \end{cases}$$

$$\Phi_0^{\beta\eta}(M,\mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Phi^{\beta\eta}(N,\mathcal{F}_1) \wedge P' \in \Phi_0^{\beta\eta}(P,\mathcal{F}_2)\} & \text{if } \Box \notin \mathcal{F} \\ \{N'P' \mid N' \in \Phi_0^{\beta\eta}(N,\mathcal{F}_1) \wedge P' \in \Phi_0^{\beta\eta}(P,\mathcal{F}_2) & \text{otherwise} \end{cases}$$

$\Box$

Lemma 6.2. *If $M \in \Lambda\eta_c$ and $n \geq 0$ then $c^n(M) \in \Lambda\eta_c$.* $\Box$

Proof. By induction on $n \geq 0$ using $(R4)$. $\Box$

Lemma 6.3.

1. *Let $c \notin FV(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. We have:*

   (a) $\Phi_0^{\beta\eta}(M,\mathcal{F}) \subseteq \Phi^{\beta\eta}(M,\mathcal{F})$.

   (b) $\forall N \in \Phi^{\beta\eta}(M,\mathcal{F})$, $FV(M) = FV(N) \setminus \{c\}$.

(c) $\Phi^{\beta\eta}(M,\mathcal{F}) \subseteq \Lambda\eta_c$.

(d) *Let* $M = Nx$ *such that* $x \notin FV(N)$ *and* $P \in \Phi_0^{\beta\eta}(M,\mathcal{F})$. *Then,* $\mathcal{R}_{\lambda x.P}^{\beta\eta} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}_P^{\beta\eta}\}$.

(e) *Let* $M = Nx$. *If* $Px \in \Phi^{\beta\eta}(Nx,\mathcal{F})$ *then* $Px \in \Phi_0^{\beta\eta}(Nx,\mathcal{F})$.

(f) $\forall N \in \Phi^{\beta\eta}(M,\mathcal{F}), \forall n \geq 0, c^n(N) \in \Phi^{\beta\eta}(M,\mathcal{F})$.

(g) $\forall N \in \Phi^{\beta\eta}(M,\mathcal{F}), |N|^c = M$.

(h) $\forall N \in \Phi^{\beta\eta}(M,\mathcal{F}), \mathcal{F} = |\mathcal{R}_N^{\beta\eta}|_{\mathcal{C}}^c$.

2. *Let* $M \in \Lambda\eta_c$. *We have:*

(a) $|\mathcal{R}_M^{\beta\eta}|_{\mathcal{C}}^c \subseteq \mathcal{R}_{|M|^c}^{\beta\eta}$ *and* $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}_M^{\beta\eta}|_{\mathcal{C}}^c)$.

(b) $(|M|^c, |\mathcal{R}_M^{\beta\eta}|_{\mathcal{C}}^c)$ *is the one and only pair* $(N,\mathcal{F})$ *such that* $c \notin FV(N)$, $\mathcal{F} \subseteq \mathcal{R}_N^{\beta\eta}$ *and* $M \in \Phi^{\beta\eta}(N,\mathcal{F})$.

□

Lemma 6.4. *Let* $M \in \Lambda$, *such that* $c \notin FV(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$, $C \in \mathcal{F}$ *and* $M \xrightarrow{C}_{\beta\eta} M'$. *Then, there exists a unique set* $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, *such that* $\forall N \in \Phi^{\beta\eta}(M,\mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M',\mathcal{F}'), \exists C' \in \mathcal{R}_N^{\beta\eta}, N \xrightarrow{C'}_{\beta\eta} N'$ *and* $|C'|_{\mathcal{C}}^c = C$. □

Definition 6.5. Let $M \in \Lambda$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$, $C \in \mathcal{F}$ and $M \xrightarrow{C}_{\beta\eta} M'$. By lemma 6.4, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that $\forall N \in \Phi^{\beta\eta}(M,\mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M',\mathcal{F}'), \exists C' \in \mathcal{R}_N^{\beta\eta}, N \xrightarrow{C'}_{\beta\eta} N'$ and $|C'|^c = C$. We call $\mathcal{F}'$ the set of **$\beta\eta$-residuals of $\mathcal{F}$ in M' relative to C.** □

Definition 6.6 ($\beta\eta$-development)**.** Let $M \in \Lambda$, where $c \notin FV(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. A one-step $\beta\eta$-development of $(M,\mathcal{F})$, denoted $(M,\mathcal{F}) \to_{\beta\eta d} (M',\mathcal{F}')$, is a $\beta\eta$-reduction $M \xrightarrow{C}_{\beta\eta} M'$ where $C \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals of $\mathcal{F}$ in M' relative to C. A **$\beta\eta$-development** is the transitive closure of a one-step $\beta\eta$-development. We write also $M \xrightarrow{\mathcal{F}}_{\beta\eta d} M'$ for the $\beta\eta$-development $(M,\mathcal{F}) \to_{\beta\eta d}^* (M',\mathcal{F}')$. □

Lemma 6.7. *Let* $M \in \Lambda$, *where* $c \notin FV(M)$, *and* $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. *Then:*

$$(M,\mathcal{F}) \to_{\beta\eta d}^* (M',\mathcal{F}') \iff \exists N \in \Phi^{\beta\eta}(M,\mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M',\mathcal{F}'), N \to_{\beta\eta}^* N'$$

and

$$(M,\mathcal{F}) \to_{\beta\eta d}^* (M',\mathcal{F}') \iff \forall N \in \Phi^{\beta\eta}(M,\mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M',\mathcal{F}'), N \to_{\beta\eta}^* N'.$$

□

Lemma 6.8. *Let* $M \in \Lambda$, *such that* $c \notin FV(M)$ *and* $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}_M^{\beta\eta}$. *If* $(M,\mathcal{F}_1) \to_{\beta\eta d} (M',\mathcal{F}_1')$ *then* $\exists \mathcal{F}_2' \subseteq \mathcal{R}_{M'}^{\beta\eta}$ *such that* $\mathcal{F}_1 \subseteq \mathcal{F}_2'$ *and* $(M,\mathcal{F}_2) \to_{\beta\eta d} (M',\mathcal{F}_2')$. □

Lemma 6.9 (confluence of the $\beta\eta$-developments)**.** *Let* $M, M_1, M_2 \in \Lambda$. *If* $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ *and* $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, *then there exists sets* $\mathcal{F}_1' \subseteq \mathcal{R}_{M_1}^{\beta\eta}$ *and* $\mathcal{F}_2' \subseteq \mathcal{R}_{M_2}^{\beta\eta}$ *and a term* $M_3 \in \Lambda$ *such that* $M_1 \xrightarrow{\mathcal{F}_1'}_{\beta\eta d} M_3$ *and* $M_2 \xrightarrow{\mathcal{F}_2'}_{\beta\eta d} M_3$. □

Notation 6.10. Let $M, M' \in \Lambda$. $M \to_1 M' \iff \exists \mathcal{F}, \mathcal{F}', (M,\mathcal{F}) \to_{\beta\eta d}^* (M',\mathcal{F}')$. □

Lemma 6.11. *Let $c \notin FV(M)$. $\forall P \in \Phi^{\beta\eta}(M, \varnothing), \mathcal{R}_P^{\beta\eta} = \varnothing$.* □

Lemma 6.12. *Let $c \notin FV(MN)$. $\forall P \in \Phi^{\beta\eta}(M, \varnothing), \forall Q \in \Phi^{\beta\eta}(N, \varnothing), \mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$.* □

Lemma 6.13. *Let $c \notin FV(M)$. If $C \in \mathcal{R}_M^{\beta\eta}$, $P \in \Phi^{\beta\eta}(M, \{C\})$ and $P \to_{\beta\eta} Q$ then $\mathcal{R}_Q^{\beta\eta} = \varnothing$.* □

Lemma 6.14. *Let $c \notin FV(M)$. If $C \in \mathcal{R}_M^{\beta\eta}$ and $M \xrightarrow{C}_{\beta\eta} M'$ then $(M, \{C\}) \to_{\beta\eta d} (M', \varnothing)$.* □

Lemma 6.15. $\to^*_{\beta\eta} = \to^*_1$. □

Lemma 6.16. *If $M \in \Lambda$ such that $c \notin FV(M)$ then $M \in \mathsf{CR}^{\beta\eta}$.* □

7 Conclusion

Reducibility is a powerful method and has been applied to prove using a single method, a number of properties of the λ-calculus (CR, SN, etc.). This paper studied two reducibilty methods which exploit the passage from typed to untyped terms. We showed that the first method [GL02] fails in its aim and we have only been able to provide a partial solution. We adapted the second method [KS08] from β to βI-reduction and we generalised it to $\beta\eta$-reduction. There are differences in the typed systems chosen and the methods of reducibility used in [GL02, KS08]. [KS08] uses system D [CDCV80], which has elimination rules for intersection types whereas [GL02] uses $\lambda\cap$ and $\lambda\cap^\Omega$ with subtyping. Moreover, [KS08] depends on the inclusion of typable λ-terms in the set of λ-terms possessing the CR property, whereas [GL02] proves the inclusion of typable terms in an arbitrary subset of the untyped λ-calculus closed by some properties. Moreover, [GL02] considers the $VAR(\mathcal{P})$, $SAT(\mathcal{P})$ and $CLO(\mathcal{P})$ whereas [KS08] uses standard reducibility methods through saturated sets. [KS08] proves the confluence of developments using the confluence of typable λ-terms in system D (the authors prove that even a simple type system is sufficient). The advantage of the proof of confluence of developments of [KS08] is that SN is not needed.

In [Gal03], Gallier considers systems D and D^Ω. He states some properties which a set of λ-terms has to satisfy to include the terms typable in D or D^Ω (under some restrictions). He states that the terms typable in D^Ω by a "weakly nontrivial type" ($WNT ::= \mathcal{A} \mid \mathsf{Type}^2 \to WNT \mid WNT \cap WNT$) are weakly head normalizable. The "weakly nontrivial types" include types in our set S_1 since, for example, the type $\alpha \to \Omega \to \alpha$, where $\alpha \in \mathcal{A}$, does not belong to S_1 but is a "weakly nontrivial type". However, unlike Gallier we only restrict functional types. There are common properties with [GL02]: we can observe some trivial correspondences: (P4w) implies $CLO(\mathcal{P}, \mathcal{P})$, (P1) and (P3s) imply $VAR(\mathcal{P}, \mathcal{P})$, $SAT(\mathcal{P}, \mathcal{P})$ implies (P5n), and $VAR(\mathcal{P}, \mathcal{P})$ implies (P1). Gallier states some others properties held by the terms typable in D^Ω under some restriction (always on the use of the type Ω), and for different conditions on the properties, in order to be adapted to different cases. It is an attractive feature of [Gal03] that all the conditions on properties have the same general shape. [Gal97] considers quantifiers and other type constructors instead of intersection types.

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Contents

A Proofs of section 2

Lemma 2.2 . 1. By induction on the length of the reduction $M \to^*_{\beta\eta} M'$. If the length is 1, use induction on $M \to_{\beta\eta} M'$.

2. By induction on the length of the reduction $M \to^*_{\beta I} M'$. If the length is 1, use induction on $M \to_{\beta I} M'$.

3. If) trivial, Only if) by induction on $\lambda x.M \to_{\beta\eta} P$.

4. If) If $M \to_\beta M'$, then by definition $\lambda x.M \to_\beta \lambda x.M'$ and so $\lambda x.M \to_{\beta i\eta} \lambda x.M'$. If $M \to_\eta M'$, then by definition $\lambda x.M \to_{i\eta} \lambda x.M'$ and so $\lambda x.M \to_{\beta i\eta} \lambda x.M'$. Only if) Since $\lambda x.M \to_{\beta\eta} P$, by 3, either ($P = \lambda x.M'$ and $M \to_{\beta\eta} M'$) or ($M = Px$ and $x \notin FV(P)$). But, since the η-head redex is not reduced, the second case is impossible.

5. By definition a direct $\beta\eta$-reduct of $(\lambda x.M)N_0 \dots N_n$ is a term $PN'_0 \dots N'_n$ such that $\lambda x.M \to^*_{\beta i\eta} P$ and $\forall i\{0,\dots,n\}, N_i \to^*_{\beta\eta} N'_i$. Then, we conclude by 4.

6a. If $M = x$ then $P = xN'_0N'_1 \dots N'_n$, where $\forall i \in \{0,\dots,n\}, N_i \to^*_r N'_i$ and so P is a direct r-reduct of $MN_0N_1 \dots N_n$, absurd. So $M = \lambda x.M'$. If $k = 0$ then $P = (\lambda x.M')N_1N_1 \dots N_n$ is a direct r-reduct of $(\lambda x.M')N_0N_1 \dots N_n$, absurd. Assume $k = 1$, we prove $P = M'[x := N_0]N_1 \dots N_n$ by induction on $n \geq 0$.

- Let $n = 0$ and $r = \beta I$. By case on $(\lambda x.M')N_0 \to_{\beta I} P$.
  - If $(\lambda x.M')N_0 \to_{\beta I} M'[x := N_0]$ then we are done.
  - If $\lambda x.M' \to_{\beta I} \lambda x.M''$ then $P = (\lambda x.M'')N_0$ is a direct βI-reduct of $(\lambda x.M')N_0$, absurd.
  - If $N_0 \to_{\beta I} N'$ then $P = (\lambda x.M')N'$ is a direct βI-reduct of $(\lambda x.M')N_0$, absurd.
- Let $n = 0$ and $r = \beta\eta$. By case on $(\lambda x.M')N_0 \to_{\beta\eta} P$.
  - If $(\lambda x.M')N_0 \to_\beta M'[x := N_0]$, then we are done.
  - If $\lambda x.M' \to_{\beta\eta} Q$ and $P = QN_0$. By lemma 2.2.3,
    - Either $Q = \lambda x.M''$ and $M' \to_{\beta\eta} M''$. Hence, $\lambda x.M' \to_{\beta i\eta} \lambda x.M''$ by lemma 2.2.4, so $P = (\lambda x.M'')N_0$ is a direct $\beta\eta$-reduct of $(\lambda x.M')N_0$, absurd.
    - Or $M' = Qx$ and $x \notin FV(Q)$. Hence, $P = QN_0 = M'[x := N_0]$ and we are done.
  - If $N_0 \to_{\beta\eta} N'$ then $P = (\lambda x.M')N'$ is a direct $\beta\eta$-reduct of $(\lambda x.M')N_0$, absurd.
- Let $n = m + 1$ where $m \geq 0$. By case on $(\lambda x.M)N_0 \dots N_{m+1} \to_r P$.
  - If $(\lambda x.M')N_0 \dots N_m \to_r Q$ and $P = QN_{m+1}$.
    - If Q is a direct r-reduct of $(\lambda x.M')N_0 \dots N_m$ then P is a direct r-reduct of $(\lambda x.M')N_0 \dots N_{m+1}$, absurd.
    - So, Q is not a direct r-reduct of $(\lambda x.M')N_0 \dots N_m$ then we are done by IH.
  - If $N_{m+1} \to_r N'_{m+1}$ then $P = (\lambda x.M')N_0 \dots N_mN'_{m+1}$ is a direct r-reduct of $(\lambda x.M')N_0 \dots N_{m+1}$, absurd.

6b. By 6a, $M = \lambda x.M'$, $k \geq 1$. We prove the statement by induction on $k \geq 1$.

- If $k = 1$ then we conclude by 6a.
- Let $(\lambda x.M')N_0 \dots N_n \to^*_r Q \to_r P$.

- * If Q is a direct r-reduct of $(\lambda x.M')N_0 \dots N_n$, then $Q = (\lambda x.M'')N'_0 \dots N'_n$, such that $M' \to_r^* M''$ (use lemma 2.2.5 if $r = \beta\eta$) and $\forall i \in \{0, \dots, n\}$, $N_i \to_r^* N'_i$. Since P is not a direct r-reduct of $(\lambda x.M')N_0 \dots N_n$, P is not a direct r-reduct of Q. Hence by 6a, $P = M''[x := N'_0]N'_1 \dots N'_n$.
  * If Q is not a direct r-reduct of $(\lambda x.M')N_0 \dots N_n$, then by IH, there exists a direct r-reduct $(\lambda x.M'')N'_0 \dots N'_n$ of $(\lambda x.M')N_0 \dots N_n$ such that $M''[x := N'_0]N'_1 \dots N'_n \to_r^* Q \to_r P$.

7. If P is a direct r-reduct of $(\lambda x.M)N_0 \dots N_n$ then $P = (\lambda x.M')N'_0 \dots N'_n$ such that $M \to_r^* M'$ (use lemma 2.2.5 if $r = \beta\eta$) and $\forall i \in \{0, \dots, n\}, N_i \to_r^* N'_i$. So $P \to_r M'[x := N'_0]N'_1 \dots N'_n$ (if $r = \beta I$, note that $x \in FV(M')$ by lemma 2.2.2) and $M[x := N_0]N_1 \dots N_n \to_r^* M'[x := N'_0]N'_1 \dots N'_n$. If P is not a direct r-reduct of $(\lambda x.M)N_0 \dots N_n$ then by lemma 6.6b, there exists a direct r-reduct, $(\lambda x.M')N'_0 \dots N'_n$, such that $M \to_r^* M'$ (use lemma 2.2.5 if $r = \beta\eta$) and $\forall i \in \{0, \dots, n\}, N_i \to_r^* N'_i$, of $(\lambda x.M)N_0 \dots N_n$. Let $P' = P$. We have $M[x := N_0]N_1 \dots N_n \to_r^* M'[x := N'_0]N'_1 \dots N'_n \to_r^* P$. □

Lemma 2.4 .

1. By induction on the structure of M.
   - Let M be a variable.
     - Let $M = x$ then $M[x := c(cx)] = c(cx) \neq x$ and for any N, $M[x := c(cx)] = c(cx) \neq Nx$ (otherwise $cx = x$ absurd).
     - Let $M = y \neq x$ then $M[x := c(cx)] = y \neq x$ and for any N, $M[x := c(cx)] = y \neq Nx$.
   - Let $M = \lambda y.P$. Since $M[x := c(cx)]$ is a λ-abstraction, $M[x := c(cx)] \neq x$ and for any N, $M[x := c(cx)] \neq Nx$.
   - Let $M = PQ$. Since $M[x := c(cx)]$ is an application, $M[x := c(cx)] \neq x$. Let $N \in \Lambda$ such that, $M[x := c(cx)] = Nx$, so $Q[x := c(cx)] = x$ and by IH, absurd.
2. By induction on the structure of M.
   - Let M be a variable.
     - Let $M = y \neq x$ then $M[y := c(cx)] = c(cx) \neq x$ and for any N, $M[y := c(cx)] = c(cx) \neq Nx$ since $cx \neq x$.
     - Let $M = z \neq x$ and $z \neq y$ then $M[y := c(cx)] = z \neq x$ and for any N, $M[y := c(cx)] = z \neq Nx$.
   - Let $M = \lambda z.P$. Since $M[y := c(cx)]$ is a λ-abstraction, $M[y := c(cx)] \neq x$ and for any N, $M[y := c(cx)] \neq Nx$.
   - Let $M = PQ$. Since $M[y := c(cx)]$ is an application, $M[y := c(cx)] \neq x$. Let $N \in \Lambda$ such that, $M[y := c(cx)] = Nx$, so $Q[y := c(cx)] = x$ and by IH, absurd.
3. By cases on the derivation of $M \in \mathcal{M}_c$.
4. By cases on the structure of M using 3.
5. By cases on the derivation of $MN \in \mathcal{M}_c$.
6. By cases on the derivation of $\lambda x.P \in \Lambda\eta_c$.
7. By cases on the derivation of $\lambda x.P \in \Lambda\mathrm{I}_c$.

8. By induction on the derivation of $M \in \mathcal{M}_c$.

   - Case (R1)1. Let $M = x$ then $M[x := N] = N \in \mathcal{M}_c$. Else $M = y \neq x$ and so $M[x := N] = M \in \mathcal{M}_c$.
   - Case (R1)2. Let $M = \lambda y.P$ where $P \in \Lambda I_c$ and $y \in FV(P)$. By IH, $P[x := N] \in \Lambda I_c$ and since $y \in FV(P[x := N])$, $M[x := N] = \lambda y.P[x := N] \in \Lambda I_c$.
   - Case (R1)3. Let $M = \lambda y.P[y := c(cy)]$ such that $P \in \Lambda\eta_c$. Then by IH, $P[x := N] \in \Lambda\eta_c$. So by (R1).3 $M[x := N] = \lambda y.P[x := N][y := c(cy)] \in \Lambda\eta_c$.
   - Case (R1)4. Let $M = \lambda y.Py$ such that $Py \in \Lambda\eta_c$, $y \notin FV(P)$ and $P \neq c$. By IH, $P[x := N]y \in \Lambda\eta_c$. By lemma 2.4.4, $P[x := N] \neq c$. Since $y \notin FV(P[x := N])$, $M[x := N] = \lambda y.P[x := N]y \in \Lambda\eta_c$.
   - Case (R2) Let $M = cM_1M_2$ such that $M_1, M_2 \in \mathcal{M}_c$. Then by IH, $M_1[x := N], M_2[x := N] \in \mathcal{M}_c$. Hence, $cM_1[x := N]M_2[x := N] \in \mathcal{M}_c$.
   - Case (R3) Let $M = M_1M_2$ such that $M_1, M_2 \in \mathcal{M}_c$ and M_1 is a λ-abstraction. Then by IH, $M_1[x := N], M_2[x := N] \in \mathcal{M}_c$. Hence, $M_1[x := N]M_2[x := N] \in \mathcal{M}_c$, since $M_1[x := N]$ is a λ-abstraction.
   - Case (R4) Let $M = cP$ such that $P \in \Lambda\eta_c$. Then by IH, $P[x := N] \in \Lambda\eta_c$ and by (R4), $M[x := N] \in \Lambda\eta_c$.

9. (a) By lemma 2.4, either $P = Nx$ where $Nx \in \Lambda\eta_c$ or $P = N[x := c(cx))]$ where $N \in \Lambda\eta_c$. In the second case, since by BC and (R4), $x \neq c$ and $c(cx) \in \Lambda\eta_c$, we get by lemma 2.4.8 that $N[x := c(cx))] \in \Lambda\eta_c$.

   (b) It is easy to show that if $P, N \in \Lambda$, then $Px \neq N[x := c(cx)]$. Hence, by lemma 2.4, $Px = Nx$ where $N, Nx \in \Lambda\eta_c$, $x \notin FV(N)$ and $N \neq c$. Since $Px = Nx$ then $P = N$.

   (c) By induction on the structure of M using lemma 2.4.
      - If M is a variable distinct from c then nothing to prove.
      - If $M = \lambda y.P[y := c(cy)]$ where $P \in \Lambda\eta_c$ then by 9a, $P[y := c(cy)] \in \Lambda\eta_c$. $M[x := c(cx)] \to_{\beta\eta} M'$ only if $M' = \lambda y.P'$ where $P[y := c(cy)][x := c(cx)] \to_{\beta\eta} P'$. So by IH, $P' = P''[x := c(cx)]$ and $P[y := c(cy))] \to_{\beta\eta} P''$. Hence $M' = \lambda y.P''[x := c(cx)] = (\lambda y.P'')[x := c(cx)]$ and $\lambda y.P[y := c(cy)] \to_{\beta\eta} \lambda y.P''$.
      - If $M = \lambda y.Py$ such that $Py \in \Lambda\eta_c$, $P \neq c$ and $y \notin FV(P)$. Let $T = M[x := c(cx)] = \lambda y.P[x := c(cx)]y$ where $y \notin FV(P[x := c(cx)])$.
        - If $T \to_\eta P[x := c(cx)]$, we are done since $M \to_\eta P$.
        - If $T \to_{\beta\eta} \lambda y.P'$ where $(Py)[x := c(cx)] = P[x := c(cx)]y \to_{\beta\eta} P'$ then $P' = P''[x := c(cx)]$ and $Py \to_{\beta\eta} P''$ by IH. Hence, $M' = \lambda y.P''[x := c(cx)] = (\lambda y.P'')[x := c(cx)]$ and $M \to_{\beta\eta} \lambda y.P''$.
      - If $M = cM_1M_2$ such that $M_1, M_2 \in \Lambda\eta_c$, then let $T = M[x := c(cx)] = cM_1[x := c(cx)]M_2[x := c(cx)]$.
        - If $T \to_{\beta\eta} cM'_1M_2[x := c(cx)]$ where $M_1[x := c(cx)] \to_{\beta\eta} M'_1$, by IH, $M'_1 = M''_1[x := c(cx)]$ and $M_1 \to_{\beta\eta} M''_1$. Hence $M' = (cM''_1M_2)[x := c(cx)]$ and $M \to_{\beta\eta} cM''_1M_2$.
        - Case $T \to_{\beta\eta} cM_1[x := c(cx)]M'_2$ where $M_2[x := c(cx)] \to_{\beta\eta} M'_2$ is similar.
      - If $M = M_1M_2$ such that $M_1, M_2 \in \Lambda\eta_c$ and M_1 is a λ-abstraction, then let $T = M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$ where $M_1[x := c(cx)]$ is a λ-abstraction. Let $M_1 = \lambda z.M_0$, so $M_1[x := c(cx)] = \lambda z.M_0[x := c(cx)]$.

- Let $T \to_{\beta\eta} M_1'M_2[x := c(cx)]$ where $M_1[x := c(cx)] \to_{\beta\eta} M_1'$. Then by IH, $M_1' = M_1''[x := c(cx)]$ and $M_1 \to_{\beta\eta} M_1''$. So $M' = M_1''[x := c(cx)]M_2[x := c(cx)] = (M_1''M_2)[x := c(cx)]$ and $M \to_{\beta\eta} M_1''M_2$.
- Case $T \to_{\beta\eta} M_1[x := c(cx)]M_2'$ where $M_2[x := c(cx)] \to_{\beta\eta} M_2'$ is similar.
- Let $T \to_{\beta} M_0[x := c(cx)][z := M_2[x := c(cx)]] = M_0[z := M_2][x := c(cx)]$. We are done since $M \to_{\beta} M_0[z := M_2]$.

• If $M = cP$ where $P \in \Lambda\eta_c$ then $M[x := c(cx)] = cP[x := c(cx)] \to_{\beta\eta} cP'$ where $P[x := c(cx)] \to_{\beta\eta} P'$. So by IH, $P' = P''[x := c(cx)]$ and $P \to_{\beta\eta} P''$. Hence $M' = cP''[x := c(cx)] = (cP'')[x := c(cx)]$ and $M \to_{\beta\eta} cP''$.

(d) By induction on n.

□

Lemma 2.5. We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V}$. Let $C \in \mathcal{R}^r_M$ so $C \in \mathcal{C}$ and $\exists R \in \mathcal{R}^r$ such that $C[R] = M$. We prove by induction on the structure of C that this is absurd, i.e. $\mathcal{R}^r_M = \varnothing$.
  - Let $C = \Box$ then $M = R$. absurd since $M \notin \mathcal{R}^r$.
  - Let $C = \lambda x.C'$ then $\lambda x.C'[R] = M$, absurd.
  - Let $C = C'N$ then $C'[R]N = M$, absurd.
  - Let $C = NC'$ then $NC'[R] = M$, absurd.
- Let $M = \lambda x.N$ and $C \in \mathcal{C}$.
  - Let $M \in \mathcal{R}^r$. We prove by induction on the structure of C that if $C \in \mathcal{R}^r_M$ then $C \in \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$.
    - ∗ Let $C = \Box$ then $\exists R \in \mathcal{R}^r$ such that $\Box[R] = R = M$ and it is done.
    - ∗ Let $C = \lambda x.C'$ then $\exists R \in \mathcal{R}^r$ such that $\lambda x.N = \lambda x.C'[R]$. So $N = C'[R]$ and by definition, $C' \in \mathcal{R}^r_N$.
    - ∗ Let $C = C'P$ then $\nexists R \in \mathcal{R}^r$ such that $\lambda x.N = C'[R]P$.
    - ∗ Let $C = PC'$ then $\nexists R \in \mathcal{R}^r$ such that $\lambda x.N = PC'[R]$.

    Let $C \in \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$, we prove that $C \in \mathcal{R}^r_M$.
    - ∗ Let $C = \Box$. Since $M \in \mathcal{R}^r$ and $C[M] = M$, by definition, $C \in \mathcal{R}^r_M$.
    - ∗ Let $C = \lambda x.C'$ such that $C' \in \mathcal{R}^r_N$. By definition $\exists R \in \mathcal{R}^r$ such that $C'[R] = N$, so $C[R] = M$.
  - Let $M \notin \mathcal{R}^r$. We prove by induction on the structure of C that if $C \in \mathcal{R}^r_M$ then $C \in \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$
    - ∗ Let $C = \Box$ then $\nexists R \in \mathcal{R}^r$ such that $\Box[R] = R = M$, since $M \notin \mathcal{R}^r$.
    - ∗ Let $C = \lambda x.C'$ then $\exists R \in \mathcal{R}^r$ such that $\lambda x.N = \lambda x.C'[R]$. So $N = C'[R]$ and by definition, $C' \in \mathcal{R}^r_N$.
    - ∗ Let $C = C'P$ then $\nexists R \in \mathcal{R}^r$ such that $\lambda x.N = C'[R]P$.
    - ∗ Let $C = PC'$ then $\nexists R \in \mathcal{R}^r$ such that $\lambda x.N = PC'[R]$.

    Let $C \in \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$, we prove that $C \in \mathcal{R}^r_M$.
    - ∗ Let $C = \lambda x.C'$ such that $C' \in \mathcal{R}^r_N$. By definition $\exists R \in \mathcal{R}^r$ such that $C'[R] = N$, so $C[R] = M$.
- Let $M = PQ$ and $C \in \mathcal{C}$.

- Let $M \in \mathcal{R}^r$. We prove by induction on the structure of C that if $C \in \mathcal{R}^r_M$ then $C \in \{\Box\} \cup \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$.
  * Let $C = \Box$ then $\exists R \in \mathcal{R}^r$ such that $\Box[R] = R = M$ and it is done.
  * Let $C = \lambda x.C'$ then $\nexists R \in \mathcal{R}^r$ such that $PQ = \lambda x.C'[R]$.
  * Let $C = C'N$ then $\exists R \in \mathcal{R}^r$ such that $PQ = C'[R]N$. So $N = Q$, $C'[R] = P$ and by definition, $C' \in \mathcal{R}^r_P$.
  * Let $C = NC'$ then $\exists R \in \mathcal{R}^r$ such that $PQ = NC'[R]$. So $N = P$, $C'[R] = Q$ and by definition, $C' \in \mathcal{R}^r_Q$.

  Let $C \in \{\Box\} \cup \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$, we prove that $C \in \mathcal{R}^r_M$.
  * Let $C = \Box$. Since $M \in \mathcal{R}^r$ and $C[M] = M$, by definition, $C \in \mathcal{R}^r_M$.
  * Let $C = C'Q$ such that $C' \in \mathcal{R}^r_P$. By definition $\exists R \in \mathcal{R}^r$ such that $C'[R] = P$, so $C[R] = M$.
  * Let $C = PC'$ such that $C' \in \mathcal{R}^r_Q$. By definition $\exists R \in \mathcal{R}^r$ such that $C'[R] = Q$, so $C[R] = M$.
- Let $M \notin \mathcal{R}^r$. We prove by induction on the structure of C that if $C \in \mathcal{R}^r_M$ then $C \in \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$.
  * Let $C = \Box$ then $\nexists R \in \mathcal{R}^r$ such that $\Box[R] = R = M$, since $M \notin \mathcal{R}^r$.
  * Let $C = \lambda x.C'$ then $\nexists R \in \mathcal{R}^r$ such that $PQ = \lambda x.C'[R]$.
  * Let $C = C'N$ then $\exists R \in \mathcal{R}^r$ such that $PQ = C'[R]N$. So $N = Q$, $C'[R] = P$ and by definition, $C' \in \mathcal{R}^r_P$.
  * Let $C = NC'$ then $\exists R \in \mathcal{R}^r$ such that $PQ = NC'[R]$. So $N = P$, $C'[R] = Q$ and by definition, $C' \in \mathcal{R}^r_Q$.

  Let $C \in \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$, we prove that $C \in \mathcal{R}^r_M$.
  * Let $C = C'Q$ such that $C' \in \mathcal{R}^r_P$. By definition $\exists R \in \mathcal{R}^r$ such that $C'[R] = P$, so $C[R] = M$.
  * Let $C = PC'$ such that $C' \in \mathcal{R}^r_Q$. By definition $\exists R \in \mathcal{R}^r$ such that $C'[R] = Q$, so $C[R] = M$.

□

Lemma 2.6. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$, by lemma 2.5, $\mathcal{R}^r_M = \varnothing$, so $\mathcal{F} = \varnothing$.
- Let $M = \lambda y.N$ then by lemma 2.5:
  - If $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$. Let $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\}$. Let $C \in \mathcal{F}'$ then $\lambda x.C \in \mathcal{F}$, so $C \in \mathcal{R}^r_N$.
    * Let $C \in \mathcal{F} \setminus \{\Box\}$ then $C = \lambda x.C'$ such that $C' \in \mathcal{R}^r_N$. So $C' \in \mathcal{F}'$ and it is done.
    * Let $C \in \{\lambda x.C \mid C \in \mathcal{F}'\}$ then $C = \lambda x.C'$ such that $C' \in \mathcal{F}'$. So $\lambda x.C' = C \in \mathcal{F} \setminus \{\Box\}$.
  - If $M \notin \mathcal{R}^r$ then $\mathcal{R}^r_M = \{\lambda x.C \mid C \in \mathcal{R}^r_N\}$. Let $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\}$. Let $C \in \mathcal{F}'$ then $\lambda x.C \in \mathcal{F}$, so $C \in \mathcal{R}^r_N$.
    * Let $C \in \mathcal{F}$ then $C = \lambda x.C'$ such that $C' \in \mathcal{R}^r_N$. So $C' \in \mathcal{F}'$ and it is done.
    * Let $C \in \{\lambda x.C \mid C \in \mathcal{F}'\}$ then $C = \lambda x.C'$ such that $C' \in \mathcal{F}'$. So $\lambda x.C' = C \in \mathcal{F}$.
- Let $M = PQ$ then by lemma 2.5:

- If $M \in \mathcal{R}^r$ then $\mathcal{R}^r_M = \{\square\} \cup \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$. Let $\mathcal{F}_1 = \{C \mid CQ \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{C \mid PC \in \mathcal{F}\}$. Let $C \in \mathcal{F}_1$ then $CQ \in \mathcal{F}$, so $C \in \mathcal{R}^r_P$. Let $C \in \mathcal{F}_2$ then $PC \in \mathcal{F}$, so $C \in \mathcal{R}^r_Q$.
  * Let $C \in \mathcal{F} \setminus \{\square\}$. Either $C = C'Q$ such that $C' \in \mathcal{R}^r_P$, so $C' \in \mathcal{F}_1$ and it is done. Or $C = PC'$ such that $C' \in \mathcal{R}^r_Q$, so $C' \in \mathcal{F}_2$ and it is done.
  * Let $C \in \{CQ \mid C \in \mathcal{F}_1\} \cup \{PC \mid C \in \mathcal{F}_2\}$. Either $C = C'Q$ such that $C' \in \mathcal{F}_1$, so $C'Q \in \mathcal{F} \setminus \{\square\}$. Or $C = PC'$ such that $C' \in \mathcal{F}_2$, so $PC' \in \mathcal{F} \setminus \{\square\}$.
- If $M \notin \mathcal{R}^r$ then $\mathcal{R}^r_M = \{CQ \mid C \in \mathcal{R}^r_P\} \cup \{PC \mid C \in \mathcal{R}^r_Q\}$. Let $\mathcal{F}_1 = \{C \mid CQ \in \mathcal{F}\}$ and $\mathcal{F}_2 = \{C \mid PC \in \mathcal{F}\}$. Let $C \in \mathcal{F}_1$ then $CQ \in \mathcal{F}$, so $C \in \mathcal{R}^r_P$. Let $C \in \mathcal{F}_2$ then $PC \in \mathcal{F}$, so $C \in \mathcal{R}^r_Q$.
  * Let $C \in \mathcal{F}$. Either $C = C'Q$ such that $C' \in \mathcal{R}^r_P$, so $C' \in \mathcal{F}_1$ and it is done. Or $C = PC'$ such that $C' \in \mathcal{R}^r_Q$, so $C' \in \mathcal{F}_2$ and it is done.
  * Let $C \in \{CQ \mid C \in \mathcal{F}_1\} \cup \{PC \mid C \in \mathcal{F}_2\}$. Either $C = C'Q$ such that $C' \in \mathcal{F}_1$, so $C'Q \in \mathcal{F}$. Or $C = PC'$ such that $C' \in \mathcal{F}_2$, so $PC' \in \mathcal{F}$.

□

Lemma 2.7.

⇒) we prove the statement by induction on M.

- $M \notin \mathcal{V}$ since by lemma 2.5, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda y.P$ so $M[x := N] = \lambda y.P[x := N]$. By lemma 2.5:
  * If $M \in \mathcal{R}^r$ then:
    · Either $C = \square$ so $M[x := N] = C[x := N][R] = \square[x := N][R] = R$. Hence, $R = M[x := N]$ and $M = \square[M]$.
    · Or $C = \lambda y.C'$ such that $C' \in \mathcal{R}^r_P$. Then, $C[x := N][R] = \lambda y.C'[x := N][R]$ and $P[x := N] = C'[x := N][R]$. By IH, $R = R'[x := N]$ and $P = C'[R']$. Hence, $M = \lambda y.P = \lambda y.C'[R'] = C[R']$.
  * If $M \notin \mathcal{R}^r$ then $C = \lambda y.C'$ such that $C' \in \mathcal{R}^r_P$. So, $C[x := N][R] = \lambda y.C'[x := N][R]$ and $P[x := N] = C'[x := N][R]$. By IH, $R = R'[x := N]$ and $P = C'[R']$. Hence, $M = \lambda y.P = \lambda y.C'[R'] = C[R']$.
- Let $M = PQ$ so $M[x := N] = P[x := N]Q[x := N]$. By lemma 2.5:
  * If $M \in \mathcal{R}^r$ then:
    · Either $C = \square$ so $M[x := N] = C[x := N][R] = \square[x := N][R] = R$. So $R = M[x := N]$ and $M = \square[M]$.
    · Or $C = C'Q$ such that $C' \in \mathcal{R}^r_P$. Then, $C[x := N][R] = C'[x := N][R]Q[x := N] = P[x := N]Q[x := N]$ and $P[x := N] = C'[x := N][R]$. By IH, $R = R'[x := N]$ and $P = C'[R']$. Hence, $M = PQ = C'[R']Q = C[R']$.
    · Or $C = PC'$ such that $C' \in \mathcal{R}^r_Q$. Then, $C[x := N][R] = P[x := N]C'[x := N][R] = P[x := N]Q[x := N]$ and $Q[x := N] = C'[x := N][R]$. By IH, $R = R'[x := N]$ and $Q = C'[R']$. Hence, $M = PQ = PC'[R'] = C[R']$.
  * If $M \notin \mathcal{R}^r$ then:
    · Either $C = C'Q$ such that $C' \in \mathcal{R}^r_P$. Then, $C[x := N][R] = C'[x := N][R]Q[x := N] = P[x := N]Q[x := N]$ and $P[x := N] = C'[x := N][R]$. By IH, $R = R'[x := N]$ and $P = C'[R']$. Hence, $M = PQ = C'[R']Q = C[R']$.

- $\cdot$ Or $C = PC'$ such that $C' \in \mathcal{R}^r_Q$. Then, $C[x := N][R] = P[x := N]C'[x := N][R] = P[x := N]Q[x := N]$ and $Q[x := N] = C'[x := N][R]$. By IH, $R = R'[x := N]$ and $Q = C'[R']$. Hence, $M = PQ = PC'[R'] = C[R']$.

$\Leftarrow$) We prove the statement by induction on the structure of M.

- $M \notin \mathcal{V}$ since by lemma 2.5, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda y.P$. By lemma 2.5:
  - $*$ Let $M \in \mathcal{R}^r$.
    - $\cdot$ Either $C = \square$ and $R = M$, so $C[x := N][R[x := N]] = \square[M[x := N]] = M[x := N]$.
    - $\cdot$ Or $C = \lambda y.C'$ such that $C' \in \mathcal{R}^r_P$. Then, $\lambda y.P = \lambda y.C'[R]$ so $P = C'[R]$. By IH, $P[x := N] = C'[x := N][R[x := N]]$. Hence, $M[x := N] = \lambda y.P[x := N] = \lambda y.C'[x := N][R[x := N]] = (\lambda y.C'[x := N])[R[x := N]] = (\lambda y.C')[x := N][R[x := N]] = C[x := N][R[x := N]]$.
  - $*$ Let $M \notin \mathcal{R}^r$, then $C = \lambda y.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_P$. Then, $\lambda y.P = \lambda y.C'[R]$ so $P = C'[R]$. By IH, $P[x := N] = C'[x := N][R[x := N]]$. Hence, $M[x := N] = \lambda y.P[x := N] = \lambda y.C'[x := N][R[x := N]] = (\lambda y.C'[x := N])[R[x := N]] = (\lambda y.C')[x := N][R[x := N]] = C[x := N][R[x := N]]$.
- Let $M = PQ$. By lemma 2.5:
  - $*$ Let $M \in \mathcal{R}^r$.
    - $\cdot$ Either $C = \square$ and $R = M$, so $C[x := N][R[x := N]] = \square[M[x := N]] = M[x := N]$.
    - $\cdot$ Or $C = C'Q$ such that $C' \in \mathcal{R}^r_P$. Then, $PQ = (C'Q)[R] = C'[R]Q$ and $P = C'[R]$. By IH, $P[x := N] = C'[x := N][R[x := N]]$. Hence, $M[x := N] = P[x := N]Q[x := N] = C'[x := N][R[x := N]]Q[x := N] = (C'[x := N]Q[x := N])[R[x := N]] = (C'Q)[x := N][R[x := N]] = C[x := N][R[x := N]]$.
    - $\cdot$ Or $C = PC'$ such that $C' \in \mathcal{R}^r_Q$. Then $PQ = (PC')[R] = PC'[R]$ and $Q = C'[R]$. By IH, $Q[x := N] = C'[x := N][R[x := N]]$. Hence, $M[x := N] = P[x := N]Q[x := N] = P[x := N]C'[x := N][R[x := N]] = (P[x := N]C'[x := N])[R[x := N]] = (PC')[x := N][R[x := N]] = C[x := N][R[x := N]]$.
  - $*$ Let $M \notin \mathcal{R}^{\beta\eta}$.
    - $\cdot$ Either $C = C'Q$ such that $C' \in \mathcal{R}^r_P$. Then, $PQ = (C'Q)[R] = C'[R]Q$ and $P = C'[R]$. By IH, $P[x := N] = C'[x := N][R[x := N]]$. Hence, $M[x := N] = P[x := N]Q[x := N] = C'[x := N][R[x := N]]Q[x := N] = (C'[x := N]Q[x := N])[R[x := N]] = (C'Q)[x := N][R[x := N]] = C[x := N][R[x := N]]$.
    - $\cdot$ Or $C = PC'$ such that $C' \in \mathcal{R}^r_Q$. Then $PQ = (PC')[R] = PC'[R]$ and $Q = C'[R]$. By IH, $Q[x := N] = C'[x := N][R[x := N]]$. Hence, $M[x := N] = P[x := N]Q[x := N] = P[x := N]C'[x := N][R[x := N]] = (P[x := N]C'[x := N])[R[x := N]] = (PC')[x := N][R[x := N]] = C[x := N][R[x := N]]$.

$\square$

Lemma 2.8. We prove the lemma by induction on the structure of C.

- Let $C = \Box$ then $C[x := N][R] = \Box[R] = R$ and $C[R][x := N] = R[x := N] = R$.
- Let $C = \lambda y.C'$. By (BC), $x \neq y$. Then, $C[x := N][R] = \lambda y.C'[x := N][R] =^{IH} \lambda y.C'[R][x := N] = C[R][x := N]$.
- Let $C = C'P$. Then, $C[x := N][R] = C'[x := N][R]P[x := N] =^{IH} C'[R][x := N]P[x := N] = (C'[R]P)[x := N] = C[R][x := N]$.
- Let $C = PC'$. Then, $C[x := N][R] = P[x := N]C'[x := N][R] =^{IH} P[x := N]C'[R][x := N] = (PC'[R])[x := N] = C[R][x := N]$. $\Box$

Lemma 2.9.

1. By case on the structure of M.
   - let $M \in \mathcal{V}$.
     - Either $M = x$ then, $M[x := c(cx)] = c(cx)$. Hence, $c(cx) \neq y$, $c(cx) \neq Py$ since $cx \neq y$ and $c(cx) \neq \lambda y.P$.
     - Or $M = z \neq x$ then $M[x := c(cx)] = z$. Hence, if $z = y$ then $M = y$, $z \neq Py$ and $z \neq \lambda y.P$.
   - Let $M = \lambda z.M'$ then $M[x := c(cx)] = \lambda z.M'[x := c(cx)]$. Hence, $\lambda z.M'[x := c(cx)] \neq y$ and $\lambda z.M'[x := c(cx)] \neq Py$. By (BC), $y \notin FV(M')$ so $M = \lambda y.M'[z := y]$ and $M[x := c(cx)] = \lambda y.M'[z := y][x := c(cx)] = \lambda y.P$. Hence, $M'[z := y][x := c(cx)] = P$
   - Let $M = M_1M_2$ then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. Hence, $M_1[x := c(cx)]M_2[x := c(cx)] \neq y$ and $M_1[x := c(cx)]M_2[x := c(cx)] \neq \lambda y.P$. If $M_1[x := c(cx)]M_2[x := c(cx)] = Py$ then $P = M_1[x := c(cx)]$ and $M_2[x := c(cx)] = y$. So $M_2 = y$.
2. By case on the structure of M.
   - Let $M \in \mathcal{V}$ then $M \notin \mathcal{R}^{\beta\eta}$ and $M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$.
   - Let $M = \lambda y.N$ then $M[x := c(cx)] = \lambda y.N[x := c(cx)]$. By (BC), $x \neq y \neq c$.
     - If $M \in \mathcal{R}^{\beta\eta}$ then $N = Py$ such that $y \notin FV(P)$. $N[x := c(cx)] = P[x := c(cx)]y$ and $y \notin FV(P[x := c(cx)])$, so $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
     - If $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ then $N[x := c(cx)] = Py$ such that $y \notin FV(P)$. By 1, $N = Qy$ and $P = Q[x := c(cx)]$. So $M = \lambda y.Qy$. Since $y \notin FV(P)$, $y \notin FV(Q)$. So $M \in \mathcal{R}^{\eta}$.
   - Let $M = M_1M_2$ then $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$.
     - If $M \in \mathcal{R}^{\beta\eta}$ then $M_1 = \lambda y.M_0$. So $M[x := c(cx)] = (\lambda y.M_0[x := c(cx)])M_2[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
     - If $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ then $M_1[x := c(cx)] = \lambda y.P$. By 1, $M_1 = \lambda y.M_0$ and $P = M_0[x := c(cx)]$. So, $M \in \mathcal{R}^{\beta\eta}$
3. $\Rightarrow$) Let $C \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$. By lemma 2.4, $\lambda x.M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$ so by lemma 2.5, $C = \lambda x.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$.

   $\Leftarrow$) Let $C \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. By lemma 2.5, $\lambda x.C \in \mathcal{R}^{\beta\eta}_{\lambda x.M[x:=c(cx)]}$.
4. $\Rightarrow$) Let $C \in \mathcal{R}^{\beta\eta}_{M[x:=c(cx)]}$. We prove the statement by induction on the structure of M
   - $M \notin \mathcal{V}$ since $\mathcal{R}^{\beta\eta}_{M[x:=c(cx)]} = \varnothing$.

– Let $M = \lambda y.N$ so $M[x := c(cx)] = \lambda y.N[x := c(cx)]$. By lemma 2.5:
  * If $M \in \mathcal{R}^{\beta\eta}$ then by 2, $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
    · Either $C = \Box$ and $C[x := c(cx)] = \Box \in \mathcal{R}^{\beta\eta}_M$.
    · Or $C = \lambda y.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}$. By IH, $C' = C''[x := c(cx)]$ and $C'' \in \mathcal{R}^{\beta\eta}_N$. Hence $C = \lambda y.C''[x := c(cx)] = (\lambda y.C'')[x := c(cx)]$ and by lemma 2.5, $\lambda y.C'' \in \mathcal{R}^{\beta\eta}_M$.
  * Or $M \notin \mathcal{R}^{\beta\eta}$ then by 2, $M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$. So, $C = \lambda y.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}$. By IH, $C' = C''[x := c(cx)]$ and $C'' \in \mathcal{R}^{\beta\eta}_N$. Hence $C = \lambda y.C''[x := c(cx)] = (\lambda y.C'')[x := c(cx)]$ and by lemma 2.5, $\lambda y.C'' \in \mathcal{R}^{\beta\eta}_M$.

– Let $M = M_1 M_2$ so $M[x := c(cx)] = M_1[x := c(cx)]M_2[x := c(cx)]$. By lemma 2.5:
  * If $M \in \mathcal{R}^{\beta\eta}$ then by 2, $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
    · Either $C = \Box$ and $C[x := c(cx)] = \Box \in \mathcal{R}^{\beta\eta}_M$.
    · Or $C = C_1 M_2[x := c(cx)]$ such that $C_1 \in \mathcal{R}^{\beta\eta}_{M_1[x:=c(cx)]}$. By IH, $C_1 = C'_1[x := c(cx)]$ and $C'_1 \in \mathcal{R}^{\beta\eta}_{M_1}$. Hence $C = (C'_1 M_2)[x := c(cx)]$ and by lemma 2.5, $C'_1 M_2 \in \mathcal{R}^{\beta\eta}_M$.
    · Or $C = M_1[x := c(cx)]C_2$ such that $C_2 \in \mathcal{R}^{\beta\eta}_{M_2[x:=c(cx)]}$. By IH, $C_2 = C'_2[x := c(cx)]$ and $C'_2 \in \mathcal{R}^{\beta\eta}_{M_1}$. Hence $C = (M_1 C'_2)[x := c(cx)]$ and by lemma 2.5, $M_1 C'_2 \in \mathcal{R}^{\beta\eta}_M$.
  * Or $M \notin \mathcal{R}^{\beta\eta}$ then by 2, $M[x := c(cx)] \notin \mathcal{R}^{\beta\eta}$.
    · Either $C = C_1 M_2[x := c(cx)]$ and $C_1 \in \mathcal{R}^{\beta\eta}_{M_1[x:=c(cx)]}$. By IH, $C_1 = C'_1[x := c(cx)]$ and $C'_1 \in \mathcal{R}^{\beta\eta}_{M_1}$. Hence $C = (C'_1 M_2)[x := c(cx)]$ and by lemma 2.5, $C'_1 M_2 \in \mathcal{R}^{\beta\eta}_M$.
    · Or $C = M_1[x := c(cx)]C_2$ and $C_2 \in \mathcal{R}^{\beta\eta}_{M_2[x:=c(cx)]}$. By IH, $C_2 = C'_2[x := c(cx)]$ and $C'_2 \in \mathcal{R}^{\beta\eta}_{M_1}$. Hence $C = (M_1 C'_2)[x := c(cx)]$ and by lemma 2.5, $M_1 C'_2 \in \mathcal{R}^{\beta\eta}_M$.

$\Leftarrow$) Let $C \in \mathcal{R}^r_M$. Then $C \in \mathcal{C}$ and $\exists R \in \mathcal{R}^{\beta\eta}$ such that $C[R] = M$. So by 2, $R[x := c(cx)] \in \mathcal{R}^{\beta\eta}$ and by lemma 2.7, $C[x := c(cx)][R[x := c(cx)]] = M[x := c(cx)]$. Hence, by definition, $C[x := c(cx)] \in \mathcal{R}^r_{M[x:=c(cx)]}$.

5. We prove this statement by induction on $n \geq 0$.

- Let $n = 0$ then trivial.
- let $n = m+1$ such that $m \geq 0$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_{c^m(M)} = \{C c^m(M) \mid C \in \mathcal{R}^{\beta\eta}_c\} \cup \{c(C) \mid C \in \mathcal{R}^{\beta\eta}_{c^m(M)}\} =^{IH} \{c^n(C) \mid C \in \mathcal{R}^{\beta\eta}_M\}$.

$\Box$

Lemma 2.10. We prove the statement by case on r.

- Either $r = \beta I$. Since $M \in \Lambda I_c$, $M \in \Lambda I$, so $\lambda x.P, Q \in \Lambda I$. Hence, $x \in FV(P)$ and $M \in \mathcal{R}^{\beta I}$.
- Or $r = \beta\eta$. Trivial. $\Box$

Lemma 2.11. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$. Nothing to prove since by lemma 2.5, $\mathcal{R}^r_M = \varnothing$.

- Let $M = \lambda x.N \in \Lambda I$. let $C \in \mathcal{R}^{\beta I}_M$ then by definition, $\exists R \in \mathcal{R}^{\beta I}$ such that $M = C[R]$. Since $M \notin \mathcal{R}^{\beta I}$, by lemma 2.5, $C = \lambda x.C'$ such that $C' \in \mathcal{R}^{\beta I}_N$. So $\lambda x.N = \lambda x.C'[R]$ and $N = C'[R]$. By IH, $R \in \Lambda I_c$.
- Let $M = \lambda x.N[x := c(cx)] \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then by definition $\exists R \in \mathcal{R}^{\beta\eta}$ such that $M = C[R]$. By lemma 2.9.3, $C = \lambda x.C'$ and $C' \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}$. By lemma 2.9.4, $C' = C''[x := c(cx)]$ and $C'' \in \mathcal{R}^{\beta\eta}_N$. Since $x \notin FV(R)$, by lemma 2.8, $\lambda x.N[x := c(cx)] = (\lambda x.C''[x := c(cx)])[R] = \lambda x.C''[x := c(cx)][R] = \lambda x.C''[R][x := c(cx)]$ and $N = C''[R]$. By IH, $R \in \Lambda\eta_c$.
- Let $M = \lambda x.Nx \in \Lambda\eta_c$ such that $Nx \in \Lambda\eta_c$, $x \notin FV(N)$ and $c \neq N$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then by definition $\exists R \in \mathcal{R}^{\beta\eta}$ such that $M = C[R]$. Since $M \in \mathcal{R}^{\beta\eta}$, by lemma 2.5:
  - Either $C = \Box$ so $\Box[R] = R = M$ and $M \in \Lambda\eta_c$.
  - Or $C = \lambda x.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{Nx}$. So $M = \lambda x.Nx = \lambda x.C'[R]$ and $Nx = C'[R]$. By IH, $R \in \Lambda\eta_c$.
- Let $M = cNP \in \mathcal{M}_c$ such that $N, P \in \mathcal{M}_c$. Let $C \in \mathcal{R}^r_M$ then by definition $\exists R \in \mathcal{R}^r$ such that $M = C[R]$. Since $M, cN \notin \mathcal{R}^r$, by lemma 2.5:
  - Either $C = cC'P$ such that $C' \in \mathcal{R}^r_N$. So $M = cNP = (cC'P)[R] = cC'[R]P$ and $N = C'[R]$. By IH, $R \in \mathcal{M}_c$.
  - Or $C = cNC'$ such that $C' \in \mathcal{R}^P_r$. So $M = cNP = (cNC')[R] = cNC'[R]$ and $P = C'[R]$. By IH, $R \in \mathcal{M}_c$.
- Let $M = (\lambda x.N)P \in \mathcal{M}_c$ such that $\lambda x.N, P \in \mathcal{M}_c$. Let $C \in \mathcal{R}^r_M$ then by definition $\exists R \in \mathcal{R}^r$ such that $M = C[R]$. Since by lemma 2.10, $M \in \mathcal{R}^r$, by lemma 2.5:
  - Either $C = \Box$ so $M = \Box[R] = R$ and $M \in \mathcal{M}_c$.
  - Or $C = C'P$ such that $C' \in \mathcal{R}^r_{\lambda x.N}$. So $M = (\lambda x.N)P = (C'P)[R] = C'[R]P$ and $\lambda x.N = C'[R]$. By IH, $R \in \mathcal{M}_c$.
  - Or $C = (\lambda x.N)C'$ such that $C' \in \mathcal{R}^r_P$. So $M = (\lambda x.N)P = ((\lambda x.N)C')[R] = (\lambda x.N)C'[R]$ and $P = C'[R]$. By IH, $R \in \mathcal{M}_c$.
- Let $M = cN \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then by definition $\exists R \in \mathcal{R}^{\beta\eta}$ such that $M = C[R]$. Since $M \notin \mathcal{R}^{\beta\eta}$, by lemma 2.5, $C = cC'$ such that $C' \in \mathcal{R}^{\beta\eta}_N$. So $M = cN = cC'[R]$ and $N = C'[R]$. By IH, $R \in \Lambda\eta_c$. $\Box$

Lemma 2.12.

1. By induction on $M \to_{\beta\eta} M'$.
   - Let $M = \lambda x.Nx \to_\eta N = M'$ where $x \notin FV(N)$. By lemma 2.4, $N \in \Lambda\eta_c$.
   - Let $M = (\lambda x.N)P \to_\beta N[x := P] = M'$. By lemmas 2.4 and 2.4.9, $N, P \in \Lambda\eta_c$. By lemma 2.4.8, $N[x := P] \in \Lambda\eta_c$.
   - Let $M = \lambda x.N \to_{\beta\eta} \lambda x.N' = M'$ such that $N \to_{\beta\eta} N'$. By lemma 2.4:
     - Either $M = \lambda x.P[x := c(cx)]$ where $P \in \Lambda\eta_c$ and $P[x := c(cx)] \to_{\beta\eta} N'$. So by lemma 2.4.9.9c, $N' = N''[x := c(cx)]$ and $P \to_{\beta\eta} N''$. By IH, $N'' \in \Lambda\eta_c$ so by BC, (R1).3, $\lambda x.N' \in \Lambda\eta_c$.
     - Or $M = \lambda x.Px$ where $P, Px \in \Lambda\eta_c$, $x \notin FV(P)$, $P \neq c$ and $Px \to_{\beta\eta} N'$. So by IH, $N' \in \Lambda\eta_c$. One of two cases holds:

    * $Px \to_{\beta\eta} P'x$ where $P \to_{\beta\eta} P'$. By IH, $P', P'x \in \Lambda\eta_c$. By lemmas 2.4.3 and 2.2.1, $P' \neq c$ and $x \notin FV(P')$. By (R1).4, $\lambda x.P'x \in \Lambda\eta_c$.
    * $P = \lambda y.P_0$ and $Px \to_\beta P_0[y := x]$. So $M \to_\beta \lambda x.P_0[y := x] = P \in \Lambda\eta_c$.

- Let $M = M_1M_2 \to_{\beta\eta} M_1'M_2 = M'$ such that $M_1 \to_{\beta\eta} M_1'$. By lemma 2.4:
  - Either $M_1 = cM_0$ and $M_0, M_2 \in \Lambda\eta_c$. Then, $M_1 = cM_0 \to_{\beta\eta} cM_0' = M_1'$ where $M_0 \to_{\beta\eta} M_0'$. By IH, $M_0' \in \Lambda\eta_c$, so by (R2), $M' \in \Lambda\eta_c$.
  - Or $M_1 = \lambda x.M_0$ and $M_1, M_2 \in \Lambda\eta_c$. By lemma 2.4.9.9a, $M_0 \in \Lambda\eta_c$ and by IH, $M_1' \in \Lambda\eta_c$.
    * Either $M = (\lambda x.M_0)M_2 \to_{\beta\eta} (\lambda x.M_0')M_2$ where $M_0 \to_{\beta\eta} M_0'$. So $M_1' = \lambda x.M_0'$ is a λ-abstraction and by (R3), $M' \in \Lambda\eta_c$.
    * Or $M = (\lambda x.M_1'x)M_2 \to_\eta M_1'M_2$ where $x \notin FV(M_1')$. Since $M_1 \in \Lambda\eta_c$, by lemma 2.4, $M_1' \neq c$ and $M_1' \in \Lambda\eta_c$. Since $M_0 = M_1'x \in \Lambda\eta_c$, again by lemma 2.4, either $M_1' = cM_1''$ such that $M_1'' \in \Lambda\eta_c$ and so by (R2) $M' \in \Lambda\eta_c$, or $M_1' \in \Lambda\eta_c$ is a λ-abstraction and so by (R3) $M' \in \Lambda\eta_c$.
- Let $M = M_1M_2 \to_{\beta\eta} M_1M_2' = M'$ such that $M_2 \to_{\beta\eta} M_2'$. By lemma 2.4, $M_2 \in \Lambda\eta_c$ so by IH, $M_2' \in \Lambda\eta_c$. By lemma 2.4, there are 3 cases:
  - $M_1 = cM_0$ where $M_0 \in \Lambda\eta_c$. Then, $M' \in \Lambda\eta_c$ by (R2).
  - $M_1 \in \Lambda\eta_c$ is a λ-abstraction. Then $M' \in \Lambda\eta_c$ by (R3).
  - $M_1 = c$. Then $M' \in \Lambda\eta_c$ by (R4).

2. By induction on $M \to_{\beta I} M'$ in a similar fashion to the above. □

Lemma 'refncstwo. We prove the statement by induction on $n \geq 0$.

- Let $n = 0$ then by definition $|c^n(M)|^c = |M|^c$.
- Let $n = m+1$ such that $m \geq 0$ then $|c^n(M)|^c = |c(c^m(M))|^c = |c^m(M)|^c =^{IH} |M|^c$. □

Lemma 2.16.

- let $P \in \mathcal{V}$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M = P$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c \neq P$.
  - Let $M = M_1M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(P)$. If $M_1 \neq c$ then $|M|^c = |M_1|^c|M_2|^c \neq P$.
- Let $P = \lambda x.Q$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M \neq \lambda x.Q$.
  - Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c$ so $|N|^c = Q$.
  - Let $M = M_1M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(\lambda x.N)$ and $|N|^c = Q$. If $M_1 \neq c$ then $|M|^c = |M_1|^c|M_2|^c \neq \lambda x.Q$.
- Let $P = P_1P_2$. We prove the statement by induction on the structure of M.
  - Let $M \in \mathcal{V}$ then $|M|^c = M \neq P_1P_2$.

– Let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c \neq P_1P_2$.

– Let $M = M_1M_2$. If $M_1 = c$ then $|M|^c = |M_2|^c$. By IH, $\exists n \geq 0$ such that $M_2 = c^n(M_2'M_2'')$, $|M_2'|^c = P_1$ and $|M_2''|^c = P_2$. If $M_1 \neq c$ then $|M|^c = |M_1|^c|M_2|^c = P_1P_2$ so $|M_1|^c = P_1$ and $|M_2|^c = P_2$.

□

Lemma 2.17. We prove the statement by induction on M.

- Let $M \in \mathcal{V}$ then by lemma 2.5, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda x.N$ then by lemma 2.5:
  - Either $M \in \mathcal{R}^r$ then:
    - Either $C = \Box = C'$ so it is done.
    - Or $C = \Box$ and $C' = \lambda x.C_0'$ such that $C_0' \in \mathcal{R}^r_N$. Nothing to prove since $\Box \neq \lambda x.|C_0'|^c_{\mathcal{C}}$.
    - Or $C = \lambda x.C_0$ and $C' = \lambda x.C_0'$ such that $C_0, C_0' \in \mathcal{R}^r_N$. By hypothesis, $\lambda x.|C_0|^c_{\mathcal{C}} = \lambda x.|C_0'|^c_{\mathcal{C}}$ so $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$.
  - Or $M \notin \mathcal{R}^r$ then $C = \lambda x.C_0$ and $C' = \lambda x.C_0'$ such that $C_0, C_0' \in \mathcal{R}^r_N$. By hypothesis, $\lambda x.|C_0|^c_{\mathcal{C}} = \lambda x.|C_0'|^c_{\mathcal{C}}$ so $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$.
- Let $M = PQ$ then by lemma 2.5:
  - Either $M \in \mathcal{R}^r$, so P is a λ-abstraction and:
    - Either $C = \Box = C'$ so it is done.
    - Or $C = \Box$ and $C' = C_0'Q$ such that $C_0' \in \mathcal{R}^r_P$. Nothing to prove since $\Box \neq |C_0'|^c_{\mathcal{C}}|Q|^c$.
    - Or $C = \Box$ and $C' = PC_0'$ such that $C_0' \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, $\Box \neq |P|^c|C_0'|^c_{\mathcal{C}}$.
    - Or $C = C_0Q$ and $C' = C_0'Q$ such that $C_0, C_0' \in \mathcal{R}^r_P$. Since by hypothesis, $|C|^c_{\mathcal{C}} = |C_0|^c_{\mathcal{C}}|Q|^c = |C_0'|^c_{\mathcal{C}}|Q|^c = |C'|^c_{\mathcal{C}}$, then $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$.
    - Or $C = C_0Q$ and $C' = PC_0'$ such that $C_0 \in \mathcal{R}^r_P$ and $C_0' \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, $|C|^c_{\mathcal{C}} = |C_0|^c_{\mathcal{C}}|Q|^c \neq |P|^c|C_0'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$.
    - Or $C = PC_0$ and $C' = PC_0'$ such that $C_0, C_0' \in \mathcal{R}^r_Q$. Since P is a λ-abstraction, by hypothesis, $|C|^c_{\mathcal{C}} = |P|^c|C_0|^c_{\mathcal{C}} = |P|^c|C_0'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$ so $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$.
  - Or $M \notin \mathcal{R}^r$, then:
    - Or $C = C_0Q$ and $C' = C_0'Q$ such that $C_0, C_0' \in \mathcal{R}^r_P$. Since by hypothesis, $|C|^c_{\mathcal{C}} = |C_0|^c_{\mathcal{C}}|Q|^c = |C_0'|^c_{\mathcal{C}}|Q|^c = |C'|^c_{\mathcal{C}}$, then $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$.
    - Or $C = C_0Q$ and $C' = PC_0'$ such that $C_0 \in \mathcal{R}^r_P$ and $C_0' \in \mathcal{R}^r_Q$. $P =\neq c$, otherwise, by lemma 2.5, $\mathcal{R}^r_P = \varnothing$. Moreover, $|C|^c_{\mathcal{C}} = |C_0|^c_{\mathcal{C}}|Q|^c \neq |P|^c|C_0'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$.
    - Or $C = PC_0$ and $C' = PC_0'$ such that $C_0, C_0' \in \mathcal{R}^r_Q$. If $P \neq c$ then, by hypothesis, $|C|^c_{\mathcal{C}} = |P|^c|C_0|^c_{\mathcal{C}} = |P|^c|C_0'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$ so $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$. If $P = c$ then, by hypothesis, $|C|^c_{\mathcal{C}} = |C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}} = |C'|^c_{\mathcal{C}}$ so $|C_0|^c_{\mathcal{C}} = |C_0'|^c_{\mathcal{C}}$. By IH, $C_0 = C_0'$ so $C = C'$.

□

Lemma 2.18. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$
  - Let $M = x$ then $|M[x := c(cx)]|^c = |c(cx)|^c = |x|^c$.
  - Let $M = y \neq x$ then $|M[x := c(cx)]|^c = |M|^c$.
- Let $M = \lambda y.N$ then $|M[x := c(cx)]|^c = \lambda y.|N[x := c(cx)]|^c =^{IH} \lambda y.|N|^c = |M|^c$.
- Let $M = NP$.
  - Either $N = c$, so $N[x := c(cx)] = c$. Then, $|M[x := c(cx)]|^c = |P[x := c(cx)]|^c =^{IH} |P|^c = |M|^c$.
  - Or $N \neq c$, so $N[x := c(cx)] \neq c$. Then, $|M[x := c(cx)]|^c = |N[x := c(cx)]|^c|P[x := c(cx)]|^c =^{IH} |N|^c|P|^c = |M|^c$.

□

Lemma 2.19. We prove the statement by induction on the structure of C.

- Let $C = \Box$ then $|C[x := c(cx)]|^c_{\mathcal{C}} = \Box = |C|^c_{\mathcal{C}}$.
- Let $C = \lambda y.C'$ then $|C[x := c(cx)]|^c_{\mathcal{C}} = \lambda y.|C'[x := c(cx)]|^c_{\mathcal{C}} =^{IH} \lambda y.|C'|^c_{\mathcal{C}} = |C|^c_{\mathcal{C}}$.
- Let $C = C'P$ then $|C[x := c(cx)]|^c_{\mathcal{C}} = |C'[x := c(cx)]|^c_{\mathcal{C}}|P[x := c(cx)]|^c =^{IH,\ 2.18} |C'|^c_{\mathcal{C}}|P|^c = |C|^c_{\mathcal{C}}$.
- Let $C = PC'$.
  - Either $P = c$, so $P[x := c(cx)] = c$. Then, $|C[x := c(cx)]|^c_{\mathcal{C}} = |C'[x := c(cx)]|^c_{\mathcal{C}} =^{IH} |C'|^c_{\mathcal{C}} = |C|^c_{\mathcal{C}}$.
  - Or $P \neq c$, so $P[x := c(cx)] \neq c$. Then, $|C[x := c(cx)]|^c_{\mathcal{C}} = |P[x := c(cx)]|^c|C'[x := c(cx)]|^c_{\mathcal{C}} =^{IH,\ 2.18} |P|^c|C'|^c_{\mathcal{C}} = |C|^c_{\mathcal{C}}$.

□

Lemma 2.20. We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$.
  - Either $M = x$ then $|M[x := N]|^c = |N|^c = M[x := |N|^c] = |M|^c[x := |N|^c]$.
  - Or $M = y \neq x$ then $|M[x := N]|^c = |M|^c = M = M[x := |N|^c] = |M|^c[x := |N|^c]$.
- Let $M = \lambda y.P \in \Lambda\mathrm{I}$. $|M[x := N]|^c = \lambda y.|P[x := N]|^c =^{IH} \lambda y.|P|^c[x := |N|^c] = |M|^c[x := |N|^c]$.
- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$. Since $y \notin FV(N)$, $|M[x := N]|^c = \lambda y.|P[y := c(cy)][x := N]|^c = \lambda y.|P[x := N][y := c(cy)]|^c =^{2.18} \lambda y.|P[x := N]|^c =^{IH} \lambda y.|P|^c[x := |N|^c] =^{2.18} |P[y := c(cy)]|^c[x := |N|^c] = |M|^c[x := |N|^c]$.
- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin FV(P)$ and $c \neq N$. $|M[x := N]|^c = \lambda y.|(Py)[x := N]|^c =^{IH} \lambda y.|Py|^c[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. $|M[x := N]|^c = |P[x := N]|^c|Q[x := N]|^c =^{IH} |P|^c[x := |N|^c]|Q|^c[x := |N|^c] = (|P|^c|Q|^c)[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. $|M[x := N]|^c = |(\lambda y.P)[x := N]|^c|Q[x := N]|^c =^{IH} |\lambda y.P|^c[x := |N|^c]|Q|^c[x := |N|^c] = (|\lambda y.P|^c|Q|^c)[x := |N|^c] = |M|^c[x := |N|^c]$.

- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. $|M[x := N]|^c = |P[x := N]|^c =^{IH} |P|^c[x := |N|^c] = |M|^c[x := |N|^c]$. □

Lemma 2.21. We prove this lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $|M|^c = M$ and $FV(M) \setminus \{c\} = \{M\} = FV(|M|^c)$.

- Let $M = \lambda y.P \in \Lambda\mathrm{I}$ then $|M|^c = \lambda y.|P|^c$. $FV(M) \setminus \{c\} = FV(P) \setminus \{y, c\} =^{IH} FV(|P|^c) \setminus \{y\} = FV(|M|^c)$.

- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$. $|M|^c = \lambda y.|P[y := c(cy)]|^c =^{2.18} \lambda y.|P|^c$. $FV(M) \setminus \{c\} = FV(P[y := c(cy)]) \setminus \{c, y\} = FV(P) \setminus \{c, y\} =^{IH} FV(|P|^c) \setminus \{y\} = FV(|M|^c)$.

- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin FV(P)$ and $c \neq N$. $|M|^c = \lambda y.|Py|^c$. $FV(M) \setminus \{c\} = FV(Py) \setminus \{c, y\} =^{IH} FV(|Py|^c) \setminus \{y\} = FV(|M|^c)$.

- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. $|M|^c = |P|^c|Q|^c$. $FV(M) \setminus \{c\} = (FV(P) \cup FV(Q)) \setminus \{c\} = (FV(P) \setminus \{c\}) \cup (FV(Q) \setminus \{c\}) =^{IH} FV(|P|^c) \cup FV(|Q|^c) = FV(|M|^c)$.

- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. $|M|^c = |\lambda y.P|^c|Q|^c$. $FV(M) \setminus \{c\} = (FV(\lambda y.P) \cup FV(Q)) \setminus \{c\} = (FV(\lambda y.P) \setminus \{c\}) \cup (FV(Q) \setminus \{c\}) =^{IH} FV(|\lambda y.P|^c) \cup FV(|Q|^c) = FV(|M|^c)$.

- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. $|M|^c = |P|^c$. $FV(M) \setminus \{c\} = FV(P) \setminus \{c\} =^{IH} FV(|P|^c) = FV(|M|^c)$. □

Lemma 2.22. We prove the lemma by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then $|M|^c = M \in \mathcal{V} \setminus \{c\} \subseteq \Lambda\mathrm{I}$.

- let $M = \lambda x.N$ then $|M|^c = \lambda x.|N|^c$. By (BC), $x \neq c$. Since $N \in \Lambda\mathrm{I}_c$, by IH, $|N|^c \in \Lambda\mathrm{I}$. Since $x \in FV(N)$, by lemma 2.21, $x \notin FV(|N|^c)$, so $|M|^c \in \Lambda\mathrm{I}$.

- Let $M = cPQ$ then $|M|^c = |P|^c|Q|^c$. Since $P, Q \in \Lambda\mathrm{I}_c$, by IH, $|P|^c, |Q|^c \in \Lambda\mathrm{I}$, hence $|M|^c \in \Lambda\mathrm{I}$.

- Let $M = (\lambda x.P)Q$ then $|M|^c = |\lambda x.P|^c|Q|^c$. Since $\lambda x.P, Q \in \Lambda\mathrm{I}_c$, by IH, $|\lambda x.P|^c, |Q|^c \in \Lambda\mathrm{I}$, hence $|M|^c \in \Lambda\mathrm{I}$. □

Lemma 2.23. We prove this lemma by case on r.

- Either $r = \beta I$, so $R = (\lambda x.M)N$ such that $x \in FV(M)$. By (BC), $x \neq c$. Since $R \in \Lambda\mathrm{I}_c$ by lemma 2.4, $(\lambda x.M), N \in \Lambda\mathrm{I}_c$ and again by lemma 2.4, $M \in \Lambda\mathrm{I}_c$. By lemma 2.21, $x \in FV(|M|^c)$, so $|R|^c = (\lambda x.|M|^c)|N|^c \in \mathcal{R}^{\beta I}$. $|M|^c[x := |N|^c] =^{2.20} |M[x := N]|^c$ is the contractum of $|R|^c$ and $M[x := N]$ is the contractum of R.

- Or $r = \beta\eta$, so $R \in \mathcal{R}^{\beta\eta}$.

- Either $R \in \mathcal{R}^\beta$, so $R = (\lambda x.M)N$. By (BC), $x \neq c$. Since $R \in \Lambda\eta_c$ by lemma 2.4, $(\lambda x.M), N \in \Lambda\eta_c$ and again by lemma 2.4, $M \in \Lambda\eta_c$. $|R|^c = (\lambda x.|M|^c)|N|^c \in \mathcal{R}^{\beta\eta}$. $|M|^c[x := |N|^c] =^{2.20} |M[x := N]|^c$ is the contractum of $|R|^c$ and $M[x := N]$ is the contractum of R.
- Or $R \in \mathcal{R}^\beta$, so $R = \lambda x.Mx$ such that $x \notin FV(M)$. By (BC), $x \neq c$. Since $R \in \Lambda\eta_c$, by lemma 2.4, $M, Mx \in \Lambda\eta_c$ and again by lemma 2.4, $M \neq c$. By lemma 2.21, $x \notin FV(|M|^c)$, so $|R|^c = \lambda x.|M|^c x \in \mathcal{R}^{\beta\eta}$. Hence, $|M|^c$ is the contractum of $|R|^c$ and M is the contractum of R.

□

Lemma 2.24. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$ then by lemma 2.5, $\mathcal{R}^r_M = \varnothing$.
- Let $M = \lambda y.P \in \Lambda\text{I}$. Let $C \in \mathcal{R}^{\beta I}_M$ then $\exists R \in \mathcal{R}^{\beta I}$ such that $M = C[R]$. Since $M \notin \mathcal{R}^{\beta I}$, by lemma 2.5, $C = \lambda y.C'$ such that $C' \in \mathcal{R}^{\beta I}_P$. So, $\lambda y.P = \lambda y.C'[R]$ and $P = C'[R]$. By IH, $|P|^c = |C'|^c_{\mathcal{C}}[|R|^c]$. Hence, $|M|^c = \lambda y.|P|^c = \lambda y.|C'|^c_{\mathcal{C}}[|R|^c] = (\lambda y.|C'|^c_{\mathcal{C}})[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.
- Let $M = \lambda y.P[y := c(cy)] \in \Lambda\eta_c$ such that $P \in \Lambda\eta_c$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then by definition, $\exists R \in \mathcal{R}^{\beta\eta}$ such that $M = C[R]$. By lemma 2.9.3, $C = \lambda y.C'$ and $C' \in \mathcal{R}^{\beta\eta}_{P[y:=c(cy)]}$. By lemma 2.9.4, $C' = C''[y := c(cy)]$ and $C'' \in \mathcal{R}^{\beta\eta}_P$. Since $y \notin FV(R)$, $\lambda y.P[y := c(cy)] = \lambda y.C''[y := c(cy)][R] =^{2.8} \lambda y.C''[R][y := c(cy)]$ and $P = C''[R]$. By IH, $|P|^c = |C''|^c_{\mathcal{C}}[|R|^c]$. Hence, $|M|^c = \lambda y.|P[y := c(cy)]|^c =^{2.18} \lambda y.|P|^c =^{IH} \lambda y.|C''|^c_{\mathcal{C}}[|R|^c] =^{2.19} \lambda y.|C''[y := c(cy)]|^c_{\mathcal{C}}[|R|^c] = (\lambda y.|C''[y := c(cy)]|^c_{\mathcal{C}})[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.
- Let $M = \lambda y.Py \in \Lambda\eta_c$ such that $Py \in \Lambda\eta_c$, $y \notin FV(P)$ and $c \neq N$. let $C \in \mathcal{R}^{\beta\eta}_M$ then by definition, $\exists R \in \mathcal{R}^{\beta\eta}$ such that $M = C[R]$. Since $M \in \mathcal{R}^{\beta\eta}$, by lemma 2.5:
  - Either $C = \Box$ then $M = C[R] = \Box[R] = R$ and $|M|^c = \Box[|M|^c] = |\Box|^c_{\mathcal{C}}[|M|^c] = |\Box|^c_{\mathcal{C}}[|R|^c]$.
  - Or $C = \lambda y.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{Py}$. So, $\lambda y.Py = \lambda y.C'[R]$ and $Py = C'[R]$. Hence, $|M|^c = \lambda y.|Py|^c =^{IH} \lambda y.|C'|^c_{\mathcal{C}}[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.
- Let $M = cPQ \in \mathcal{M}_c$ such that $P, Q \in \mathcal{M}_c$. let $C \in \mathcal{R}^r_M$ then by definition, $\exists R \in \mathcal{R}^R$ such that $M = C[R]$. Since $M, cP \notin \mathcal{R}^r$, by lemma 2.5:
  - Either $C = cC'Q$ such that $C' \in \mathcal{R}^r_P$. So, $cPQ = cC'[R]Q$ and $P = C'[R]$. Hence, $|M|^c = |P|^c|Q|^c =^{IH} |C'|^c_{\mathcal{C}}[|R|^c]|Q|^c = (|C'|^c_{\mathcal{C}}|Q|^c)[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.
  - Or $C = cPC'$ such that $C' \in \mathcal{R}^r_Q$. So, $cPQ = cPC'[R]$ and $Q = C'[R]$. Hence, $|M|^c = |P|^c|Q|^c =^{IH} |P|^c|C'|^c_{\mathcal{C}}[|R|^c] = (|P|^c|C'|^c_{\mathcal{C}})[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.
- Let $M = (\lambda y.P)Q \in \mathcal{M}_c$ such that $\lambda y.P, Q \in \mathcal{M}_c$. Let $C \in \mathcal{R}^r_M$ then by definition, $\exists R \in \mathcal{R}^r$ such that $M = C[R]$. Since by lemma 2.11, $M \in \mathcal{R}^r$, by lemma 2.5:
  - Either $C = \Box$ then $M = C[R] = \Box[R] = R$ and $|M|^c = \Box[|M|^c] = |\Box|^c_{\mathcal{C}}[|M|^c] = |\Box|^c_{\mathcal{C}}[|R|^c]$.
  - Or $C = C'Q$ such that $C' \in \mathcal{R}^r_{\lambda y.P}$. So, $(\lambda y.P)Q = C'[R]Q$ and $\lambda y.P = C'[R]$. Hence,
    $|M|^c = |\lambda y.P|^c|Q|^c =^{IH} |C'|^c_{\mathcal{C}}[|R|^c]|Q|^c = (|C'|^c_{\mathcal{C}}|Q|^c)[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.

– Or $C = (\lambda y.P)C'$ such that $C' \in \mathcal{R}^r_Q$. So, $(\lambda y.P)Q = (\lambda y.P)C'[R]$ and $Q = C'[R]$. Now, $|M|^c = |\lambda y.P|^c|Q|^c =^{IH}$ $|\lambda y.P|^c|C'|^c_{\mathcal{C}}[|R|^c] = (|\lambda y.P|^c|C'|^c_{\mathcal{C}})[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$.

- Let $M = cP \in \Lambda\eta_c$ such that $N \in \Lambda\eta_c$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then $\exists R \in \mathcal{R}^{\beta\eta}$ such that $M = C[R]$. Since $M \notin \mathcal{R}^{\beta\eta}$, by lemma 2.5, $C = cC'$ such that $C' \in \mathcal{R}^{\beta\eta}_P$. $cP = cC'[R]$ and $P = C'[R]$. $|M|^c = |P|^c =^{IH} |C'|^c_{\mathcal{C}}[|R|^c] = |C|^c_{\mathcal{C}}[|R|^c]$. □

Lemma 2.25. Since $C \in \mathcal{R}^r_M$, then by definition, $\exists R \in \mathcal{R}^r$ such that $C[R] = M$. By lemma 2.11, $R \in \mathcal{M}_c$. By lemma 2.23, $|R|^c \in \mathcal{R}^r$. By lemma 2.24, $|M|^c = |C|^c_{\mathcal{C}}[|R|^c]$. So by definition, $|C|^c_{\mathcal{C}} \in \mathcal{R}^r_M$ and $|C|^c_{\mathcal{C}}[|R|^c] \overset{|C|^c_{\mathcal{C}}}{\to}_r |C|^c_{\mathcal{C}}[R'']$ such that R'' is the contractum of $|R|^c$. So, by lemma 2.23, $R'' = |R'|^c$ and R' is the contractum of R. By lemma 2.24, $|C|^c_{\mathcal{C}}[|R'|^c] = |C[R']|^c$. □

Lemma 2.26. Let $C \in \mathcal{R}^r_M$, then by definition, $\exists R \in \mathcal{R}^r$ such that $M = C[R]$. So $M' = C[R']$ such that R' is the contractum of R. By lemma 2.25, $|M|^c = |C[R]|^c \overset{|C|^c_{\mathcal{C}}}{\to}_r |C[R']|^c = |M'|^c$. □

Lemma 2.27. By (BC), $x \neq c$. The proof is by induction on the structure of M_1.

- Let $M_1 \in \mathcal{V}$. Then $M_1 = |M_1|^c = |M_2|^c = M_2$.
  - Either $M_1 = x$, then $M_1[x := N_1] = N_1$ and $M_2[x := N_2] = N_2$. By hypothesis $|\mathcal{R}^r_{N_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{N_2}|^c_{\mathcal{C}}$
  - Or $M_1 = y \neq x$ then $M_1[x := N_1] = y = M_2[x := N_2]$.
- Let $M_1 = \lambda y.M_1' \in \Lambda\mathrm{I}_c$ then $|M_1|^c = \lambda y.M_1' = |M_2|^c$. By lemma 2.16 and since $M_2 \in \Lambda\mathrm{I}_c$, $M_2 = \lambda y.M_2'$ such that $|M_2'|^c = |M_1'|^c$. Since $M_1, M_2 \in \Lambda\mathrm{I}_c$ and are λ-abstractions, $M_1N_1, M_2N_2 \in \Lambda\mathrm{I}_c$. Since $|M_1|^c = \lambda y.|M_1'|^c = \lambda y.|M_2'|^c = |M_2|^c$, $|M_1'|^c = |M_2'|^c$. By lemma 2.5, $\mathcal{R}^{\beta I}_{M_1} = \{\lambda y.C \mid C \in \mathcal{R}^{\beta I}_{M_1'}\}$ and $\mathcal{R}^{\beta I}_{M_2} = \{\lambda y.C \mid C \in \mathcal{R}^{\beta I}_{M_2'}\}$. So, $|\mathcal{R}^{\beta I}_{M_1}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta I}_{M_1'}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta I}_{M_2}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta I}_{M_2'}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta I}_{M_1'}|^c_{\mathcal{C}}$, then $\lambda y.C \in |\mathcal{R}^{\beta I}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^{\beta I}_{M_2'}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta I}_{M_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{M_2'}|^c_{\mathcal{C}}$. By IH, $|\mathcal{R}^{\beta I}_{M_1'[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}$. Since $M_1[x := N_1] = \lambda y.M_1'[x := N_1]$ and $M_2[x := N_2] = \lambda y.M_2'[x := N_2]$, by lemma 2.5, $\mathcal{R}^{\beta I}_{M_1[x:=N_1]} = \{\lambda y.C \mid C \in \mathcal{R}^{\beta I}_{M_1'[x:=N_1]}\}$ and $\mathcal{R}^{\beta I}_{M_2[x:=N_2]} = \{\lambda y.C \mid C \in \mathcal{R}^{\beta I}_{M_2'[x:=N_2]}\}$. So $|\mathcal{R}^{\beta I}_{M_1[x:=N_1]}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta I}_{M_1'[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta I}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta I}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta I}_{M_1[x:=N_1]}|^c_{\mathcal{C}}$ then $C = \lambda y.C'$ such that $C' \in |\mathcal{R}^{\beta I}_{M_1'[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^{\beta I}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
- Let $M_1 = \lambda y.M_1'[y := c(cy)] \in \Lambda\eta_c$ such that $M_1' \in \Lambda\eta_c$, then $|M_1|^c =^{2.18} \lambda y.|M_1'|^c$. We prove the statement by induction on the structure of M_2.
  - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq \lambda y.|M_1'|^c$.
  - Let $M_2 = \lambda y.M_2'[y := c(cy)]$ such that $M_2' \in \Lambda\eta_c$, so $M_1N_1, M_2N_2 \in \Lambda\eta_c$. Since $|M_1|^c = \lambda y.|M_1'[y := c(cy)]|^c = \lambda y.|M_2'[y := c(cy)]|^c = |M_2|^c$, $|M_1'[y := c(cy)]|^c = |M_2'[y := c(cy)]|^c$. $\mathcal{R}^{\beta\eta}_{M_1} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}\}$. $\mathcal{R}^{\beta\eta}_{M_2} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\}$.
    So $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}|^c_{\mathcal{C}}$ then $\lambda y.C \in rdbeEM_1 \subseteq$

$|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}|^c_{\mathcal{C}}$.
By IH, $|\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)][x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)][x:=N_2]}|^c_{\mathcal{C}}$.
Since $M_1[x := N_1] = \lambda y.M_1'[y := c(cy)][x := N_1] = \lambda y.M_1'[x := N_1][y := c(cy)]$ and $M_2[x := N_2] = \lambda y.M_2'[y := c(cy)][x := N_2] = \lambda y.M_2'[x := N_2][y := c(cy)]$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)][x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)][x:=N_2]}\}$. So $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)][x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)][x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta I}_{M_1[x:=N_1]}|^c_{\mathcal{C}}$ then $C = \lambda y.C' \in |\mathcal{R}^{\beta I}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$ and $C' \in |\mathcal{R}^{\beta I}_{M_1'[y:=c(cy)][x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{M_2'[y:=c(cy)][x:=N_2]}|^c_{\mathcal{C}}$.

- Let $M_2 = \lambda y.M_2'y$ such that $M_2'y \in \Lambda\eta_c$, $y \notin FV(M_2')$ and $M_2' \neq c$, so $M_1N_1, M_2N_2 \in \Lambda\eta_c$. Since $|M_1|^c = \lambda y.|M_1'[y := c(cy)]|^c = \lambda y.|M_2'y|^c = |M_2|^c$, $|M_1'[y := c(cy)]|^c = |M_2'y|^c$. $\mathcal{R}^{\beta\eta}_{M_1} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}\}$.
  Since $M_2 \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_2'y}\}$. So $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_2'y}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}|^c_{\mathcal{C}}$ then $\lambda y.C \in rdbeEM_1 \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{M_2'y}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2'y}|^c_{\mathcal{C}}$.
  By IH, $|\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)][x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}|^c_{\mathcal{C}}$. Since $M_1[x := N_1] = \lambda y.M_1'[y := c(cy)][x := N_1] = \lambda y.M_1'[x := N_1][y := c(cy)]$, $M_2[x := N_2] = \lambda y.(M_2'y)[x := N_2] = \lambda y.M_2'[x := N_2]y$ and $y \notin FV(N_2)$, we have $M_2[x := N_2] \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)][x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\}$.
  So $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}} = \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)][x:=N_1]}|^c_{\mathcal{C}}\}$ and
  $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}|^c_{\mathcal{C}}\}$.
  Let $C \in |\mathcal{R}^{\beta I}_{M_1[x:=N_1]}|^c_{\mathcal{C}}$ then $C = \lambda y.C'$ such that
  $C' \in |\mathcal{R}^{\beta I}_{M_1'[y:=c(cy)][x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{(M_2'y)[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^{\beta I}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
- Let $M_2 = cP_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$, then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda y.|M_1'|^c$.
- Let $M_2 = P_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$ and P_2 is a λ-abstraction, then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda y.|M_1'|^c$.
- Let $M_2 = cM_2'$ such that $M_2' \in \Lambda\eta_c$. So $|M_2|^c = |M_2'|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$. Again by lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \mathcal{R}^{\beta\eta}_{cM_2'[x:=N_2]} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\}$, so $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}$. Since $(\lambda x.M_2')N_2 \in \Lambda\eta_c$, $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

- Let $M_1 = \lambda y.M_1'y \in \Lambda\eta_c$ such that $M_1'y \in \Lambda\eta_c$, $M_1' \neq c$ and $y \notin FV(M_1')$, then $|M_1|^c = \lambda y.|M_1'y|^c$. We prove the statement by induction on the structure of M_2.
  - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq \lambda y.|M_1'y|^c$.
  - Let $M_2 = \lambda y.M_2'[y := c(cy)]$ such that $M_2' \in \Lambda\eta_c$. Since $M_1 \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_1'[y:=c(cy)]}\}$. Moreover,
    $\mathcal{R}^{\beta\eta}_{M_2} =^{2.9.3} \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[y:=c(cy)]}\}$, so $\Box \in \mathcal{R}^{\beta\eta}_{M_1}$ but $\Box \notin \mathcal{R}^{\beta\eta}_{M_2}$.
  - Let $M_2 = \lambda y.M_2'y$ such that $M_2'y \in \Lambda\eta_c$, $y \notin FV(M_2')$ and $M_2' \neq c$, so $M_1N_1, M_2N_2 \in \Lambda\eta_c$. Since $|M_1|^c = \lambda y.|M_1'y|^c = \lambda y.|M_2'y|^c = |M_2|^c$,

$|M_1'y|^c = |M_2'y|^c$. Since $M_1, M_2 \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M_1} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_1'y}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{M_2'y}\}$. So $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_1'y}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{M_2'y}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{M_1'y}|^c_{\mathcal{C}}$ then $\lambda y.C \in rdbeEM_1 \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{M_2'y}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{M_1'y}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2'y}|^c_{\mathcal{C}}$. By IH, $|\mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}|^c_{\mathcal{C}}$. Since $M_1[x := N_1] = \lambda y.(M_1'y)[x := N_1] = \lambda y.M_1'[x := N_1]y$, $M_2[x := N_2] = \lambda y.(M_2'y)[x := N_2] = \lambda y.M_2'[x := N_2]y$ and $y \notin FV(N_1) \cup FV(N_2)$, we have $M_1[x := N_1], M_2[x := N_2] \in \mathcal{R}^{\beta\eta}$, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}\}$ and $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \{\Box\} \cup \{\lambda y.C \mid C \in \mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}\}$. So $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{(M_1'y)[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda y.C \mid C \in |\mathcal{R}^{\beta\eta}_{(M_2'y)[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta I}_{M_1[x:=N_1]}|^c_{\mathcal{C}}$ then either $C = \Box \in |\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$ or $C = \lambda y.C'$ such that $C' \in |\mathcal{R}^{\beta I}_{(M_1'y)[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{(M_2'y)[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^{\beta I}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

  - Let $M_2 = cP_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$, then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda y.|M_1'y|^c$.
  - Let $M_2 = P_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$ and P_2 is a λ-abstraction, then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda y.|M_1'y|^c$.
  - Let $M_2 = cM_2'$ such that $M_2' \in \Lambda\eta_c$. So $|M_2|^c = |M_2'|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$. Again by lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \mathcal{R}^{\beta\eta}_{cM_2'[x:=N_2]} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\}$, so $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}$. Since $(\lambda x.M_2')N_2 \in \Lambda\eta_c$, $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

- Let $M_1 = cP_1Q_1$ then $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. $M_1 \notin \mathcal{R}^r$. We prove the statement by induction on the structure of M_2:
  - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = \lambda y.M_2' \in \Lambda I_c$ then $|M_2|^c = \lambda y.|M_2'|^c \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = \lambda y.M_2'[x := c(cx)] \in \Lambda\eta_c$ then $|M_2|^c = \lambda y.|M_2'[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = \lambda y.M_2'y \in \Lambda\eta_c$ then $|M_2|^c = \lambda y.|M_2'y|^c \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = P_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$ and P_2 is a λ-abstraction, then $|P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. By lemma 2.10, since $M_2 \in \mathcal{M}_c$, $M_2 \in \mathcal{R}^r$. By lemma 2.4.8, $M_2[x := N_2] \in \mathcal{M}_c$ and by lemma 2.10, $M_2[x := N_2] \in \mathcal{R}^r$. By lemma 2.5, $\mathcal{R}^r_{M_1} = \{cCQ_1 \mid C \in \mathcal{R}^{P_1}_r\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^{M_2}_r = \{\Box\} \cup \{CQ_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{P_2C \mid C \in \mathcal{R}^r_{Q_2}\}$. So $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^cC \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^cC \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}$ then $C|Q_1|^c = C|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. Let $C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}$ then $|P_1|^cC = |P_2|^cC \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. Since $x \in FV(M_1)$:
    - ∗ Either $x \in FV(P_1)$ and $x \in FV(Q_1)$. By lemma 2.21, $x \in FV(P_2)$ and $x \in FV(Q_2)$. Since $P_1, Q_1, P_2, Q_2 \in \mathcal{M}_c$ then $(\lambda x.P_1)N_1, (\lambda x.Q_1)N_1, (\lambda x.P_2)N_2, (\lambda x.Q_2)N_2 \in \mathcal{M}_c$. Hence, by IH, $|\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$ and $|\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|P_1[x := N_1]|^c = |P_1|^c[x := |N_1|^c] = |P_2|^c[x := |N_2|^c] = |P_2[x := N_2]|^c$ and $|Q_1[x := N_1]|^c = |Q_1|^c[x := |N_1|^c] =$

$|Q_2|^c[x := |N_2|^c] = |Q_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{cP_1[x:=N_1]Q_1[x:=N_1]} = \{cCQ_1[x := N_1] \mid C \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{cP_1[x := N_1]C \mid C \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{P_2[x:=N_2]Q_2[x:=N_2]} = \{\Box\} \cup \{CQ_2[x := N_2] \mid C \in \mathcal{R}^r_{P_2[x:=N_2]}\} \cup \{P_2[x := N_2]C \mid C \in \mathcal{R}^r_{Q_2[x:=N_2]}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_1[x:=N_1]Q_1[x:=N_1]}|^c_{\mathcal{C}} = \{C|Q_1[x := N_1]|^c \mid C \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}}\} \cup \{|P_1[x := N_1]|^c C \mid C \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_2[x:=N_2]Q_2[x:=N_2]}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2[x := N_2]|^c \mid C \in |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}\} \cup \{|P_2[x := N_2]|^c C \mid C \in |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:

  - Either $C = C'|Q_1[x := N_1]|^c = C'|Q_2[x := N_2]|^c$ such that $C' \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
  - Or $C = |P_1[x := N_1]|^c C' = |P_2[x := N_2]|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

* Or $x \in FV(P_1)$ and $x \notin FV(Q_1)$. By lemma 2.21, $x \in FV(P_2)$ and $x \notin FV(Q_2)$. Since $P_1, P_2 \in \mathcal{M}_c$ then $(\lambda x.P_1)N_1, (\lambda x.P_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|P_1[x := N_1]|^c = |P_1|^c[x := |N_1|^c] = |P_2|^c[x := |N_2|^c] = |P_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{cP_1[x:=N_1]Q_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{cP_1[x := N_1]C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{P_2[x:=N_2]Q_2} = \{\Box\} \cup \{CQ_2 \mid C \in \mathcal{R}^r_{P_2[x:=N_2]}\} \cup \{P_2[x := N_2]C \mid C \in \mathcal{R}^r_{Q_2}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_1[x:=N_1]Q_1}|^c_{\mathcal{C}} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}}\} \cup \{|P_1[x := N_1]|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_2[x:=N_2]Q_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}\} \cup \{|P_2[x := N_2]|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:
  - Either $C = C'|Q_1|^c = C'|Q_2|^c$ such that $C' \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
  - Or $C = |P_1[x := N_1]|^c C' = |P_2[x := N_2]|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

* Or $x \notin FV(P_1)$ and $x \in FV(Q_1)$. By lemma 2.21, $x \notin FV(P_2)$ and $x \in FV(Q_2)$. Since $Q_1, Q_2 \in \mathcal{M}_c$ then $(\lambda x.Q_1)N_1, (\lambda x.Q_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|Q_1[x := N_1]|^c = |Q_1|^c[x := |N_1|^c] = |Q_2|^c[x := |N_2|^c] = |Q_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{cP_1Q_1[x:=N_1]} = \{cCQ_1[x := N_1] \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{P_2Q_2[x:=N_2]} = \{\Box\} \cup \{CQ_2[x := N_2] \mid C \in \mathcal{R}^r_{P_2}\} \cup \{P_2C \mid C \in \mathcal{R}^r_{Q_2[x:=N_2]}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_1Q_1[x:=N_1]}|^c_{\mathcal{C}} = \{C|Q_1[x := N_1]|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^c C \mid C \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_2Q_2[x:=N_2]}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2[x := N_2]|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:
  - Either $C = C'|Q_1[x := N_1]|^c = C'|Q_2[x := N_2]|^c$ such that $C' \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
  - Or $C = |P_1|^c C' = |P_2|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

– Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$, then $|cP_2|^c = |P_2|^c = |P_1|^c$ and $|Q_1|^c = |Q_2|^c$. Since $M_2 \notin \mathcal{R}^r$, by lemma 2.5, $\mathcal{R}^r_{M_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2} = \{cCQ_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{cP_2C \mid C \in \mathcal{R}^r_{Q_2}\}$. So $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup$

$\{|P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}$ then $C|Q_1|^c = C|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. Let $C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}$ then $|P_1|^c C = |P_2|^c C \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. Since $x \in FV(M_1)$:

- ∗ Either $x \in FV(P_1)$ and $x \in FV(Q_1)$. By lemma 2.21, $x \in FV(P_2)$ and $x \in FV(Q_2)$. Since $P_1, Q_1, P_2, Q_2 \in \mathcal{M}_c$ then $(\lambda x.P_1)N_1, (\lambda x.Q_1)N_1, (\lambda x.P_2)N_2, (\lambda x.Q_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$ and $|\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20,
  $|P_1[x := N_1]|^c = |P_1|^c[x := |N_1|^c] = |P_2|^c[x := |N_2|^c] = |P_2[x := N_2]|^c$ and $|Q_1[x := N_1]|^c = |Q_1|^c[x := |N_1|^c] = |Q_2|^c[x := |N_2|^c] = |Q_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{cP_1[x:=N_1]Q_1[x:=N_1]} =$
  $\{cCQ_1[x := N_1] \mid C \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup$
  $\{cP_1[x := N_1]C \mid C \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and
  $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{cP_2[x:=N_2]Q_2[x:=N_2]} =$
  $\{cCQ_2[x := N_2] \mid C \in \mathcal{R}^r_{P_2[x:=N_2]}\} \cup$
  $\{cP_2[x := N_2]C \mid C \in \mathcal{R}^r_{Q_2[x:=N_2]}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$
  $= |\mathcal{R}^r_{cP_1[x:=N_1]Q_1[x:=N_1]}|^c_{\mathcal{C}} = \{C|Q_1[x := N_1]|^c \mid C \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}}\} \cup$
  $\{|P_1[x := N_1]|^c C \mid C \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}}\}$ and
  $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_2[x:=N_2]Q_2[x:=N_2]}|^c_{\mathcal{C}} = \{C|Q_2[x := N_2]|^c \mid C \in |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}\} \cup \{|P_2[x := N_2]|^c C \mid C \in |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:
  - · Either $C = C'|Q_1[x := N_1]|^c = C'|Q_2[x := N_2]|^c$ such that $C' \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
  - · Or $C = |P_1[x := N_1]|^c C' = |P_2[x := N_2]|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
- ∗ Or $x \in FV(P_1)$ and $x \notin FV(Q_1)$. By lemma 2.21, $x \in FV(P_2)$ and $x \notin FV(Q_2)$. Since $P_1, P_2 \in \mathcal{M}_c$ then $(\lambda x.P_1)N_1, (\lambda x.P_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|P_1[x := N_1]|^c = |P_1|^c[x := |N_1|^c] = |P_2|^c[x := |N_2|^c] = |P_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{cP_1[x:=N_1]Q_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{cP_1[x := N_1]C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{cP_2[x:=N_2]Q_2} = \{cCQ_2 \mid C \in \mathcal{R}^r_{P_2[x:=N_2]}\} \cup \{cP_2[x := N_2]C \mid C \in \mathcal{R}^r_{Q_2}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_1[x:=N_1]Q_1}|^c_{\mathcal{C}} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}}\} \cup \{|P_1[x := N_1]|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_2[x:=N_2]Q_2}|^c_{\mathcal{C}}$ $= \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}\} \cup \{|P_2[x := N_2]|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:
  - · Either $C = C'|Q_1|^c = C'|Q_2|^c$ such that $C' \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
  - · Or $C = |P_1[x := N_1]|^c C' = |P_2[x := N_2]|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
- ∗ Or $x \notin FV(P_1)$ and $x \in FV(Q_1)$. By lemma 2.21, $x \notin FV(P_2)$ and $x \in FV(Q_2)$. Since $Q_1, Q_2 \in \mathcal{M}_c$ then $(\lambda x.Q_1)N_1, (\lambda x.Q_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20,
  $|Q_1[x := N_1]|^c = |Q_1|^c[x := |N_1|^c] = |Q_2|^c[x := |N_2|^c] = |Q_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{cP_1Q_1[x:=N_1]} = \{cCQ_1[x := N_1] \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{cP_2Q_2[x:=N_2]} = \{cCQ_2[x := N_2] \mid C \in \mathcal{R}^r_{P_2}\} \cup \{cP_2C \mid C \in \mathcal{R}^r_{Q_2[x:=N_2]}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_1Q_1[x:=N_1]}|^c_{\mathcal{C}} = \{C|Q_1[x :=$

$N_1]|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^c C \mid C \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{cP_2Q_2[x:=N_2]}|^c_{\mathcal{C}} = \{cC|Q_2[x := N_2]|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{c|P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:

  - Either $C = C'|Q_1[x := N_1]|^c = C'|Q_2[x := N_2]|^c$ such that $C' \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.
  - Or $C = |P_1|^c C' = |P_2|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

- Let $M_2 = cM_2'$ such that $M_2' \in \Lambda\eta_c$. So $|M_2|^c = |M_2'|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$. Again by lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \mathcal{R}^{\beta\eta}_{cM_2'[x:=N_2]} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\}$, so $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}$. Since $(\lambda x.M_2')N_2 \in \Lambda\eta_c$, $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

- Let $M_1 = P_1Q_1 \in \mathcal{M}_c$ such that $P_1, Q_1 \in \mathcal{M}_c$ and P_1 is a λ-abstraction. Then $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. By lemma 2.10, since $M_1 \in \mathcal{M}_c$, $M_1 \in \mathcal{R}^r$. By lemma 2.4.8, $M_1[x := N_1] \in \mathcal{M}_c$ and by lemma 2.10, $M_1[x := N_1] \in \mathcal{R}^r$. So by lemma 2.5, $\Box \in \mathcal{R}^r_{M_1}$, so $\Box \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}}$. We prove the statement by induction on the structure of M_2.

  - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = \lambda y.M_2' \in \Lambda I_c$ then $|M_2|^c = \lambda y.|M_2'|^c \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = \lambda y.M_2'[y := c(cy)] \in \Lambda\eta_c$ then $|M_2|^c = \lambda y.|M_2'[y := c(cy)]|^c \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = \lambda y.M_2'y \in \Lambda\eta_c$ then $|M_2|^c = \lambda y.|M_2'y|^c \neq |P_1|^c|Q_1|^c$.
  - Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$, so $M_2 \notin \mathcal{R}^r$. Hence, by lemma 2.5, $\Box \notin \mathcal{R}^r_{M_2}$, so $\Box \notin |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \not\subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$.
  - Let $M_2 = P_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$ and P_2 is a λ-abstraction, then $|P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. By lemma 2.10, since $M_2 \in \mathcal{M}_c$, $M_2 \in \mathcal{R}^r$. By lemma 2.4.8, $M_2[x := N_2] \in \mathcal{M}_c$ and by lemma 2.10, $M_2[x := N_2] \in \mathcal{R}^r$.By lemma 2.5, $\mathcal{R}^r_{M_1} = \{\Box\} \cup \{CQ_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{P_1C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2} = \{\Box\} \cup \{CQ_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{P_2C \mid C \in \mathcal{R}^r_{Q_2}\}$. So, $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}$ then $C|Q_1|^c = C|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. let $C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}$ then $|P_1|^c C = |P_2|^c C \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So, $C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. Since $x \in FV(M_1)$:
    * Either $x \in FV(P_1)$ and $x \in FV(Q_1)$. By lemma 2.21, $x \in FV(P_2)$ and $x \in FV(Q_2)$. Since $P_1, Q_1, P_2, Q_2 \in \mathcal{M}_c$ then $(\lambda x.P_1)N_1, (\lambda x.Q_1)N_1, (\lambda x.P_2)N_2, (\lambda x.Q_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$ and $|\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|P_1[x := N_1]|^c = |P_1|^c[x := |N_1|^c] = |P_2|^c[x := |N_2|^c] = |P_2[x := N_2]|^c$ and $|Q_1[x := N_1]|^c = |Q_1|^c[x := |N_1|^c] = |Q_2|^c[x := |N_2|^c] = |Q_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{P_1[x:=N_1]Q_1[x:=N_1]} = \{\Box\} \cup \{CQ_1[x := N_1] \mid C \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{P_1[x := N_1]C \mid C \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{P_2[x:=N_2]Q_2[x:=N_2]} = \{\Box\} \cup \{CQ_2[x := N_2] \mid C \in \mathcal{R}^r_{P_2[x:=N_2]}\} \cup \{P_2[x := N_2]C \mid C \in \mathcal{R}^r_{Q_2[x:=N_2]}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_1[x:=N_1]Q_1[x:=N_1]}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_1[x := N_1]|^c \mid C \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}}\} \cup \{|P_1[x := N_1]|^c C \mid C \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} =$

$|\mathcal{R}^r_{P_2[x:=N_2]Q_2[x:=N_2]}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2[x := N_2]|^c \mid C \in |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}\}$ $\cup \{|P_2[x := N_2]|^c C \mid C \in |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:

· Either $C = \Box \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

· Or $C = C'|Q_1[x := N_1]|^c = C'|Q_2[x := N_2]|^c$ such that $C' \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

· Or $C = |P_1[x := N_1]|^c C' = |P_2[x := N_2]|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

$*$ Or $x \in FV(P_1)$ and $x \notin FV(Q_1)$. By lemma 2.21, $x \in FV(P_2)$ and $x \notin FV(Q_2)$. Since $P_1, P_2 \in \mathcal{M}_c$ then $(\lambda x.P_1)N_1, (\lambda x.P_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|P_1[x := N_1]|^c = |P_1|^c[x := |N_1|^c] = |P_2|^c[x := |N_2|^c] = |P_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{P_1[x:=N_1]Q_1} = \{\Box\} \cup \{CQ_1 \mid C \in \mathcal{R}^r_{P_1[x:=N_1]}\} \cup \{P_1[x := N_1]C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{P_2[x:=N_2]Q_2} = \{\Box\} \cup \{CQ_2 \mid C \in \mathcal{R}^r_{P_2[x:=N_2]}\} \cup \{P_2[x := N_2]C \mid C \in \mathcal{R}^r_{Q_2}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_1[x:=N_1]Q_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}}\} \cup \{|P_1[x := N_1]|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_2[x:=N_2]Q_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}\} \cup \{|P_2[x := N_2]|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:

· Either $C = \Box \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

· Or $C = C'|Q_1|^c = C'|Q_2|^c$ such that $C' \in |\mathcal{R}^r_{P_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

· Or $C = |P_1[x := N_1]|^c C' = |P_2[x := N_2]|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

$*$ Or $x \notin FV(P_1)$ and $x \in FV(Q_1)$. By lemma 2.21, $x \notin FV(P_2)$ and $x \in FV(Q_2)$. Since $Q_1, Q_2 \in \mathcal{M}_c$ then $(\lambda x.Q_1)N_1, (\lambda x.Q_2)N_2 \in \mathcal{M}_c$. So by IH, $|\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. By lemma 2.20, $|Q_1[x := N_1]|^c = |Q_1|^c[x := |N_1|^c] = |Q_2|^c[x := |N_2|^c] = |Q_2[x := N_2]|^c$. By lemma 2.5, $\mathcal{R}^r_{M_1[x:=N_1]} = \mathcal{R}^r_{P_1Q_1[x:=N_1]} = \{\Box\} \cup \{CQ_1[x := N_1] \mid C \in \mathcal{R}^r_{P_1}\} \cup \{P_1C \mid C \in \mathcal{R}^r_{Q_1[x:=N_1]}\}$ and $\mathcal{R}^r_{M_2[x:=N_2]} = \mathcal{R}^r_{P_2Q_2[x:=N_2]} = \{\Box\} \cup \{CQ_2[x := N_2] \mid C \in \mathcal{R}^r_{P_2}\} \cup \{P_2C \mid C \in \mathcal{R}^r_{Q_2[x:=N_2]}\}$. So, $|\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_1Q_1[x:=N_1]}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_1[x := N_1]|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^c C \mid C \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^r_{P_2Q_2[x:=N_2]}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2[x := N_2]|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M_1[x:=N_1]}|^c_{\mathcal{C}}$:

· Either $C = \Box \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

· Or $C = C'|Q_1[x := N_1]|^c = C'|Q_2[x := N_2]|^c$ such that $C' \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

· Or $C = |P_1|^c C' = |P_2|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1[x:=N_1]}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2[x:=N_2]}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

– Let $M_2 = cM_2'$ such that $M_2' \in \Lambda\eta_c$. So $|M_2|^c = |M_2'|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$. Again by lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]} = \mathcal{R}^{\beta\eta}_{cM_2'[x:=N_2]} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}\}$, so $|\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}}$. Since $(\lambda x.M_2')N_2 \in \Lambda\eta_c$, $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^c_{\mathcal{C}}$ $\subseteq^{IH} |\mathcal{R}^{\beta\eta}_{M_2'[x:=N_2]}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^c_{\mathcal{C}}$.

- Let $M_1 = cM_1' \in \Lambda\eta_c$ such that $M_1' \in \Lambda\eta_c$. So $|M_1|^c = |M_1'|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_1} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M_1'}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$. Again by lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]} = \mathcal{R}^{\beta\eta}_{cM'_1[x:=N_1]} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{M'_1[x:=N_1]}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^{c}_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M'_1[x:=N_1]}|^{c}_{\mathcal{C}}$. Since $(\lambda x.M'_1)N_1 \in \Lambda\eta_c$, $|\mathcal{R}^{\beta\eta}_{M_1[x:=N_1]}|^{c}_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M'_1[x:=N_1]}|^{c}_{\mathcal{C}} =\subseteq^{IH} |\mathcal{R}^{\beta\eta}_{M_2[x:=N_2]}|^{c}_{\mathcal{C}}$. □

Lemma 2.28. Since $M_1 \overset{C_1}{\to}_r M'_1$, $C_1 \in \mathcal{R}^{r}_{M_1}$ and $\exists R_1 \in \mathcal{R}^r$ such that $M_1 = C_1[R_1]$. So $M'_1 = C_1[R'_1]$ such that R'_1 is the contractum of R_1. Since $M_2 \overset{C_2}{\to}_r M'_2$, $C_2 \in \mathcal{R}^{r}_{M_2}$ and $\exists R_2 \in \mathcal{R}^r$ such that $M_2 = C_2[R_2]$. So $M'_2 = C_2[R'_2]$ such that R'_2 is the contractum of R_2. We prove this lemma by induction on the structure of M_1.

1. Let $M_1 \in \mathcal{V} \setminus \{c\}$ then nothing to prove since M_1 does not reduce.

2. Let $M_1 = \lambda x.N_1 \in \Lambda I_c$. So $|M_1|^c = \lambda x.|N_1|^c = |M_2|^c$. By lemma 2.16, since $M_2 \in \Lambda I_c$ and by lemma 2.4, $M_2 = \lambda x.N_2$ and $|N_2|^c = |N_1|^c$. Since $M_1, M_2 \notin \mathcal{R}^{\beta I}$, by lemma 2.5, $\mathcal{R}^{\beta I}_{M_1} = \{\lambda x.C \mid C \in \mathcal{R}^{\beta I}_{N_1}\}$ and $\mathcal{R}^{\beta I}_{M_2} = \{\lambda x.C \mid C \in \mathcal{R}^{\beta I}_{N_2}\}$ so $|\mathcal{R}^{\beta I}_{M_1}|^{c}_{\mathcal{C}} = \{|\lambda x.C|^{c}_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta I}_{N_1}\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta I}_{N_1}|^{c}_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta I}_{M_2}|^{c}_{\mathcal{C}} = \{|\lambda x.C|^{c}_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta I}_{N_2}\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta I}_{N_2}|^{c}_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta I}_{N_1}|^{c}_{\mathcal{C}}$ then $\lambda x.C \in |\mathcal{R}^{\beta I}_{M_1}|^{c}_{\mathcal{C}}$, so by hypothesis, $\lambda x.C \in |\mathcal{R}^{\beta I}_{M_2}|^{c}_{\mathcal{C}}$. Hence, $C \in |\mathcal{R}^{\beta I}_{N_2}|^{c}_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta I}_{N_1}|^{c}_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{N_2}|^{c}_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^{\beta I}_{M_1}$, $C_1 = \lambda x.C'_1$ such that $C'_1 \in \mathcal{R}^{\beta I}_{N_1}$. Since $C_2 \in \mathcal{R}^{\beta I}_{M_2}$, $C_2 = \lambda x.C'_2$ such that $C'_2 \in \mathcal{R}^{\beta I}_{N_2}$. Since $|C_1|^{c}_{\mathcal{C}} = |C_2|^{c}_{\mathcal{C}}$, $|C'_1|^{c}_{\mathcal{C}} = |C'_2|^{c}_{\mathcal{C}}$. Hence, $M_1 = \lambda x.N_1 = \lambda x.C'_1[R_1] \overset{C_1}{\to}_{\beta I} \lambda x.C'_1[R'_1] = \lambda x.N'_1 = M'_1$, $M_2 = \lambda x.N_2 = \lambda x.C'_2[R_2] \overset{C_2}{\to}_{\beta I} \lambda x.C'_2[R'_2] = \lambda x.N'_2 = M'_2$, $N_1 \overset{C'_1}{\to}_{\beta I} N'_1$ and $N_2 \overset{C'_2}{\to}_{\beta I} N'_2$. By IH, $|\mathcal{R}^{\beta I}_{N'_1}|^{c}_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta I}_{N'_2}|^{c}_{\mathcal{C}}$. By lemma 2.5, $\mathcal{R}^{\beta I}_{M'_1} = \{\lambda x.C \mid C \in \mathcal{R}^{\beta I}_{N'_1}\}$ and $\mathcal{R}^{\beta I}_{M'_2} = \{\lambda x.C \mid C \in \mathcal{R}^{\beta I}_{N'_2}\}$, so $|\mathcal{R}^{\beta I}_{M'_1}|^{c}_{\mathcal{C}} = \{|\lambda x.C|^{c}_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta I}_{N'_1}\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta I}_{N'_1}|^{c}_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta I}_{M'_2}|^{c}_{\mathcal{C}} = \{|\lambda x.C|^{c}_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta I}_{N'_2}\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta I}_{N'_2}|^{c}_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta I}_{M'_1}|^{c}_{\mathcal{C}}$, then $C = \lambda x.C'$ such that $C' \in |\mathcal{R}^{\beta I}_{N'_1}|^{c}_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^{\beta I}_{N'_2}|^{c}_{\mathcal{C}}$, so $\lambda x.C' \in |\mathcal{R}^{\beta I}_{M'_2}|^{c}_{\mathcal{C}}$.

3. Let $M_1 = \lambda x.N_1[x := c(cx)] \in \Lambda\eta_c$ such that $N_1 \in \Lambda\eta_c$ then $|M_1|^c = \lambda x.|N_1[x := c(cx)]|^c =^{2.18} \lambda x.|N_1|^c$. We prove the statement by induction on the structure of M_2:

   - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq \lambda x.|N_1|^c$.
   - Let $M_2 = \lambda x.N_2[x := c(cx)]$ such that $N_2 \in \Lambda\eta_c$. Since $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c =^{2.18} \lambda x.|N_2|^c$, $|N_1|^c = |N_2|^c$. $\mathcal{R}^{\beta\eta}_{M_1} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_1[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.C[x := c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_2[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.C[x := c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$. So, $|\mathcal{R}^{\beta\eta}_{M_1}|^{c}_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_1}|^{c}_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^{c}_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_2}|^{c}_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{N_1}|^{c}_{\mathcal{C}}$ then $\lambda x.C \in |\mathcal{R}^{\beta\eta}_{M_1}|^{c}_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^{c}_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{N_2}|^{c}_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{N_1}|^{c}_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2}|^{c}_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, $C_1 = \lambda x.C'_1[x := c(cx)]$ such that $C'_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, $C_2 = \lambda x.C'_2[x := c(cx)]$ such that $C'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. Since $\lambda x.|C'_1|^{c}_{\mathcal{C}} =^{2.19} |C_1|^{c}_{\mathcal{C}} = |C_2|^{c}_{\mathcal{C}} =^{2.19} \lambda x.|C'_2|^{c}_{\mathcal{C}}$, $|C'_1|^{c}_{\mathcal{C}} = |C'_2|^{c}_{\mathcal{C}}$. So $M_1 = \lambda x.N_1[x := c(cx)] = \lambda x.C'_1[x := c(cx)][R_1] =^{2.8} \lambda x.C'_1[R_1][x := c(cx)] \overset{C_1}{\to}_{\beta\eta} \lambda x.C'_1[R'_1][c := c(cx)] = \lambda x.N'_1[x := c(cx)] = M'_1$, $M_2 = \lambda x.N_2[x := c(cx)] = \lambda x.C'_2[x := c(cx)][R_2] =^{2.8} \lambda x.C'_2[R_2][x := c(cx)] \overset{C_2}{\to}_{\beta\eta} \lambda x.C'_2[R'_2][c := c(cx)] = \lambda x.N'_2[x := c(cx)] = M'_2$, $N_1 = C'_1[R_1] \overset{C'_1}{\to}_{\beta\eta} C'_1[R'_1] = N'_1$ and $N_2 = C'_2[R_2] \overset{C'_2}{\to}_{\beta\eta} C'_2[R'_2] = N'_2$. By IH, $|\mathcal{R}^{\beta\eta}_{N'_1}|^{c}_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^{c}_{\mathcal{C}}$. Hence, $\mathcal{R}^{\beta\eta}_{M'_1} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_1[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.C[x :=$

$c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{M_2'} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_2'[x:=c(cx)]}\}$ $=^{2.9.4} \{\lambda x.C[x := c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. So, $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_1'}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}}$ then $C = \lambda x.C'$ such that $C' \in |\mathcal{R}^{\beta\eta}_{N_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$.

- Let $M_2 = \lambda x.N_2 x$ such that $N_2 x \in \Lambda\eta_c$, $x \notin FV(N_2)$ and $N_2 \neq c$, then $M_2 \in \mathcal{R}^{\beta\eta}$. Since $|M_2|^c = \lambda x.|N_2 x|^c$, $|N_1|^c = |N_2 x|^c$. $\mathcal{R}^{\beta\eta}_{M_1} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_1[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.C[x := c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} =^{2.5} \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_2 x}\}$. So, $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_2 x}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}}$ then $\lambda x.C \in |\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{N_2 x}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2 x}|^c_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, $C_1 = \lambda x.C_1'[x := c(cx)]$ such that $C_1' \in \mathcal{R}^{\beta\eta}_{N_1}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $C_2 = \lambda x.C_2'$ such that $C_2' \in \mathcal{R}^{\beta\eta}_{N_2 x}$. Since $\lambda x.|C_1'|^c_{\mathcal{C}} =^{2.19} |C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}} = \lambda x.|C_2'|^c_{\mathcal{C}}$, $|C_1'|^c_{\mathcal{C}} = |C_2'|^c_{\mathcal{C}}$. So $M_1 = \lambda x.N_1[x := c(cx)] = \lambda x.C_1'[x := c(cx)][R_1] =^{2.8} \lambda x.C_1'[R_1][x := c(cx)] \overset{C_1}{\rightarrow}_{\beta\eta} \lambda x.C_1'[R_1'][c := c(cx)] = \lambda x.N_1'[x := c(cx)] = M_1'$, $M_2 = \lambda x.N_2 x = \lambda x.C_2'[R_2] \overset{C_2}{\rightarrow}_{\beta\eta} \lambda x.C_2'[R_2'] = \lambda x.N_2' = M_2'$, $N_1 = C_1'[R_1] \overset{C_1'}{\rightarrow}_{\beta\eta} C_1'[R_1'] = N_1'$ and $N_2 x = C_2'[R_2] \overset{C_2'}{\rightarrow}_{\beta\eta} C_2'[R_2'] = N_2'$. By IH, $|\mathcal{R}^{\beta\eta}_{N_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}}$. Hence, $\mathcal{R}^{\beta\eta}_{M_1'} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_1'[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.C[x := c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_1'}\}$ and $\mathcal{R}^{\beta\eta}_{M_2'} \setminus \{\Box\} =^{2.5} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_2'}\}$. So, $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_1'}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}} \setminus \{\Box\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}}$ then $C = \lambda x.C'$ such that $C' \in |\mathcal{R}^{\beta\eta}_{N_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}} \setminus \{\Box\}$, i.e. $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$.
- Let $M_2 = cP_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$ then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda x.|N_1|^c$.
- Let $M_2 = P_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$ and P_2 is a λ-abstraction then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda x.|N_1|^c$.
- Let $M_2 = cN_2$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, $C_2 = cC_2'$ such that $C_2' \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 = cC_2'[R_2] \overset{cC_2'}{\rightarrow}_{\beta\eta} cC_2'[R_2'] = cN_2' = M_2'$ and $N_2 = C_2'[R_2] \overset{C_2'}{\rightarrow}_{\beta\eta} C_2'[R_2'] = N_2'$. Since $|C_2'|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}} = |C_1|^c_{\mathcal{C}}$, by IH, $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}}$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2'} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N_2'}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1'}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2'}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_2'}|^c_{\mathcal{C}}$.

4. Let $M_1 = \lambda x.N_1 x \in \Lambda\eta_c$ such that $N_1 x \in \Lambda\eta_c$, $x \notin FV(N_1)$ and $N_1 \neq c$, then $M_1 \in \mathcal{R}^{\beta\eta}$ and $|M_1|^c = \lambda x.|N_1 x|^c = \lambda x.|N_1|^c x$. We prove the statement by induction on the structure of M_2:

   (a) Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq \lambda x.|N_1 x|^c$.

   (b) Let $M_2 = \lambda x.N_2[x := c(cx)]$ such that $N_2 \in \Lambda\eta_c$. $\mathcal{R}^{\beta\eta}_{M_1} =^{2.5} \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_1 x}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} =^{2.9.3} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_2[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.C[x := c(cx)] \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$. So, $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_1 x}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}\}$. Hence, $\Box \in |\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}}$ but $\Box \notin |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$.

(c) Let $M_2 = \lambda x.N_2x$ such that $N_2x \in \Lambda\eta_c$, $x \notin FV(N_2)$ and $N_2 \neq c$, then $M_2 \in \mathcal{R}^{\beta\eta}$. Since $|M_2|^c = \lambda x.|N_2x|^c = \lambda x.|N_2|^c x$, $|N_1x|^c = |N_2x|^c$ and $|N_1|^c = |N_2|^c$. $\mathcal{R}^{\beta\eta}_{M_1} =^{2.5} \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_1x}\}$ and $\mathcal{R}^{\beta\eta}_{M_2} =^{2.5} \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N_2x}\}$. So, $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}}$ then $\lambda x.C \in |\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}}$. Moreover, $\mathcal{R}^{\beta\eta}_{N_1x} \setminus \{\Box\} =^{2.5} \{Cx \mid C \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and $\mathcal{R}^{\beta\eta}_{N_2x} \setminus \{\Box\} =^{2.5} \{Cx \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}} \setminus \{\Box\} = \{Cx \mid C \in |\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}} \setminus \{\Box\} = \{Cx \mid C \in |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}}$ then $Cx \in |\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}} \setminus \{\Box\} \subseteq |\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^{\beta\eta}_{M_1}$:

- Either $C_1 = \Box$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $C_2 = \Box$. So $M_1 \xrightarrow{\Box}_{\beta\eta} N_1$ and $M_2 \xrightarrow{\Box}_{\beta\eta} N_2$. It is done since $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$.
- $C_1 = \lambda x.C'_1$ such that $C'_1 \in \mathcal{R}^{\beta\eta}_{N_1x}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $C_2 = \lambda x.C'_2$ such that $C'_2 \in \mathcal{R}^{\beta\eta}_{N_2x}$. Since $\lambda x.|C'_1|^c_{\mathcal{C}} = |C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}} = \lambda x.|C'_2|^c_{\mathcal{C}}$, $|C'_1|^c_{\mathcal{C}} = |C'_2|^c_{\mathcal{C}}$. So $M_1 = \lambda x.N_1x = \lambda x.C'_1[R_1] \xrightarrow{C_1}_{\beta\eta} \lambda x.C'_1[R'_1] = \lambda x.N'_1 = M'_1$, $M_2 = \lambda x.N_2x = \lambda x.C'_2[R_2] \xrightarrow{C_2}_{\beta\eta} \lambda x.C'_2[R'_2] = \lambda x.N'_2 = M'_2$, $N_1x = C'_1[R_1] \xrightarrow{C'_1}_{\beta\eta} C'_1[R'_1] = N'_1$ and $N_2x = C'_2[R_2] \xrightarrow{C'_2}_{\beta\eta} C'_2[R'_2] = N'_2$. By IH, $|\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$.
  - Either $M'_1 \in \mathcal{R}^{\beta\eta}$, then $M'_1 = \lambda x.Px$ such that $x \notin FV(P)$. We prove the statement by case on the belonging of N_1x in $\mathcal{R}^{\beta\eta}$.
    - ∗ Either $N_1x \in \mathcal{R}^{\beta\eta}$, so by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_1x} = \{\Box\} \cup \{Cx \mid C \in \mathcal{R}^{\beta\eta}_{N_1}\}$ and so $N_1 = \lambda y.P_1$. Since $|\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}}$, $\Box \in \mathcal{R}^{\beta\eta}_{N_2x}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_2x} = \{\Box\} \cup \{Cx \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$ and so $N_2 = \lambda y.P_2$.
      - · Let $C'_1 = \Box$. Since $|C'_1|^c_{\mathcal{C}} = |C'_2|^c_{\mathcal{C}}$, $C'_2 = \Box$. So $M_1 = \lambda x.(\lambda y.P_1)x = \lambda x.\Box[R_1] \xrightarrow{C_1}_{\beta\eta} \lambda x.\Box[R'_1] = \lambda x.P_1[y := x] = M'_1$, $M_2 = \lambda x.(\lambda y.P_2)x = \lambda x.\Box[R_2] \xrightarrow{C_2}_{\beta\eta} \lambda x.\Box[R'_2] = \lambda x.P_2[y := x] = M'_2$. Since $x \notin FV(N_1) \cup FV(N_2)$, $M'_1 = N_1$ and $M'_2 = N_2$. It is done since $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$.
      - · Let $C'_1 = C''_1x$ such that $C''_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Since $|C'_1|^c_{\mathcal{C}} = |C'_2|^c_{\mathcal{C}}$, $C'_2 = C''_2x$ such that $C''_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So $M_1 = \lambda x.N_1x = \lambda x.C''_1[R_1]x \xrightarrow{C_1}_{\beta\eta} \lambda x.C''_1[R'_1]x = \lambda x.N''_1x = \lambda x.N'_1 = M'_1$, $M_2 = \lambda x.N_2x = \lambda x.C''_2[R_2]x \xrightarrow{C_2}_{\beta\eta} \lambda x.C''_2[R'_2]x = \lambda x.N''_2x = \lambda x.N'_2 = M'_2$, $N_1 = C''_1[R_1] \xrightarrow{C''_1}_{\beta\eta} C''_1[R'_1] = N''_1$ and $N_2 = C''_2[R_2] \xrightarrow{C''_2}_{\beta\eta} C''_2[R'_2] = N''_2$. Since $x \notin FV(N_1) \cup FV(N_2)$, by lemma 2.2.1, $x \notin FV(N''_1) \cup FV(N''_2)$. So, $M'_1, M'_2 \in \mathcal{R}^{\beta\eta}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M'_1} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_1}\}$, $\mathcal{R}^{\beta\eta}_{M'_2} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_2}\}$, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}}\}$, $|\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}\}$. Since $|\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}}\} \subseteq \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}\} = |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$.
    - ∗ Else by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_1x} = \{Cx \mid C \in \mathcal{R}^{\beta\eta}_{N_1}\}$. Since $|\mathcal{R}^{\beta\eta}_{N_1x}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N_2x}|^c_{\mathcal{C}}$, $\Box \notin \mathcal{R}^{\beta\eta}_{N_2x}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{N_2x} = \{Cx \mid C \in$

$\mathcal{R}^{\beta\eta}_{N_2}\}$ and so $N_2 = \lambda y.P_2$. Let $C'_1 = C''_1 x$ such that $C''_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. $C'_2 = C''_2 x$ such that $C''_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So $M_1 = \lambda x.N_1 x = \lambda x.C''_1[R_1]x \stackrel{C_1}{\to}_{\beta\eta} \lambda x.C''_1[R'_1]x = \lambda x.N''_1 x = \lambda x.N'_1 = M'_1$, $M_2 = \lambda x.N_2 x = \lambda x.C''_2[R_2]x \stackrel{C_2}{\to}_{\beta\eta} \lambda x.C''_2[R'_2]x = \lambda x.N''_2 x = \lambda x.N'_2 = M'_2$, $N_1 = C''_1[R_1] \stackrel{C''_1}{\to}_{\beta\eta} C''_1[R'_1] = N''_1$ and $N_2 = C''_2[R_2] \stackrel{C''_2}{\to}_{\beta\eta} C''_2[R'_2] = N''_2$. Since $x \notin FV(N_1) \cup FV(N_2)$, by lemma 2.2.1, $x \notin FV(N''_1) \cup FV(N''_2)$. So, $M'_1, M'_2 \in \mathcal{R}^{\beta\eta}$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_{M'_1} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_1}\}$, $\mathcal{R}^{\beta\eta}_{M'_2} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_2}\}$, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}}\}$, $|\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}\}$. Since $|\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}}\} \subseteq \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}\} = |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$.

– Else, $\mathcal{R}^{\beta\eta}_{M'_1} =^{2.5} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_1}\}$ and $\mathcal{R}^{\beta\eta}_{M'_2} \setminus \{\Box\} =^{2.5} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{N'_2}\}$. So, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}} \setminus \{\Box\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}}$ then $C = \lambda x.C'$ such that $C' \in |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$, so $C \in |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}} \setminus \{\Box\}$, i.e. $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$.

(d) Let $M_2 = cP_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$ then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda x.|N_1 x|^c$.

(e) Let $M_2 = P_2Q_2$ such that $P_2, Q_2 \in \Lambda\eta_c$ and P_2 is a λ-abstraction then $|M_2|^c = |P_2|^c|Q_2|^c \neq \lambda x.|N_1 x|^c$.

(f) Let $M_2 = cN_2$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, $C_2 = cC'_2$ such that $C'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 = cC'_2[R_2] \stackrel{cC'_2}{\to}_{\beta\eta} cC'_2[R'_2] = cN'_2 = M'_2$ and $N_2 = C'_2[R_2] \stackrel{C'_2}{\to}_{\beta\eta} C'_2[R'_2] = N'_2$. Since $|C'_2|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}} = |C_1|^c_{\mathcal{C}}$, by IH, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M'_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N'_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$.

5. Let $M_1 = cP_1Q_1 \in \mathcal{M}_c$ such that $P_1, P_2 \in \mathcal{M}_c$. So $|M_1|^c = |P_1|^c|Q_1|^c = |M_2|^c$. We prove the statement by induction on the structure of M_2:

- Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = \lambda x.N_2 \in \Lambda\mathrm{I}$ then $|M_2|^c = \lambda x.|N_2|^c \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = \lambda x.N_2[x := c(cx)] \in \Lambda\eta_c$ then $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = \lambda x.N_2 x \in \Lambda\eta_c$ then $|M_2|^c = \lambda x.|N_2 x|^c \neq |P_1|^c|Q_1|^c$.
- Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$, then $|cP_2|^c = |P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. Since $M_1 \notin \mathcal{R}^r$, by lemma 2.5, $\mathcal{R}^r_{M_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q_1}\}$. So, $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} = \{|cCQ_1|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{P_1}\} \cup \{|cP_1C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{Q_1}\} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$. Again by lemma 2.5, since $M_2 \notin \mathcal{R}^r$, $\mathcal{R}^r_{M_2} = \{cCQ_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{cP_2C \mid C \in \mathcal{R}^r_{Q_2}\}$. So, $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{|cCQ_2|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{P_2}\} \cup \{|cP_2C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{Q_2}\} = \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}$ then $C|Q_1|^c = C|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. Hence, $C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. Let $C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}$ then $|P_1|^c C =$

$|P_2|^c C \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. Hence, $C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^r_{M_1}$:

- Either $C_1 = cC'_1Q_1$ such that $C'_1 \in \mathcal{R}^r_{P_1}$. $|C_1|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}|Q_1|^c = |C'_1|^c_{\mathcal{C}}|Q_2|^c = |C_2|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^r_{M_2}$, $|C_2|^c_{\mathcal{C}} \in |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, so $|C'_1|^c_{\mathcal{C}} \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$ and $C_2 = cC'_2Q_2$ such that $|C'_2|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}$ and $C'_2 \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = cP_1Q_1 = cC'_1[R_1]Q_1 \stackrel{C_1}{\to}_r cC'_1[R'_1]Q_1 = cP'_1Q_1 = M'_1$, $M_2 = cP_2Q_2 = cC'_2[R_2]Q_2 \stackrel{C_2}{\to}_r cC'_2[R'_2]Q_2 = cP'_2Q_2 = M'_2$, $P_1 = C'_1[R_1] \stackrel{C'_1}{\to}_r C'_1[R'_1] = P'_1$ and $P_2C'_2[R_2] \stackrel{C'_2}{\to}_r C'_2[R'_2] = P'_2$. By lemma 2.26, $|M_1|^c = \stackrel{|C_1|^c_{\mathcal{C}}}{\to}_r |M'_1|^c = |P'_1|^c|Q_1|^c$ and $|M_2|^c \stackrel{|C_2|^c_{\mathcal{C}}}{\to}_r |M'_2|^c = |P'_2|^c|Q_2|^c$. Since $|M_1|^c = |M_2|^c$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $|P'_1|^c = |P'_2|^c$. By IH, $|\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P'_1}\} \cup \{cP'_1C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M'_2} = \{cCQ_2 \mid C \in \mathcal{R}^r_{P'_2}\} \cup \{cP'_2C \mid C \in \mathcal{R}^r_{Q_2}\}$, so $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}}\} \cup \{|P'_1|^cC \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}} = \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}\} \cup \{|P'_2|^cC \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}}$. Either $C = C'|Q_1|^c = C'|Q_2|^c$ such that $C' \in |\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = |P'_1|^cC' = |P'_2|^cC'$ such that $C' \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.
- Or $C_1 = cP_1C'_1$ such that $C'_1 \in \mathcal{R}^r_{Q_1}$. $|C_1|^c_{\mathcal{C}} = |P_1|^c|C'_1|^c_{\mathcal{C}} = |P_2|^c|C'_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^r_{M_2}$, $|C_2|^c_{\mathcal{C}} \in |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, so $|C'_1|^c_{\mathcal{C}} \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$ and $C_2 = cP_2C'_2$ such that $|C'_2|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}$. Hence, $M_1 = cP_1Q_1 = cP_1C'_1[R_1] \stackrel{C_1}{\to}_r cP_1C'_1[R'_1] = cP_1Q'_1 = M'_1$, $M_2 = cP_2Q_2 = cP_2C'_2[R_2] \stackrel{C_2}{\to}_r cP_2C'_2[R'_2] = cP_2Q'_2 = M'_2$, $Q_1 \stackrel{C'_1}{\to}_r Q'_1$ and $Q_2 \stackrel{C'_2}{\to}_r Q'_2$. By lemma 2.26, $|M_1|^c = \stackrel{|C_1|^c_{\mathcal{C}}}{\to}_r |M'_1|^c = |P_1|^c|Q'_1|^c$ and $|M_2|^c \stackrel{|C_2|^c_{\mathcal{C}}}{\to}_r |M'_2|^c = |P_2|^c|Q'_2|^c$. Since $|M_1|^c = |M_2|^c$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $|Q'_1|^c = |Q'_2|^c$. By IH, $|\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{cCQ'_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} = \{cCQ'_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{cP_2C \mid C \in \mathcal{R}^r_{Q'_2}\}$, so $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} = \{C|Q'_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^cC \mid C \in |\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}} = \{C|Q'_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^cC \mid C \in |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}}$. Either $C = C'|Q'_1|^c = C'|Q'_2|^c$ such that $C' \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = |P_1|^cC' = |P_2|^cC'$ such that $C' \in |\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.

- Let $M_2 = P_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$ and P_2 is a λ-abstraction. Then $|P_2|^c = |P_1|^c$ and $|Q_2|^c = |Q_1|^c$. Since $M_1 \notin \mathcal{R}^r$, by lemma 2.5, $\mathcal{R}^r_{M_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q_1}\}$. So, $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} = \{|cCQ_1|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{P_1}\} \cup \{|cP_1C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{Q_1}\} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^cC \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$. Again by lemma 2.5, since $M_2 \in \mathcal{R}^r$ by lemma 2.10, $\mathcal{R}^r_{M_2} = \{\Box\} \cup \{CQ_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{P_2C \mid C \in \mathcal{R}^r_{Q_2}\}$. So, $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{|CQ_2|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{P_2}\} \cup \{|P_2C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^r_{Q_2}\} = \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^cC \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}$ then $C|Q_1|^c = C|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. Hence, $C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. Let $C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}$ then $|P_1|^cC = |P_2|^cC \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. Hence, $C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^r_{M_1}$:
  - Either $C_1 = cC'_1Q_1$ such that $C'_1 \in \mathcal{R}^r_{P_1}$. $|C_1|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}|Q_1|^c = |C'_1|^c_{\mathcal{C}}|Q_2|^c = |C_2|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^r_{M_2}$, $|C_2|^c_{\mathcal{C}} \in |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, so $|C'_1|^c_{\mathcal{C}} \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$ and $C_2 = C'_2Q_2$ such that $|C'_2|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}$ and $C'_2 \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = cP_1Q_1 = cC'_1[R_1]Q_1 \stackrel{C_1}{\to}_r cC'_1[R'_1]Q_1 = cP'_1Q_1 = M'_1$, $M_2 = P_2Q_2 = C'_2[R_2]Q_2 \stackrel{C_2}{\to}_r C'_2[R'_2]Q_2 = P'_2Q_2 = M'_2$, $P_1 \stackrel{C'_1}{\to}_r P'_1$

and $P_2 \overset{C'_2}{\to}_r P'_2$. By lemma 2.26, $|M_1|^c = \overset{|C_1|^c_{\mathcal{C}}}{\to}_r |M'_1|^c = |P'_1|^c|Q_1|^c$ and $|M_2|^c \overset{|C_2|^c_{\mathcal{C}}}{\to}_r |M'_2|^c = |P'_2|^c|Q_2|^c$. Since $|M_1|^c = |M_2|^c$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $|P'_1|^c = |P'_2|^c$. By IH, $|\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{cCQ_1 \mid C \in \mathcal{R}^r_{P'_1}\} \cup \{cP'_1C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M'_2} \setminus \{\Box\} = \{CQ_2 \mid C \in \mathcal{R}^r_{P'_2}\} \cup \{P'_2C \mid C \in \mathcal{R}^r_{Q_2}\}$, so $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} = \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}}\} \cup \{|P'_1|^cC \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}} \setminus \{\Box\} = \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}\} \cup \{|P'_2|^cC \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}}$. Either $C = C'|Q_1|^c = C'|Q_2|^c$ such that $C' \in |\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = |P'_1|^cC' = |P'_2|^cC'$ such that $C' \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.

– Or $C_1 = cP_1C'_1$ such that $C'_1 \in \mathcal{R}^r_{Q_1}$. $|C_1|^c_{\mathcal{C}} = |P_1|^c|C'_1|^c_{\mathcal{C}} = |P_2|^c|C'_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^r_{M_2}$, $|C_2|^c_{\mathcal{C}} \in |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, so $|C'_1|^c_{\mathcal{C}} \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$ and $C_2 = P_2C'_2$ such that $|C'_2|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}$. Hence, $M_1 = cP_1Q_1 = cP_1C'_1[R_1] \overset{C_1}{\to}_r cP_1C'_1[R'_1] = cP_1Q'_1 = M'_1$, $M_2 = P_2Q_2 = P_2C'_2[R_2] \overset{C_2}{\to}_r P_2C'_2[R'_2] = P_2Q'_2 = M'_2$, $Q_1 \overset{C'_1}{\to}_r Q'_1$ and $Q_2 \overset{C'_2}{\to}_r Q'_2$. By lemma 2.26, $|M_1|^c = \overset{|C_1|^c_{\mathcal{C}}}{\to}_r |M'_1|^c = |P_1|^c|Q'_1|^c$ and $|M_2|^c \overset{|C_2|^c_{\mathcal{C}}}{\to}_r |M'_2|^c = |P_2|^c|Q'_2|^c$. Since $|M_1|^c = |M_2|^c$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $|Q'_1|^c = |Q'_2|^c$. By IH, $|\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}$. By lemma 2.5, $\mathcal{R}^r_{M'_1} = \{cCQ'_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{cP_1C \mid C \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} \setminus \{\Box\} = \{CQ'_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{P_2C \mid C \in \mathcal{R}^r_{Q'_2}\}$, so $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} = \{C|Q'_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^cC \mid C \in |\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}} \setminus \{\Box\} = \{C|Q'_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^cC \mid C \in |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}}$. Either $C = C'|Q'_1|^c = C'|Q'_2|^c$ such that $C' \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = |P_1|^cC' = |P_2|^cC'$ such that $C' \in |\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.

- Let $M_2 = cN_2 \in \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, $C_2 = cC'_2$ such that $C'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 = cC'_2[R_2] \overset{cC'_2}{\to}_{\beta\eta} cC'_2[R'_2] = cN'_2 = M'_2$ and $N_2 = C'_2[R_2] \overset{C'_2}{\to}_{\beta\eta} C'_2[R'_2] = N'_2$. Since $|C'_2|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}} = |C_1|^c_{\mathcal{C}}$, by IH, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M'_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N'_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$.

6. Let $M_1 = (\lambda x.P_1)Q_1 \in \mathcal{M}_c$ such that $\lambda x.P_1, Q_1 \in \mathcal{M}_c$.
So $|M_1|^c = |\lambda x.P_1|^c|Q_1|^c = |M_2|^c$. By lemma 2.10, $M_1 \in \mathcal{R}^r$, so by lemma 2.5, $\mathcal{R}^r_{M_1} = \{\Box\} \cup \{CQ_1 \mid C \in \mathcal{R}^r_{\lambda x.P_1}\} \cup \{(\lambda x.P_1)C \mid C \in \mathcal{R}^r_{Q_1}\} = \{\Box\} \cup \{(\lambda x.C)Q_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{(\lambda x.P_1)C \mid C \in \mathcal{R}^r_{Q_1}\}$ and so $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_1|^c \mid C \in |\mathcal{R}^r_{\lambda x.P_1}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P_1|^cC \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\} = \{\Box\} \cup \{(\lambda x.C)|Q_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P_1|^cC \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$. We prove this statement by induction on the structure of M_2:

   - Let $M_2 \in \mathcal{V} \setminus \{c\}$ then $|M_2|^c = M_2 \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2 \in \Lambda I$ then $|M_2|^c = \lambda x.|N_2|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2[x := c(cx)] \in \Lambda\eta_c$ then $|M_2|^c = \lambda x.|N_2[x := c(cx)]|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = \lambda x.N_2x \in \Lambda\eta_c$ then $|M_2|^c = \lambda x.|N_2x|^c \neq |P_1|^c|Q_1|^c$.
   - Let $M_2 = cP_2Q_2 \in \mathcal{M}_c$ such that $P_2, Q_2 \in \mathcal{M}_c$. By lemma 2.5, $\mathcal{R}^r_{M_2} = \{cCQ_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{cP_2C \mid C \in \mathcal{R}^r_{Q_2}\}$, so $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|P_2|^cC \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Since $\Box \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}}$ and $\Box \notin rdGEM_2r$, $|\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \not\subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$.

- Let $M_2 = (\lambda x.P_2)Q_2 \in \mathcal{M}_c$ such that $\lambda x.P_2, Q_2 \in \mathcal{M}_c$, then $|P_1|^c = |P_2|^c$ and $|Q_1|^c = |Q_2|^c$. By lemma 2.5, $\mathcal{R}^r_{M_2} = \{\Box\} \cup \{CQ_2 \mid C \in \mathcal{R}^r_{\lambda x.P_2}\} \cup \{(\lambda x.P_2)C \mid C \in \mathcal{R}^r_{Q_2}\} = \{\Box\} \cup \{(\lambda x.C)Q_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{(\lambda x.P_2)C \mid C \in \mathcal{R}^r_{Q_2}\}$, so $|\mathcal{R}^r_{M_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{C|Q_2|^c \mid C \in |\mathcal{R}^r_{\lambda x.P_2}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\} = \{\Box\} \cup \{(\lambda x.C)|Q_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P_2|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. let $C \in |\mathcal{R}^r_{\lambda x.P_1}|^c_{\mathcal{C}}$ then $C|Q_1|^c = C|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{\lambda x.P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{\lambda x.P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{\lambda x.P_2}|^c_{\mathcal{C}}$. Let $C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}$ then $(\lambda x.C)|Q_1|^c = (\lambda x.C)|Q_2|^c \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. let $C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}$ then $|\lambda x.P_1|^c C = |\lambda x.P_2|^c C \in |\mathcal{R}^r_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$, i.e. $|\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^r_{M_1}$:
  - Either $C_1 = \Box$, so $C_2 = \Box$. Hence, $M_1 = (\lambda x.P_1)Q_1 \stackrel{\Box}{\to}_r P_1[x := Q_1] = M'_1$ and $M_2 = (\lambda x.P_2)Q_2 \stackrel{\Box}{\to}_r P_2[x := Q_2] = M'_2$. By lemma 2.27, $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.
  - Or $C_1 = (\lambda x.C'_1)Q_1$ such that $C'_1 \in \mathcal{R}^r_{P_1}$. $|C_1|^c_{\mathcal{C}} = (\lambda x.|C'_1|^c_{\mathcal{C}})|Q_1|^c = (\lambda x.|C'_1|^c_{\mathcal{C}})|Q_2|^c = |C_2|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^r_{M_2}$, $|C_2|^c_{\mathcal{C}} \in |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, so $|C'_1|^c_{\mathcal{C}} \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$ and $C_2 = (\lambda x.C'_2)Q_2$ such that $|C'_2|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}$ and $C'_2 \in \mathcal{R}^r_{P_2}$. Hence, $M_1 = (\lambda x.P_1)Q_1 = (\lambda x.C'_1[R_1])Q_1 \stackrel{C_1}{\to}_r$
    $(\lambda x.C'_1[R'_1])Q_1 = (\lambda x.P'_1)Q_1 = M'_1$,
    $M_2 = (\lambda x.P_2)Q_2 = (\lambda x.C'_2[R_2])Q_2 \stackrel{C_2}{\to}_r (\lambda x.C'_2[R'_2])Q_2 = (\lambda x.P'_2)Q_2$
    $= M'_2$, $P_1 \stackrel{C'_1}{\to}_r P'_1$ and $P_2 \stackrel{C'_2}{\to}_r P'_2$. By lemma 2.26, $|M_1|^c =\stackrel{|C_1|^c_{\mathcal{C}}}{\to}_r |M'_1|^c = |\lambda x.P'_1|^c|Q_1|^c$ and $|M_2|^c \stackrel{|C_2|^c_{\mathcal{C}}}{\to}_r |M'_2|^c = |\lambda x.P'_2|^c|Q_2|^c$. Since $|M_1|^c = |M_2|^c$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $|P'_1|^c = |P'_2|^c$. By IH, $|\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}$. Since $M_1, M_2 \in \mathcal{M}_c$, by lemma 2.12, $M'_1, M'_2 \in \mathcal{M}_c$. By lemma 2.5 and lemma 2.10, $\mathcal{R}^r_{M'_1} = \{\Box\} \cup \{(\lambda x.C)Q_1 \mid C \in \mathcal{R}^r_{P'_1}\} \cup \{(\lambda x.P'_1)C \mid C \in \mathcal{R}^r_{Q_1}\}$ and $\mathcal{R}^r_{M'_2} = \{\Box\} \cup \{(\lambda x.C)Q_2 \mid C \in \mathcal{R}^r_{P'_2}\} \cup \{(\lambda x.P'_2)C \mid C \in \mathcal{R}^r_{Q_2}\}$, so $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{(\lambda x.C)|Q_1|^c \mid C \in |\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P'_1|^c C \mid C \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}}\}$ and
    $|\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{(\lambda x.C)|Q_2|^c \mid C \in |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P'_2|^c C \mid C \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}}$. Either $C = \Box$ then $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = (\lambda x.C')|Q_1|^c = (\lambda x.C')|Q_2|^c$ such that $C' \in |\mathcal{R}^r_{P'_1}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^r_{P'_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = |\lambda x.P'_1|^c C' = |\lambda x.P'_2|^c C'$ such that $C' \in |\mathcal{R}^r_{Q_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.
  - Or $C_1 = (\lambda x.P_1)C'_1$ such that $C'_1 \in \mathcal{R}^r_{Q_1}$. $|C_1|^c_{\mathcal{C}} = |\lambda x.P_1|^c|C'_1|^c_{\mathcal{C}} = |\lambda x.P_2|^c|C'_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^r_{M_2}$, $|C_2|^c_{\mathcal{C}} \in |\mathcal{R}^r_{M_2}|^c_{\mathcal{C}}$, so $|C'_1|^c_{\mathcal{C}} \in |\mathcal{R}^r_{Q_2}|^c_{\mathcal{C}}$ and $C_2 = (\lambda x.P_2)C'_2$ such that $|C'_2|^c_{\mathcal{C}} = |C'_1|^c_{\mathcal{C}}$. Hence, $M_1 = (\lambda x.P_1)Q_1 = (\lambda x.P_1)C'_1[R_1] \stackrel{C_1}{\to}_r (\lambda x.P_1)C'_1[R'_1] = (\lambda x.P_1)Q'_1 = M'_1$,
    $M_2 = (\lambda x.P_2)Q_2 = (\lambda x.P_2)C'_2[R_2] \stackrel{C_2}{\to}_r (\lambda x.P_2)C'_2[R'_2] = (\lambda x.P_2)Q'_2$
    $= M'_2$, $Q_1 \stackrel{C'_1}{\to}_r Q'_1$ and $Q_2 \stackrel{C'_2}{\to}_r Q'_2$. By lemma 2.26, $|M_1|^c =\stackrel{|C_1|^c_{\mathcal{C}}}{\to}_r |M'_1|^c = |\lambda x.P_1|^c|Q'_1|^c$ and $|M_2|^c \stackrel{|C_2|^c_{\mathcal{C}}}{\to}_r |M'_2|^c = |\lambda x.P_2|^c|Q'_2|^c$. Since $|M_1|^c = |M_2|^c$ and $|C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, $|Q'_1|^c = |Q'_2|^c$. By IH, $|\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}$. Since $M_1, M_2 \in \mathcal{M}_c$, by lemma 2.12, $M'_1, M'_2 \in \mathcal{M}_c$. By lemma 2.5 and lemma 2.10, $\mathcal{R}^r_{M'_1} = \{\Box\} \cup \{(\lambda x.C)Q'_1 \mid C \in \mathcal{R}^r_{P_1}\} \cup \{(\lambda x.P_1)C \mid C \in \mathcal{R}^r_{Q'_1}\}$ and $\mathcal{R}^r_{M'_2} = \{\Box\} \cup \{(\lambda x.C)Q'_2 \mid C \in \mathcal{R}^r_{P_2}\} \cup \{(\lambda x.P_2)C \mid C \in \mathcal{R}^r_{Q'_2}\}$, so $|\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}} = \{\Box\} \cup \{(\lambda x.C)|Q'_1|^c \mid C \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P_1|^c C \mid C \in |\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}}\}$ and
    $|\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}} = \{\Box\} \cup \{(\lambda x.C)|Q'_2|^c \mid C \in |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}\} \cup \{|\lambda x.P_2|^c C \mid C \in$

$|\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}\}$. Let $C \in |\mathcal{R}^r_{M'_1}|^c_{\mathcal{C}}$. Either $C = (\lambda x.C')|Q'_1|^c = (\lambda x.C')|Q'_2|^c$ such that $C' \in |\mathcal{R}^r_{P_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^r_{P_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$. Or $C = |\lambda x.P_1|^c C' = |\lambda x.P_2|^c C'$ such that $C' \in |\mathcal{R}^r_{Q'_1}|^c_{\mathcal{C}} \subseteq^{IH} |\mathcal{R}^r_{Q'_2}|^c_{\mathcal{C}}$. So $C \in |\mathcal{R}^r_{M'_2}|^c_{\mathcal{C}}$.

- Let $M_2 = cN_2 \in \Lambda\eta_c$ such that $N_2 \in \Lambda\eta_c$. So $|N_2|^c = |M_2|^c = |M_1|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$. Since $C_2 \in \mathcal{R}^{\beta\eta}_{M_2}$, $C_2 = cC'_2$ such that $C'_2 \in \mathcal{R}^{\beta\eta}_{N_2}$. So, $M_2 = cN_2 = cC'_2[R_2] \stackrel{cC'_2}{\rightarrow}_{\beta\eta} cC'_2[R'_2] = cN'_2 = M'_2$ and $N_2 = C'_2[R_2] \stackrel{C'_2}{\rightarrow}_{\beta\eta} C'_2[R'_2] = N'_2$. Since $|C'_2|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}} = |C_1|^c_{\mathcal{C}}$, by IH, $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M'_2} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N'_2}\}$, so $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$.

7. Let $M_1 = cN_1 \in \Lambda\eta_c$ such that $N_1 \in \Lambda\eta_c$. So $|N_1|^c = |M_1|^c = |M_2|^c$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M_1} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N_1}\}$, so $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{M_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M_2}|^c_{\mathcal{C}}$. Since $C_1 \in \mathcal{R}^{\beta\eta}_{M_1}$, $C_1 = cC'_1$ such that $C'_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. So, $M_1 = cN_1 = cC'_1[R_1] \stackrel{cC'_1}{\rightarrow}_{\beta\eta} cC'_1[R'_1] = cN'_1 = M'_1$ and $N_1 \stackrel{C'_1}{\rightarrow}_{\beta\eta} N'_1$. Since $|C'_1|^c_{\mathcal{C}} = |C_1|^c_{\mathcal{C}} = |C_2|^c_{\mathcal{C}}$, by IH, $|\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$. By lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{M'_1} = \{cC \mid C \in \mathcal{R}^{\beta\eta}_{N'_1}\}$, so $|\mathcal{R}^{\beta\eta}_{M'_1}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{N'_1}|^c_{\mathcal{C}} \subseteq |\mathcal{R}^{\beta\eta}_{M'_2}|^c_{\mathcal{C}}$. □

B Proofs of section 5

Lemma 5.2. 1. (a) By induction on the structure of $M \in \Lambda I$.

- Let $M = x \neq c$. Then $\Phi^{\beta I}(x, \mathcal{F}) = x$, $\mathcal{F} = \varnothing$ and $FV(x) = FV(x) \setminus \{c\}$.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. $FV(M) = FV(N) \setminus \{x\} =^{IH} FV(\Phi^{\beta I}(N, \mathcal{F}')) \setminus \{c, x\} = FV(\lambda x.\Phi^{\beta I}(N, \mathcal{F}')) \setminus \{c\} = \Phi^{\beta I}(M, \mathcal{F})$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{C \mid M_1C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $\Box \in \mathcal{F}$ then, $\Phi^{\beta I}(M, \mathcal{F}) = \Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)$.
  - Else, $\Phi^{\beta I}(M, \mathcal{F}) = c\Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)$.

  In both cases, $FV(M) = FV(M_1) \cup FV(M_2) =^{IH}$ $(FV(\Phi^{\beta I}(M_1, \mathcal{F}_1)) \setminus \{c\}) \cup (FV(\Phi^{\beta I}(M_2, \mathcal{F}_2)) \setminus \{c\}) =$ $FV(\Phi^{\beta I}(M, \mathcal{F})) \setminus \{c\}$.

(b) By induction on the structure of $M \in \Lambda I$.

- Let $M \in \mathcal{V}$, then $M \neq c$. So $\mathcal{F} = \varnothing$ and $\Phi^{\beta I}(M, \mathcal{F}) = M \in \Lambda I_c$.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_N$. By IH, $\Phi^{\beta I}(N, \mathcal{F}') \in \Lambda I_c$. Since by (BC), $x \neq c$, by lemma 5.2.1a, $x \in FV(\Phi^{\beta I}(N, \mathcal{F}'))$. Hence, $\Phi^{\beta I}(M, \mathcal{F}) = \lambda x.\Phi^{\beta I}(N, \mathcal{F}') \in \Lambda I_c$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\mathcal{F}_2 = \{C \mid M_1C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta I}_{M_2}$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta I}(M, \mathcal{F}) = \Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)$. By IH, $\Phi^{\beta I}(M_1, \mathcal{F}_1), \Phi^{\beta I}(M_2, \mathcal{F}_2) \in \Lambda I_c$ and as M_1 is a λ-abstraction, $\Phi^{\beta I}(M_1, \mathcal{F}_1)$ is a λ-abstraction. Hence $\Phi^{\beta I}(M, \mathcal{F}) \in \Lambda I_c$.
  - Else, $\Phi^{\beta I}(M, \mathcal{F}) = c\Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)$. By IH, $\Phi^{\beta I}(M_1, \mathcal{F}_1), \Phi^{\beta I}(M_2, \mathcal{F}_2) \in \Lambda I_c$, hence, $\Phi^{\beta I}(M, \mathcal{F}) \in \Lambda I_c$.

(c) By induction on $M \in \Lambda\text{I}$.

- Let $M = x \neq c$. Then, $\mathcal{F} = \varnothing$ and $\Phi^{\beta I}(x, \mathcal{F}) = x = |x|^c$.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$. $|\Phi^{\beta I}(M, \mathcal{F})|^c = |\lambda x.\Phi^{\beta I}(N, \mathcal{F}')|^c = \lambda x.|\Phi^{\beta I}(N, \mathcal{F}')|^c =^{IH} \lambda x.N$.
- Let $M = M_1 M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}_{M_1}^{\beta I}$ and $\mathcal{F}_2 = \{C \mid M_1 C \in \mathcal{F}\} \subseteq \mathcal{R}_{M_2}^{\beta I}$.
  - If $\Box \in \mathcal{F}$ then M_1 is a λ-abstraction, hence, $\Phi^{\beta I}(M_1, \mathcal{F}_1)$ is a λ-abstraction. So, $|\Phi^{\beta I}(M, \mathcal{F})|^c = |\Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)|^c = |\Phi^{\beta I}(M_1, \mathcal{F}_1)|^c|\Phi^{\beta I}(M_2, \mathcal{F}_2)|^c =^{IH} M_1 M_2 = M$.
  - Else, $|\Phi^{\beta I}(M, \mathcal{F})|^c = |c\Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)|^c = |\Phi^{\beta I}(M_1, \mathcal{F}_1)|^c|\Phi^{\beta I}(M_2, \mathcal{F}_2)|^c =^{IH} M_1 M_2 = M$.

(d) By induction on $M \in \Lambda\text{I}$.

- If $M = x \neq c$ then $\Phi^{\beta I}(M, \mathcal{F}) = M$ and $\mathcal{F} = \varnothing = |\mathcal{R}_M^{\beta I}|_{\mathcal{C}}^{c}$.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$. $\mathcal{F} =^{2.6} \{\lambda x.C \mid C \in \mathcal{F}'\} =^{IH} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(P,\mathcal{F}')}|_{\mathcal{C}}^{c}\} = \{\lambda x.|C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(P,\mathcal{F}')}\}$
  $= \{|\lambda x.C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(P,\mathcal{F}')}\} =^{2.5} |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F})}|_{\mathcal{C}}^{c}$.
- Let $M = M_1 M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}_{M_1}^{\beta I}$ and $\mathcal{F}_2 = \{C \mid M_1 C \in \mathcal{F}\} \subseteq \mathcal{R}_{M_2}^{\beta I}$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta I}(M, \mathcal{F}) = \Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction then $\Phi^{\beta I}(M_1, \mathcal{F}_1)$ too. By lemma 5.2.1b, $\Phi^{\beta I}(M, \mathcal{F}) \in \Lambda\text{I}_c$ then $\Phi^{\beta I}(M, \mathcal{F}) \in \mathcal{R}^{\beta I}$.
    $\mathcal{F} =^{2.6} \{\Box\} \cup \{CM_2 \mid C \in \mathcal{F}_1\} \cup \{M_1 C \mid C \in \mathcal{F}_2\} =^{IH} \{\Box\} \cup \{CM_2 \mid C \in |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\mathcal{F}_1)}|_{\mathcal{C}}^{c}\} \cup \{M_1 C \mid C \in |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\mathcal{F}_2)}|_{\mathcal{C}}^{c}\} = \{\Box\} \cup \{|C|_{\mathcal{C}}^{c} M_2 \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\mathcal{F}_1)}\} \cup \{M_1 |C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\mathcal{F}_2)}\}$
    $=^{5.2.1c} \{\Box\} \cup \{|C\Phi^{\beta I}(M_2, \mathcal{F}_2)|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\mathcal{F}_1)}\} \cup$
    $\{|\Phi^{\beta I}(M_1, \mathcal{F}_1)C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\mathcal{F}_2)}\} =^{2.5} |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F})}|_{\mathcal{C}}^{c}$.
  - Else, $\Phi^{\beta I}(M, \mathcal{F}) = c\Phi^{\beta I}(M_1, \mathcal{F}_1)\Phi^{\beta I}(M_2, \mathcal{F}_2)$.
    $\mathcal{F} =^{2.6} \{CM_2 \mid C \in \mathcal{F}_1\} \cup \{M_1 C \mid C \in \mathcal{F}_2\} =^{IH} \{CM_2 \mid C \in |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\mathcal{F}_1)}|_{\mathcal{C}}^{c}\} \cup \{M_1 C \mid C \in |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\mathcal{F}_2)}|_{\mathcal{C}}^{c}\} = \{|C|_{\mathcal{C}}^{c} M_2 \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\mathcal{F}_1)}\} \cup \{M_1 |C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\mathcal{F}_2)}\} =^{5.2.1c}$
    $\{|cC\Phi^{\beta I}(M_2, \mathcal{F}_2)|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\mathcal{F}_1)}\} \cup$
    $\{|c\Phi^{\beta I}(M_1, \mathcal{F}_1)C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\mathcal{F}_2)}\} =^{2.5} |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F})}|_{\mathcal{C}}^{c}$.

2. (a) By induction on the construction of $M \in \Lambda\text{I}_c$. By lemma 2.22, $|M|^c \in \Lambda\text{I}$

- Let $M \in \mathcal{V} \setminus \{c\}$. Hence $|M|^c = M$, by lemma 2.5, $|\mathcal{R}_M^{\beta I}|_{\mathcal{C}}^{c} = \varnothing = \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^{\beta I}(|M|^c, |\mathcal{R}_M^{\beta I}|_{\mathcal{C}}^{c})$.
- Let $M = \lambda x.P$ where $P \in \Lambda\text{I}_c$ and $x \in FV(P)$. $|M|^c = \lambda x.|P|^c$. By IH, $|\mathcal{R}_P^{\beta I}|_{\mathcal{C}}^{c} \subseteq \mathcal{R}^{\beta I}_{|P|^c}$ and $P = \Phi^{\beta I}(|P|^c, |\mathcal{R}_P^{\beta I}|_{\mathcal{C}}^{c})$. $|\mathcal{R}_M^{\beta I}|_{\mathcal{C}}^{c} =^{2.5} \{|\lambda x.C|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}_P^{\beta I}\} = \{\lambda x.C \mid C \in |\mathcal{R}_P^{\beta I}|_{\mathcal{C}}^{c}\} \subseteq \{\lambda x.C \mid C \in \mathcal{R}^{\beta I}_{|P|^c}\} =^{2.5} \mathcal{R}^{\beta I}_{|M|^c}$. Moreover, $M = \Phi^{\beta I}(|M|^c, |\mathcal{R}_M^{\beta I}|_{\mathcal{C}}^{c})$.
- Let $M = cPQ$ where $P, Q \in \Lambda\text{I}_c$. Let $|M|^c = |P|^c|Q|^c$. By IH, $|\mathcal{R}_P^{\beta I}|_{\mathcal{C}}^{c} \subseteq \mathcal{R}^{\beta I}_{|P|^c}$, $|\mathcal{R}_Q^{\beta I}|_{\mathcal{C}}^{c} \subseteq \mathcal{R}^{\beta I}_{|Q|^c}$, $P = \Phi^{\beta I}(|P|^c, |\mathcal{R}_P^{\beta I}|_{\mathcal{C}}^{c})$ and $Q = \Phi^{\beta I}(|Q|^c, |\mathcal{R}_Q^{\beta I}|_{\mathcal{C}}^{c})$. $|\mathcal{R}_M^{\beta I}|_{\mathcal{C}}^{c} =^{2.5} \{|cCQ|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}_P^{\beta I}\} \cup \{|cPC|_{\mathcal{C}}^{c} \mid C \in \mathcal{R}_Q^{\beta I}\} = \{C|Q|^c \mid C \in |\mathcal{R}_P^{\beta I}|_{\mathcal{C}}^{c}\} \cup \{|P|^c C \mid C \in |\mathcal{R}_Q^{\beta I}|_{\mathcal{C}}^{c}\} \subseteq \{C|Q|^c \mid C \in$

$\mathcal{R}^{\beta I}_{|P|^c}\} \cup \{|P|^c C \mid C \in \mathcal{R}^{\beta I}_{|Q|^c}\} \subseteq^{2.5} \mathcal{R}^{\beta I}_{|M|^c}$.
Moreover $M = \Phi^{\beta I}(|M|^{\beta I}, |\mathcal{R}^{\beta I}_M|^c_{\mathcal{C}})$.

- Let $M = PQ$ where $P, Q \in \Lambda\mathrm{I}_c$ and P is a λ-abstraction. Let $|M|^c = |P|^c|Q|^c$, where $|P|^c$ is a λ-abstraction. By IH, $|\mathcal{R}^{\beta I}_P|^c_{\mathcal{C}} \subseteq \mathcal{R}^{\beta I}_{|P|^c}$, $|\mathcal{R}^{\beta I}_Q|^c_{\mathcal{C}} \subseteq \mathcal{R}^{\beta I}_{|Q|^c}$, $P = \Phi^{\beta I}(|P|^c, |\mathcal{R}^{\beta I}_P|^c_{\mathcal{C}})$ and $Q = \Phi^{\beta I}(|Q|^c, |\mathcal{R}^{\beta I}_Q|^c_{\mathcal{C}})$. $|\mathcal{R}^{\beta I}_M|^c_{\mathcal{C}} =^{2.5} \{\Box\} \cup \{|CQ|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta I}_P\} \cup \{|PC|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta I}_Q\} = \{\Box\} \cup \{C|Q|^c \mid C \in |\mathcal{R}^{\beta I}_P|^c_{\mathcal{C}}\} \cup \{|P|^c C \mid C \in |\mathcal{R}^{\beta I}_Q|^c_{\mathcal{C}}\} \subseteq \{\Box\} \cup \{C|Q|^c \mid C \in \mathcal{R}^{\beta I}_{|P|^c}\} \cup \{|P|^c C \mid C \in \mathcal{R}^{\beta I}_{|Q|^c}\} =^{2.5} \mathcal{R}^{\beta I}_{|M|^c}$.
  Moreover $M = \Phi^{\beta I}(|M|^{\beta I}, |\mathcal{R}^{\beta I}_M|^c_{\mathcal{C}})$.

(b) By lemma 2.22, $|M|^c \in \Lambda\mathrm{I}$. By lemma 2.21 $c \notin FV(|M|^c)$. By lemma 5.2.2a, $|\mathcal{R}^{\beta I}_M|^c_{\mathcal{C}} \subseteq \mathcal{R}^{\beta I}_{|M|^c}$ and $M = \Phi^{\beta I}(|M|^c, |\mathcal{R}^{\beta I}_M|^c_{\mathcal{C}})$. To prove unicity, assume that $(N', \mathcal{F}')$ is another such pair. So $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{N'}$ and $M = \Phi^{\beta I}(N', \mathcal{F}')$. Then, $|M|^c = |\Phi^{\beta I}(N', \mathcal{F}')|^c =^{5.2.1c} N'$ and $\mathcal{F}' =^{5.2.1d} |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(N', \mathcal{F}')}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta I}_M|^c_{\mathcal{C}}$. □

Lemma 5.3. By lemma 5.2.1c and lemma 2.17, there exists a unique $C' \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F})}$, such that $|C'|^c_{\mathcal{C}} = C$. By definition $\exists R \in \mathcal{R}^{\beta I}$ such that $\Phi^{\beta I}(M, \mathcal{F}) = C'[R]$. By lemma 5.2.1c, $|C'[R]|^c = M$. By lemma 2.25, $|C'[R]|^c \stackrel{|C'|^c_{\mathcal{C}}}{\rightarrow}_{\beta I} |C'[R']|^c$ such that R' is the contractum of R. So $M \stackrel{C}{\rightarrow}_{\beta I} |C'[R']|^c$, then $M' = |C'[R']|^c$. Let $\mathcal{F}' = |\mathcal{R}^{\beta I}_{C'[R']}|^c_{\mathcal{C}}$. Since, $\Phi^{\beta I}(M, \mathcal{F}) = C'[R] \stackrel{C'}{\rightarrow}_{\beta I} C'[R']$, by lemma 2.12 and lemma 5.2.1b, $C'[R'] \in \Lambda\mathrm{I}_c$. By lemma 5.2.2a, $C'[R'] = \Phi^{\beta I}(M', \mathcal{F}')$ and $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$. By lemma 5.2.2b, $\mathcal{F}'$ is unique. □

Lemma 5.6. It sufficient to prove:

$$(M, \mathcal{F}) \rightarrow_{\beta Id} (M', \mathcal{F}') \iff \Phi^{\beta I}(M, \mathcal{F}) \rightarrow_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$$

- $\Rightarrow$) let $(M, \mathcal{F}) \rightarrow_{\beta Id} (M', \mathcal{F}')$. Then by definition 5.5, $\exists C \in \mathcal{F}$ such that $M \stackrel{C}{\rightarrow}_{\beta I} M'$ and $\mathcal{F}'$ is the set of βI-residuals in M' relative to C. By definition 5.4 we obtain $\Phi^{\beta I}(M, \mathcal{F}) \rightarrow_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$.

- $\Leftarrow$) Let $\Phi^{\beta I}(M, \mathcal{F}) \stackrel{C}{\rightarrow}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$ such that $C \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F})}$. Since, by lemma 5.2.1b, $\Phi^{\beta I}(M, \mathcal{F}) \in \Lambda\mathrm{I}_c$, by lemma 2.26 and lemma 5.2.1c, $M = |\Phi^{\beta I}(M, \mathcal{F})|^c \stackrel{|C|^c_{\mathcal{C}}}{\rightarrow}_{\beta I} |\Phi^{\beta I}(M', \mathcal{F}')|^c = M'$. By definition 5.4, $\mathcal{F}'$ is the set of βI-residuals of $\mathcal{F}$ in M' relative to $|C|^c_{\mathcal{C}}$. By definition 5.5 we obtain $(M, \mathcal{F}) \rightarrow_{\beta d} (M', \mathcal{F}')$. □

Lemma 5.7. By lemma 5.2.1b, $\Phi^{\beta I}(M, \mathcal{F}_1), \Phi^{\beta I}(M, \mathcal{F}_2) \in \Lambda\mathrm{I}_c$.
By lemma 5.2.1c, $|\Phi^{\beta I}(M, \mathcal{F}_1)|^c = |\Phi^{\beta I}(M, \mathcal{F}_2)|^c$. By lemma 5.2.1d, $|\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F}_1)}|^c_{\mathcal{C}} = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F}_2)}|^c_{\mathcal{C}}$.

If $(M, \mathcal{F}_1) \rightarrow_{\beta Id} (M', \mathcal{F}'_1)$ then by lemma 5.6, $\Phi^{\beta I}(M, \mathcal{F}_1) \rightarrow_{\beta I} \Phi^{\beta I}(M', \mathcal{F}'_1)$. Let $\Phi^{\beta I}(M, \mathcal{F}_1) \stackrel{C_1}{\rightarrow}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}'_1)$ such that $C_1 \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F}_1)}$. Let $C_0 = |C_1|^c_{\mathcal{C}}$, so by lemma 5.2.1d, $C_0 \in \mathcal{F}_1$. By lemma 2.26 and lemma 5.2.1c, $M \stackrel{C_0}{\rightarrow}_{\beta I} M'$.

By lemma 5.3 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^{\beta I}(M, \mathcal{F}_1) \stackrel{C'}{\rightarrow}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$ and $|C'|^c_{\mathcal{C}} = C_0$ where $C' \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F}_1)}$. Since $C', C_1 \in \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\mathcal{F}_1)}$, by lemma 2.17, $C' = C_1$. So, $\Phi^{\beta I}(M', \mathcal{F}') = \Phi^{\beta I}(M', \mathcal{F}'_1)$. By lemma 5.2.1d, $\mathcal{F}' = \mathcal{F}'_1$. By lemma 5.2.1c, $\mathcal{F}'_1 = |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M',\mathcal{F}'_1)}|^c_{\mathcal{C}}$.

By lemma 5.3 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^{\beta I}(M, \mathcal{F}_2) \stackrel{C_2}{\rightarrow}_{\beta I} \Phi^{\beta I}(M', \mathcal{F}'_2)$ and $|C_2|^c_{\mathcal{C}} = C_0$ where $C_2 \in \Phi^{\beta I}(M, \mathcal{F}_2)$. By lemma 5.2.1c, $\mathcal{F}'_2 = |\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M', \mathcal{F}'_2)}|^c_{\mathcal{C}}$.

Hence, by lemma 2.28, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 5.6, $(M, \mathcal{F}_2) \rightarrow_{\beta Id} (M', \mathcal{F}'_2)$. □

Lemma 5.9. 1. By induction on $\Gamma \vdash^{\beta I} M : \sigma$. 2. By induction on $\Gamma \vdash^{\beta\eta} M : \sigma$.
3. First prove (*): if $\Gamma \vdash^{r} M : \sigma$, and $\sigma \sqsubseteq \sigma'$ then $\Gamma \vdash^{r} M : \sigma'$ by induction on $\sigma \sqsubseteq \sigma'$. Then, do the proof of 3. by induction on $\Gamma \vdash^{r} M : \sigma$. For the latter we do:

- Case (ax): If $\Gamma, x : \sigma \vdash^{\beta\eta} x : \sigma$, $\Gamma', x : \sigma' \sqsubseteq \Gamma, x : \sigma$ and $\sigma \sqsubseteq \sigma''$ then $\sigma' \sqsubseteq \sigma$ and so $\sigma' \sqsubseteq \sigma''$. By (ax) $\Gamma', x : \sigma' \vdash^{\beta\eta} x : \sigma'$. By (*), $\Gamma', x : \sigma' \vdash^{\beta\eta} x : \sigma''$.

- Case $(\rightarrow_{E^I})$: If $\frac{\Gamma \vdash^{\beta I} M : \sigma \rightarrow \tau \quad \Delta \vdash^{\beta I} N : \sigma}{\Gamma \sqcap \Delta \vdash^{\beta I} MN : \tau}$, $\Gamma = \Gamma_1, \Gamma_2$, $\Delta = \Delta_1, \Delta_2$, $\Gamma \sqcap \Delta = \Gamma_3, \Gamma_2, \Delta_2$, $\Gamma' = \Gamma'_3, \Gamma'_2, \Delta'_2 \sqsubseteq \Gamma$ where, $\Gamma_1 = (x_i : \sigma_i)_n$, $\Gamma_2 = (y_j, \tau_j)_m$, $\Gamma_3 = (x_i : \sigma_i \cap \sigma'_i)_n$, $\Delta_1 = (x_i : \sigma'_i)_n$, $\Delta_2 = (z_l, \rho_l)_k$, $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Delta_2) = \varnothing$, $\Gamma'_3 = (x_i : \overline{\sigma}_i)_n$, $\Gamma'_2 = (y_j, \overline{\tau}_j)_m$, $\Delta'_2 = (z_l, \overline{\rho}_l)_k$, $\overline{\sigma_i} \sqsubseteq \sigma_i \cap \sigma'_i$, $\overline{\tau_j} \sqsubseteq \tau_j$ and $\overline{\rho_l} \sqsubseteq \rho_l$ then $\Gamma'_3, \Gamma'_2 \sqsubseteq \Gamma$ and $\Gamma'_3, \Delta'_2 \sqsubseteq \Delta$. By IH, $\Gamma'_3, \Gamma'_2 \vdash^{\beta I} M : \sigma \rightarrow \tau$ and $\Gamma'_3, \Delta'_2 \vdash^{\beta I} N : \sigma$, so by $(\rightarrow_{E^I})$, $\Gamma'_3 \sqcap \Gamma'_3, \Gamma'_2, \Delta'_2 \vdash^{\beta I} MN : \tau$. By (*), and since $\Gamma'_3 \sqcap \Gamma'_3 = \Gamma'_3$, we have: $\Gamma'_3, \Gamma'_2, \Delta'_2 \vdash^{\beta I} MN : \tau$. □

Lemma 5.10. When $M \rightarrow^*_r N$ and $M \rightarrow^*_r P$, we write $M \rightarrow^*_r \{N, P\}$.

1. By induction on $\sigma \in \mathsf{Type}^1$.
   - If $\sigma \in \mathcal{A}$ then $CR^r_0 \subseteq CR^r$.
   - If $\sigma = \tau \cap \rho$ then by IH, $CR^r_0 \subseteq [\![\tau]\!]^r, [\![\rho]\!]^r \subseteq CR^r$, so $CR^r_0 \subseteq [\![\tau \cap \rho]\!]^r \subseteq CR^r$.
   - If $\sigma = \tau \rightarrow \rho$ then by IH, $CR^r_0 \subseteq [\![\tau]\!]^r, [\![\rho]\!]^r \subseteq CR^r$ and $[\![\sigma]\!]^r \subseteq CR^r$ by definition. Let $M \in CR^r_0$, so $M = xN_1 \ldots N_n$, where $n \geq 0$ and $N_1, \ldots, N_n \in CR^r$. Let $P \in [\![\tau]\!]^r$ so $P \in CR^r$, hence, $MP \in CR^r_0 \subseteq [\![\rho]\!]^r$ and $M \in [\![\sigma]\!]^r$.

2. Let $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta I}$ where $n \geq 0$, $x \in FV(M)$, and $(\lambda x.M)NN_1 \ldots N_n \rightarrow^*_{\beta I} \{M_1, M_2\}$. By lemma 2.2.7, $\exists M'_1$ and M'_2 such that $M_1 \rightarrow^*_{\beta I} M'_1$, $M[x := N]N_1 \ldots N_n \rightarrow^*_{\beta I} M'_1$, $M_2 \rightarrow^*_{\beta I} M'_2$ and $M[x := N]N_1 \ldots N_n \rightarrow^*_{\beta I} M'_2$. Then we conclude using $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta I}$.

3. Let $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta\eta}$ where $n \geq 0$ and $(\lambda x.M)NN_1 \ldots N_n \rightarrow^*_{\beta\eta} \{M_1, M_2\}$. By lemma 2.2.7, $\exists M'_1$ and M'_2 such that $M_1 \rightarrow^*_{\beta\eta} M'_1$, $M[x := N]N_1 \ldots N_n \rightarrow^*_{\beta\eta} M'_1$, $M_2 \rightarrow^*_{\beta\eta} M'_2$ and $M[x := N]N_1 \ldots N_n \rightarrow^*_{\beta\eta} M'_2$. Then we conclude using $M[x := N]N_1 \ldots N_n \in \mathsf{CR}^{\beta\eta}$.

4. By induction on σ.
   - If $\sigma \in \mathcal{A}$, then the statement is true by 2.
   - If $\sigma = \tau \cap \rho$, then by IH, $[\![\tau]\!]^{\beta I}$ and $[\![\rho]\!]^{\beta I}$ are I-saturated. Let M, N, $N_1, \ldots, N_n \in \Lambda$, $x \in FV(M)$, $n \geq 0$, and $M[x := N]N_1 \ldots N_n \in [\![\sigma]\!]^{\beta I} = [\![\tau]\!]^{\beta I} \cap [\![\rho]\!]^{\beta I}$. Then by I-saturation, $(\lambda x.M)NN_1 \ldots N_n \in [\![\tau]\!]^{\beta I}$ and $(\lambda x.M)NN_1 \ldots N_n \in [\![\rho]\!]^{\beta I}$. Done.
   - If $\sigma = \tau \rightarrow \rho$, then by IH, $[\![\tau]\!]^{\beta I}$ and $[\![\rho]\!]^{\beta I}$ are I-saturated. Let $n \geq 0$, $M, N, N_1, \ldots, N_n \in \Lambda$, $x \in FV(M)$, and $M[x := N]N_1 \ldots N_n \in [\![\sigma]\!]^{\beta I}$. Let $P \in [\![\tau]\!]^{\beta I} \neq \varnothing$, then $M[x := N]N_1 \ldots N_n P \in [\![\rho]\!]^{\beta I}$. By I-saturation, $(\lambda x.M)NN_1 \ldots N_n P \in [\![\rho]\!]^{\beta I}$ so $(\lambda x.M)NN_1 \ldots N_n \in [\![\tau]\!]^{\beta I} \Rightarrow [\![\rho]\!]^{\beta I}$. Since, $M[x := N]N_1 \ldots N_n \in [\![\sigma]\!]^{\beta I} \subseteq CR^{\beta I}$ and $CR^{\beta I}$ is saturated by 2, then $(\lambda x.M)NN_1 \ldots N_n \in CR^{\beta I}$.

5. By induction on σ.

   - If $\sigma \in \mathcal{A}$, then the statement is true by 3.
   - If $\sigma = \tau \cap \rho$, then by IH, $[\![\tau]\!]^{\beta\eta}$ and $[\![\rho]\!]^{\beta\eta}$ are saturated.
     Let $M[x := N]N_1 \ldots N_n \in [\![\sigma]\!]^{\beta\eta} = [\![\tau]\!]^{\beta\eta} \cap [\![\rho]\!]^{\beta\eta}$. Then by saturation, $(\lambda x.M)NN_1 \ldots N_n \in [\![\tau]\!]^{\beta\eta}$ and $(\lambda x.M)NN_1 \ldots N_n \in [\![\rho]\!]^{\beta\eta}$. Done.
   - If $\sigma = \tau \to \rho$, then by IH, $[\![\tau]\!]^{\beta\eta}$ and $[\![\rho]\!]^{\beta\eta}$ are saturated. Let $n \geq 0$, $M, N, N_1, \ldots, N_n \in \Lambda$, $x \in \mathcal{V}$, and $M[x := N]N_1 \ldots N_n \in [\![\sigma]\!]^{\beta\eta}$. Let $P \in [\![\tau]\!]^{\beta\eta} \neq \varnothing$, then $M[x := N]N_1 \ldots N_n P \in [\![\rho]\!]^{\beta\eta}$. By saturation, $(\lambda x.M)NN_1 \ldots N_n P \in [\![\rho]\!]^{\beta\eta}$ so $(\lambda x.M)NN_1 \ldots N_n \in [\![\tau]\!]^{\beta\eta} \Rightarrow [\![\rho]\!]^r$. Since, $M[x := N]N_1 \ldots N_n \in [\![\sigma]\!]^{\beta\eta} \subseteq CR^{\beta\eta}$ and $CR^{\beta\eta}$ is saturated by 3, then $(\lambda x.M)NN_1 \ldots N_n \in CR^{\beta\eta}$.

□

Lemma 5.11. By induction on $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash^r M : \sigma$.

- If the last rule is (ax) or (ax^I), use the hypothesis.
- If the last rule is $(\to_{E^I})$. Let $\Gamma_1 \sqcap \Gamma_2 = (x_i : \sigma_i \cap \sigma'_i)_n, (y_i : \tau_i)_p, (z_i : \rho_i)_q$ such that $\Gamma_1 = (x_i : \sigma_i)_n, (y_i : \tau_i)_p$ and $\Gamma_2 = (x_i : \sigma'_i)_n, (z_i : \rho_i)_q$. Let $\forall i \in \{1, \ldots, n\}, N_i \in [\![\sigma_i \cap \sigma'_i]\!]^{\beta I}$ so $N_i \in [\![\sigma_i]\!]^{\beta I}$ and $N_i \in [\![\sigma'_i]\!]^{\beta I}$, $\forall i \in \{1, \ldots, p\}, P_i \in [\![\tau_i]\!]^{\beta I}$ and $\forall i \in \{1, \ldots, q\}, P'_i \in [\![\rho_i]\!]^{\beta I}$. So by IH, $M[(x_i := N_i)_n, (y_i := P_i)_p] \in [\![\sigma \to \tau]\!]^{\beta I}$ and $N[(x_i := N_i)_n, (z_i := P'_i)_q] \in [\![\sigma]\!]^{\beta I}$. Hence, $(MN)[(x_i := N_i)_n, (y_i := P_i)_p, (z_i := P'_i)_q] \in [\![\tau]\!]^{\beta I}$.
- If the last rule is $(\to_E)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \ldots, n\}, N_i \in [\![\sigma_i]\!]^{\beta\eta}$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \to \tau]\!]^{\beta\eta}$ and $N[(x_i := N_i)_n] \in [\![\sigma]\!]^{\beta\eta}$. Hence, $(MN)[(x_i := N_i)_n] \in [\![\tau]\!]^{\beta\eta}$.
- If the last rule is $(\to_I)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \ldots, n\}, N_i \in [\![\sigma_i]\!]^r$. Let $P \in [\![\sigma]\!]^r \neq \varnothing$. So by IH, $M[(x_i := N_i)_n, x := P] \in [\![\tau]\!]^r$. Moreover $((\lambda x.M)[(x_i := N_i)_n])P = (\lambda x.M[(x_i := N_i)_n])P$.
  - For $\vdash^{\beta I}$, since $x \in FV(M)$ by lemma 2.2.2, $(\lambda x.M[(x_i := N_i)_n]) \to_{\beta I} M[(x_i := N_i)_n, x := P]$ and since by lemma 5.10, $[\![\tau]\!]^{\beta I}$ is I-saturated, $((\lambda x.M)[(x_i := N_i)_n])P \in [\![\tau]\!]^{\beta I}$.
  - For $\vdash^{\beta\eta}$, $(\lambda x.M[(x_i := N_i)_n]) \to_\beta M[(x_i := N_i)_n, x := P]$ and since by lemma 5.10, $[\![\tau]\!]^{\beta\eta}$ is saturated, $((\lambda x.M)[(x_i := N_i)_n])P \in [\![\tau]\!]^{\beta\eta}$.

  So $(\lambda x.M)[(x_i := N_i)_n] \in [\![\sigma]\!]^r \Rightarrow [\![\tau]\!]^r$. Since $x \in [\![\sigma]\!]^r$, $M[(x_i := N_i)_n] \in [\![\tau]\!]^r \subseteq CR^r$, so $\lambda x.M[(x_i := N_i)_n] = (\lambda x.M)[(x_i := N_i)_n] \in CR^r$.
- If the last rule is $(\cap_I)$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \ldots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\tau]\!]^r$ and $M[(x_i := N_i)_n] \in [\![\rho]\!]^r$. So $M[(x_i := N_i)_n] \in [\![\sigma]\!]^r$.
- If the last rule is $(\cap_{E1})$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \ldots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \cap \tau]\!]^r$, so $M[(x_i := N_i)_n] \in [\![\sigma]\!]^r$.
- If the last rule is $(\cap_{E2})$. Let $\Gamma = (x_i : \sigma_i)_n$ and $\forall i \in \{1, \ldots, n\}, N_i \in [\![\sigma_i]\!]^r$. So by IH, $M[(x_i := N_i)_n] \in [\![\sigma \cap \tau]\!]^r$, so $M[(x_i := N_i)_n] \in [\![\tau]\!]^r$. □

Lemma 5.13. By induction on M. Note that by Lemma 2.4, $M \neq c$.

- Let $M = x \neq c$. Then $\Gamma = \Gamma_1, x : \tau$, $\Gamma' = x : \tau$, $\Gamma' \vdash^{\beta I} x : \tau$ and $\forall \sigma$, $\Gamma_1, x : \tau, c : \sigma \vdash^{\beta\eta} x : \tau$.

- Let $M = \lambda x.N \in \Lambda I_c$ then by lemma 2.4, $N \in \Lambda I_c$ and $x \in FV(N)$. $\forall \rho$:
  - If $c \in FV(M)$ then $c \in FV(N)$ and by IH, $\exists \sigma, \tau$ where $\Gamma', x : \rho, c : \sigma \vdash^{\beta I} N : \tau$, hence $\Gamma', c : \sigma \vdash^{\beta I} \lambda x.N : \rho \to \tau$.
  - If $c \notin FV(M)$ then by IH, $\exists \tau$ where $\Gamma', x : \rho \vdash^{\beta I} N : \tau$, hence $\Gamma' \vdash^{\beta I} \lambda x.N : \tau$.
- Let $M = \lambda x.N \in \Lambda\eta_c$ then by lemma 2.4.9.9a, $N \in \Lambda\eta_c$. By IH, $\forall \rho$, $\exists \sigma, \tau$ such that $\Gamma, x : \rho, c : \sigma \vdash^{\beta\eta} N : \tau$. Hence, $\Gamma, c : \sigma \vdash^{\beta\eta} \lambda x.N : \tau$.
- Let $M = cNP$ where $N, P \in \Lambda I_c$. Let $\Gamma'_1 = \Gamma \upharpoonright FV(N)$ and $\Gamma'_2 = \Gamma \upharpoonright FV(P)$. Note that $\Gamma' = \Gamma \upharpoonright FV(cNP) = \Gamma'_1 \sqcap \Gamma'_2$.
  - If $c \notin FV(N) \cup FV(P)$ then by IH, $\exists \tau_1, \tau_2$ such that $\Gamma'_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and $\sigma = \tau_1 \to \tau_2 \to \rho$. By $(\to_{E_I})$ twice, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} cNP : \rho$.
  - If $c \in FV(N)$ and $c \notin FV(P)$ then by IH, $\exists \sigma_1, \tau_1, \tau_2$ such that $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\tau_1 \to \tau_2 \to \rho)$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta I} c : \tau_1 \to \tau_2 \to \rho$. By lemma 5.9.3, $\Gamma'_1, c : \sigma \vdash^{\beta I} N : \tau_1$. By $(\to_{E_I})$ twice, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} cNP : \rho$.
  - If $c \in FV(N) \cap FV(P)$ then by IH, $\exists \sigma_1, \sigma_2, \tau_1, \tau_2$ such that $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} N : \tau_1$ and $\Gamma'_2, c : \sigma_2 \vdash^{\beta I} N : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\sigma_2 \cap (\tau_1 \to \tau_2 \to \rho))$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta I} c : \tau_1 \to \tau_2 \to \rho$. By lemma 5.9.3, $\Gamma'_1, c : \sigma \vdash^{\beta I} N : \tau_1$, and $\Gamma'_2, c : \sigma \vdash^{\beta I} P : \tau_2$. By $(\to_{E_I})$ twice, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} cNP : \rho$.
- Let $M = cNP$ where $N, P \in \Lambda\eta_c$. by IH, $\exists \sigma_1, \sigma_2, \tau_1, \tau_2$ such that $\Gamma, c : \sigma_1 \vdash^{\beta\eta} N : \tau_1$ and $\Gamma, c : \sigma_2 \vdash^{\beta\eta} N : \tau_2$. Let $\rho \in \mathsf{Type}^1$ and let $\sigma = \sigma_1 \cap (\sigma_2 \cap (\tau_1 \to \tau_2 \to \rho))$. By (ax^I) and $(\cap_E)$, $c : \sigma \vdash^{\beta\eta} c : \tau_1 \to \tau_2 \to \rho$. By lemma 5.9.3, $\Gamma, c : \sigma \vdash^{\beta\eta} N : \tau_1$, and $\Gamma, c : \sigma \vdash^{\beta\eta} P : \tau_2$. By $(\to_{E_I})$ twice, $\Gamma, c : \sigma \vdash^{\beta\eta} cNP : \rho$.
- Let $M = NP$ where $N, P \in \Lambda I_c$ and $N = \lambda x.N_0$. So $N_0 \in \Lambda I_c$ and $x \in FV(N_0)$. Let $\Gamma'_1 = \Gamma \upharpoonright FV(N)$ and $\Gamma'_2 = \Gamma \upharpoonright FV(P)$. Note that $\Gamma' = \Gamma \upharpoonright FV(NP) = \Gamma'_1 \sqcap \Gamma'_2$. By BC, $x \neq c$ and $x \notin FV(P)$.x
  - If $c \notin FV(\lambda x.N_0) \cup FV(P)$ then by IH, $\exists \tau_2$ such that $\Gamma'_2 \vdash^{\beta I} P : \tau_2$ and again by IH, $\exists \tau_1$ such that $\Gamma'_1, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$ and $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2 \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.
  - If $c \in FV(\lambda x.N_0)$ and $c \notin FV(P)$ then by IH, $\exists \tau_2$ such that $\Gamma'_2 \vdash^{\beta I} P : \tau_2$. Again by IH, $\exists \sigma, \tau_1$ such that $\Gamma'_1, c : \sigma, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$ and $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.
  - If $c \in FV(\lambda x.N_0) \cap FV(P)$, then by IH, $\exists \sigma_2, \tau_2$ such that $\Gamma'_2, c : \sigma_2 \vdash^{\beta I} P : \tau_2$ and again by IH, $\exists \sigma_1, \tau_1$ such that $\Gamma'_1, c : \sigma_1, x : \tau_2 \vdash^{\beta I} N_0 : \tau_1$. By $(\to_I)$, $\Gamma'_1, c : \sigma_1 \vdash^{\beta I} \lambda x N_0 : \tau_2 \to \tau_1$. By $(\to_{E_I})$, $\Gamma'_1 \sqcap \Gamma'_2, c : \sigma_1 \cap \sigma_2 \vdash^{\beta I} (\lambda x.N_0)P : \tau_1$.
- Let $M = NP$ where $N, P \in \Lambda\eta_c$ and $N = \lambda x.N_0$ then by lemma 2.4.9.9a, $N_0 \in \Lambda\eta_c$. By IH, $\exists \sigma_2, \tau_2$ such that $\Gamma, c : \sigma_2 \vdash^{\beta\eta} P : \tau_2$ and again by IH, $\exists \sigma_1, \tau_1$ such that $\Gamma, c : \sigma_1, x : \tau_2 \vdash^{\beta\eta} N_0 : \tau_1$. By $(\to_I)$, $\Gamma, c : \sigma_1 \vdash^{\beta\eta} \lambda x.N_0 : \tau_2 \to \tau_1$. Let $\sigma = \sigma_1 \cap \sigma_2$. By Lemma 5.9.3, $\Gamma, c : \sigma \vdash^{\beta\eta} \lambda x.N_0 : \tau_2 \to \tau_1$ and $\Gamma, c : \sigma \vdash^{\beta\eta} P : \tau_2$. Hence, by $(\to_E)$, $\Gamma, c : \sigma \vdash^{\beta\eta} (\lambda x.N_0)P : \tau_1$.
- Let $M = cN$ where $N \in \Lambda\eta_c$. By IH, $\exists \sigma, \tau$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} N : \tau$. Let $\rho \in \mathsf{Type}^1$ and $\sigma' = \sigma \cap (\tau \to \rho)$. By Lemma 5.9.3, $\Gamma, c : \sigma' \vdash^{\beta\eta} N : \tau$ and $\Gamma, c : \sigma' \vdash^{\beta\eta} c : \tau \to \rho$. Hence, by $(\to_E)$, $\Gamma, c : \sigma' \vdash^{\beta\eta} cN : \rho$. □

Lemma 5.14. If $M \stackrel{\mathcal{F}_1}{\rightarrow}_{\beta Id} M_1$ and $M \stackrel{\mathcal{F}_2}{\rightarrow}_{\beta Id} M_2$, then $\exists \mathcal{F}_1'', \mathcal{F}_2''$ such that $(M, \mathcal{F}_1) \rightarrow^*_{\beta Id} (M_1, \mathcal{F}_1'')$ and $(M, \mathcal{F}_2) \rightarrow^*_{\beta Id} (M_2, \mathcal{F}_2'')$. Note that by definition 5.5 and lemma 2.2.2, $M_1, M_2 \in \Lambda\mathrm{I}$. By lemma 5.7, $\exists \mathcal{F}_1''' \subseteq \mathcal{R}^{\beta I}_{M_1}$ and $\exists \mathcal{F}_2''' \subseteq \mathcal{R}^{\beta I}_{M_2}$ such that $(M, \mathcal{F}_1 \cup \mathcal{F}_2) \rightarrow^*_{\beta Id} (M_1, \mathcal{F}_1'' \cup \mathcal{F}_1''')$ and $(M, \mathcal{F}_1 \cup \mathcal{F}_2) \rightarrow^*_{\beta Id} (M_2, \mathcal{F}_2'' \cup \mathcal{F}_2''')$. By lemma 5.6 there exist $T, T_1, T_2 \in \Lambda\mathrm{I}_c$ such that

$$T = \Phi^{\beta I}(M, \mathcal{F}_1 \cup \mathcal{F}_2), T_1 = \Phi^{\beta I}(M_1, \mathcal{F}_1'' \cup \mathcal{F}_1'''), T_2 = \Phi^{\beta I}(M_2, \mathcal{F}_2'' \cup \mathcal{F}_2''')$$

and $T \rightarrow^*_{\beta I} T_1$ and $T \rightarrow^*_{\beta I} T_2$. Since by lemma 5.2.1b, $T \in \Lambda\mathrm{I}_c$ and by lemma 5.13.1, T is typable in the type system DI, so $T \in \mathsf{CR}^{\beta I}$ by corollary 5.12. So, by lemma 2, there exists $T_3 \in \Lambda\mathrm{I}_c$, such that $T_1 \rightarrow^*_{\beta I} T_3$ and $T_2 \rightarrow^*_{\beta I} T_3$. Let $\mathcal{F}_3 = |\mathcal{R}^{\beta I}_{T_3}|^c_{\mathcal{C}}$ and $M_3 = |T_3|^{\beta I}$, then by lemma 5.2.2b, $T_3 = \Phi^{\beta I}(M_3, \mathcal{F}_3)$. Hence, by lemma 5.6, $(M_1, \mathcal{F}_1'' \cup \mathcal{F}_1''') \rightarrow^*_{\beta Id} (M_3, \mathcal{F}_3)$ and $(M_2, \mathcal{F}_2'' \cup \mathcal{F}_2''') \rightarrow^*_{\beta Id} (M_3, \mathcal{F}_3)$, i.e., $M_1 \stackrel{\mathcal{F}_1'' \cup \mathcal{F}_1'''}{\rightarrow}_{\beta Id} M_3$ and $M_2 \stackrel{\mathcal{F}_2'' \cup \mathcal{F}_2'''}{\rightarrow}_{\beta Id} M_3$. □

Lemma 5.16. Note that $\varnothing \subseteq \mathcal{R}^{\beta I}_M$. We prove this statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then $\Phi^{\beta I}(M, \varnothing) = M$ and $\mathcal{R}^{\beta I}_M = \varnothing$ by lemma 2.5.
- Let $M = \lambda x.N$ then $\Phi^{\beta I}(M, \varnothing) = \lambda x.\Phi^{\beta I}(N, \varnothing)$. By IH, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(N,\varnothing)} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)} = \varnothing$.
- Let $M = M_1 M_2$ then $\Phi^{\beta I}(M, \varnothing) = c\Phi^{\beta I}(M_1, \varnothing)\Phi^{\beta I}(M_2, \varnothing)$.
  By IH, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\varnothing)} = \varnothing$ and $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\varnothing)} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)} = \varnothing$. □

Lemma 5.17. We prove the statement by induction on the structure of M.

- let $M \in \mathcal{V}$, then $\Phi^{\beta I}(M, \varnothing) = M$.
  - Either $M = x$, then $\Phi^{\beta I}(M, \varnothing)[x := \Phi^{\beta I}(N, \varnothing)] = \Phi^{\beta I}(N, \varnothing)$ and by lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(N,\varnothing)} = \varnothing$.
  - Or $M \neq x$, then $\Phi^{\beta I}(M, \varnothing)[x := \Phi^{\beta I}(N, \varnothing)] = M$ and by lemma 2.5, $\mathcal{R}^{\beta I}_M = \varnothing$.
- Let $M = \lambda y.M'$ then $\Phi^{\beta I}(M, \varnothing) = \lambda y.\Phi^{\beta I}(M', \varnothing)$. So, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]}$ $= \mathcal{R}^{\beta I}_{\lambda y.\Phi^{\beta I}(M',\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]}$. By IH, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M',\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \varnothing$. By lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \varnothing$.
- Let $M = M_1 M_2$ then $\Phi^{\beta I}(M, \varnothing) = c\Phi^{\beta I}(M_1, \varnothing)\Phi^{\beta I}(M_2, \varnothing)$.
  So, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \mathcal{R}^{\beta I}_{c\Phi^{\beta I}(M_1,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]\Phi^{\beta I}(M_2,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]}$.
  By IH, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M,\varnothing)[x:=\Phi^{\beta I}(N,\varnothing)]} = \varnothing$. □

Lemma 5.18. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then by lemma 2.5, $\mathcal{R}^{\beta I}_M = \varnothing$.
- Let $M = \lambda x.N$ then by lemma 2.5, $\mathcal{R}^{\beta I}_M = \{\lambda x.C \mid C \in \mathcal{R}^{\beta I}_N\}$. Let $C \in \mathcal{R}^{\beta I}_M$, then $C = \lambda x.C'$ such that $C' \in \mathcal{R}^{\beta I}_N$. $\Phi^{\beta I}(M, \{C\}) = \lambda x.\Phi^{\beta I}(N, \{C'\}) \rightarrow_{\beta I} \lambda x.N' = M'$ such that $\Phi^{\beta I}(N, \{C'\}) \rightarrow_{\beta I} N'$. By IH, $\mathcal{R}^{\beta I}_{N'} = \varnothing$, so by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

- Let $M = M_1M_2$.
  - Let $M \in \mathcal{R}^{\beta I}$, then $M_1 = \lambda x.M_0$ and by lemma 2.5, $\mathcal{R}^{\beta I}_M = \{\Box\} \cup \{CM_2 \mid C \in \mathcal{R}^{\beta I}_{M_1}\} \cup \{M_1C \mid C \in \mathcal{R}^{\beta I}_{M_2}\}$.
    - Either $C = \Box$ then $\Phi^{\beta I}(M, \{C\}) = \Phi^{\beta I}(M_1, \varnothing)\Phi^{\beta I}(M_2, \varnothing)$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\varnothing)} = \mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\varnothing)} = \varnothing$. Since $\Phi^{\beta I}(M_1, \varnothing) = \lambda x.\Phi^{\beta I}(M_0, \varnothing)$, $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} \Phi^{\beta I}(M_0, \varnothing)[x := \Phi^{\beta I}(M_2, \varnothing)]$. By lemma 5.17, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_0,\varnothing)[x:=\Phi^{\beta I}(M_2,\varnothing)]} = \varnothing$.
    - Or $C = C'M_2$ such that $C' \in \mathcal{R}^{\beta I}_{M_1}$. So, $\Phi^{\beta I}(M, \{C\}) = c\Phi^{\beta I}(M_1, \{C'\})\Phi^{\beta I}(M_2, \varnothing)$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\varnothing)} = \varnothing$. So, if $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} M'$ then $M' = cM_1'\Phi^{\beta I}(M_2, \varnothing)$ and $\Phi^{\beta I}(M_1, \{C'\}) \to_{\beta I} M_1'$. By IH, $\mathcal{R}^{\beta I}_{M_1'} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.
    - Or $C = M_1C'$ such that $C' \in \mathcal{R}^{\beta I}_{M_2}$. So, $\Phi^{\beta I}(M, \{C\}) = c\Phi^{\beta I}(M_1, \varnothing)\Phi^{\beta I}(M_2, \{C'\})$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\varnothing)} = \varnothing$. So, if $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} M'$ then $M' = c\Phi^{\beta I}(M_1, \varnothing)M_2'$ and $\Phi^{\beta I}(M_2, \{C'\}) \to_{\beta I} M_2'$. By IH, $\mathcal{R}^{\beta I}_{M_2'} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.
  - Let $M \notin \mathcal{R}^{\beta I}$, then by lemma 2.5, $\mathcal{R}^{\beta I}_M = \{CM_2 \mid C \in \mathcal{R}^{\beta I}_{M_1}\} \cup \{M_1C \mid C \in \mathcal{R}^{\beta I}_{M_2}\}$.
    - Either $C = C'M_2$ such that $C' \in \mathcal{R}^{\beta I}_{M_1}$. So, $\Phi^{\beta I}(M, \{C\}) = c\Phi^{\beta I}(M_1, \{C'\})\Phi^{\beta I}(M_2, \varnothing)$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_2,\varnothing)} = \varnothing$. So, if $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} M'$ then $M' = cM_1'\Phi^{\beta I}(M_2, \varnothing)$ and $\Phi^{\beta I}(M_1, \{C'\}) \to_{\beta I} M_1'$. By IH, $\mathcal{R}^{\beta I}_{M_1'} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.
    - Or $C = M_1C'$ such that $C' \in \mathcal{R}^{\beta I}_{M_2}$. So, $\Phi^{\beta I}(M, \{C\}) = c\Phi^{\beta I}(M_1, \varnothing)\Phi^{\beta I}(M_2, \{C'\})$. By lemma 5.16, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M_1,\varnothing)} = \varnothing$. So, if $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} M'$ then $M' = c\Phi^{\beta I}(M_1, \varnothing)M_2'$ and $\Phi^{\beta I}(M_2, \{C'\}) \to_{\beta I} M_2'$. By IH, $\mathcal{R}^{\beta I}_{M_2'} = \varnothing$ and by lemma 2.5, $\mathcal{R}^{\beta I}_{M'} = \varnothing$.

□

Lemma 5.19. By lemma 5.3, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta I}_{M'}$, such that $\Phi^{\beta I}(M, \{C\}) \to_{\beta I} \Phi^{\beta I}(M', \mathcal{F}')$. By lemma 5.18, $\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M',\mathcal{F}')} = \varnothing$, so $|\mathcal{R}^{\beta I}_{\Phi^{\beta I}(M',\mathcal{F}')}|^c_{\mathcal{C}} = \varnothing$ and by lemma 5.2.1d, $\mathcal{F}' = \varnothing$. Finally, by lemma 5.6, $(M, \{C\}) \to_{\beta Id} (M', \varnothing)$. □

Lemma 5.20. It is obvious that $\to^*_{1I} \subseteq \to^*_{\beta I}$. We only prove that $\to^*_{\beta I} \subseteq \to^*_{1I}$. Let $M, M' \in \Lambda I$ such that $M \to^*_{\beta I} M'$. We prove this claim by induction on the length of $M \to^*_{\beta I} M'$.

- Let $M = M'$ then it is done since $(M, \mathcal{F}) \to^*_{\beta Id} (M, \mathcal{F})$ for some $\mathcal{F}$.
- Let $M \to^*_{\beta I} M'' \to_{\beta I} M'$. By IH, $M \to^*_{1I} M''$. If $M'' = C[R] \to_{\beta I} C[R'] = M'$ such that R' is the contractum of R then by lemma 5.19 $(M'', \{C\}) \to_{\beta Id} (M', \varnothing)$, so $M'' \to_{1I} M'$. Hence $M \to^*_{1I} M'' \to_{1I} M'$. □

Lemma 5.21. Assume $M \to^*_{\beta I} M_1$ and $M \to^*_{\beta I} M_2$. Then by lemma 5.20, $M \to^*_{1I} M_1$ and $M \to^*_{1I} M_2$. We prove the statement by induction on the length of $M \to^*_{1I} M_1$.

- Let $M = M_1$. Hence $M_1 \rightarrow^*_{1I} M_2$ and $M_2 \rightarrow^*_{1I} M_2$.
- Let $M \rightarrow^*_{1I} M'_1 \rightarrow_{1I} M_1$. By IH, $\exists M'_3, M'_1 \rightarrow^*_{1I} M'_3$ and $M_2 \rightarrow^*_{1I} M'_3$. We prove that $\exists M_3, M_1 \rightarrow^*_{1I} M_3$ and $M'_3 \rightarrow_{1I} M_3$, by induction on $M'_1 \rightarrow^*_{1I} M'_3$.
  - let $M'_1 = M'_3$, hence $M'_3 \rightarrow_{1I} M_1$ and $M_1 \rightarrow^*_{1I} M_1$.
  - Let $M'_1 \rightarrow^*_{1I} M''_3 \rightarrow_{1I} M'_3$. By IH, $\exists M'''_3, M_1 \rightarrow^*_{1I} M'''_3$ and $M''_3 \rightarrow_{1I} M'''_3$. By lemma 2.2.2, $c \notin FV M''_3$. Since $M''_3 \rightarrow_{1I} M'_3$ and $M''_3 \rightarrow_{1I} M'''_3$, by lemma 5.14, $\exists M_3, M'_3 \rightarrow_{1I} M_3$ and $M'''_3 \rightarrow_{1I} M_3$.

□

C Proofs of section 6

Lemma 6.3. 1. (a) By induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$, then $\mathcal{F} = \varnothing$ and $\Phi^{\beta\eta}_0(M, \varnothing) = \{M\} = \{c^0(M)\} \subseteq \Phi^{\beta\eta}(M, \varnothing)$.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}_0(M, \mathcal{F}) = \{\lambda x.N' \mid N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')\} = \{c^0(\lambda x.N') \mid N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')\} \subseteq \Phi^{\beta\eta}(M, \mathcal{F})$.
  - Else $\Phi^{\beta\eta}_0(M, \mathcal{F}) = \{\lambda x.N'[x := c(cx)] \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\} = \{c^0(\lambda x.N'[x := c(cx)]) \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\} \subseteq \Phi^{\beta\eta}(M, \mathcal{F})$.
- Let $M = NP$, $\mathcal{F}_1 = \{C \mid CP \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_P$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}_0(M, \mathcal{F}) = \{N'P' \mid N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}_0(P, \mathcal{F}_2)\} = \{c^0(N'P') \mid N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}_0(P, \mathcal{F}_2)\}$. By IH, $\Phi^{\beta\eta}_0(P, \mathcal{F}_2) \subseteq \Phi^{\beta\eta}(P, \mathcal{F}_2)$, so by definition, $\Phi^{\beta\eta}_0(M, \mathcal{F}) \subseteq \Phi^{\beta\eta}(M, \mathcal{F})$.
  - Else $\Phi^{\beta\eta}_0(M, \mathcal{F}) = \{cN'P' \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}_0(P, \mathcal{F}_2)\} = \{c^0(cN'P') \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}_0(P, \mathcal{F}_2)\}$. By IH, $\Phi^{\beta\eta}(P, \mathcal{F}_2) \in \Phi^{\beta\eta}(P, \mathcal{F}_2)$, so by definition, $\Phi^{\beta\eta}_0(M, \mathcal{F}) \subseteq \Phi^{\beta\eta}(M, \mathcal{F})$.

(b) By induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$, then $\mathcal{F} = \varnothing$, $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(M) \mid n \geq 0\}$ and $\forall N \in \Phi^{\beta\eta}(M, \mathcal{F}), FV(M) = \{M\} = FV(N) \setminus \{c\}$
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')\}$. Let $P \in \Phi^{\beta\eta}(M, \mathcal{F})$, so $\exists n \geq 0$ and $N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')$ such that $P = c^n(\lambda x.N')$. By (BC), $x \neq c$. Hence, $FV(M) = FV(N) \setminus \{x\} =^{IH,1a} FV(N') \setminus \{c, x\} = FV(P) \setminus \{c\}$.
  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\}$. Let $P \in \Phi^{\beta\eta}(M, \mathcal{F})$, so $\exists n \geq 0$ and $\exists N' \in \Phi^{\beta\eta}(N, \mathcal{F}')$ such that, $P = c^n(\lambda x.N'[x := c(cx)])$. By (BC), $x \neq c$. Hence, $FV(M) = FV(N) \setminus \{x\} =^{IH} FV(N') \setminus \{c, x\} = FV(P) \setminus \{c\}$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{C \mid M_1C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.

- If $\Box \in \mathcal{F}$ then, $\Phi^{\beta\eta}(M, \mathcal{F}) =$ $\{c^n(N'P') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(M_1, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)\}$. Let $P \in \Phi^{\beta\eta}(M, \mathcal{F})$, so $\exists n \geq 0$, $N' \in \Phi_0^{\beta\eta}(M_1, \mathcal{F}_1)$ and $P' \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)$ such that $P = c^n(N'P')$.
  Hence, $FV(M) = FV(M_1) \cup FV(M_2) =^{IH,1a} (FV(N') \setminus \{c\}) \cup (FV(P') \setminus \{c\}) = (FV(N') \cup FV(P')) \setminus \{c\} = FV(P) \setminus \{c\}$.
- Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(M_1, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)\}$. Let $P \in \Phi^{\beta\eta}(M, \mathcal{F})$, so $\exists n \geq 0$, $N' \in \Phi^{\beta\eta}(M_1, \mathcal{F}_1)$ and $P' \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)$ such that $P = c^n(cN'P')$.
  Hence, $FV(M) = FV(M_1) \cup FV(M_2) =^{IH} (FV(N') \cup FV(P')) \setminus \{c\} = FV(P) \setminus \{c\}$.

(c) By induction on the structure of M.

- If $M \in \mathcal{V} \setminus \{c\}$ then $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(M) | n \geq 0\}$. Use lemma 6.2.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$.
  - If $\Box \in \mathcal{F}$, then $N = Px$ such that $x \notin FV(P)$ and $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}')\}$. Let $\mathcal{F}'' = \{C \mid Cx \in \mathcal{F}'\} \subseteq \mathcal{R}_P^{\beta\eta}$.
    * If $\Box \in \mathcal{F}'$ then, $\Phi_0^{\beta\eta}(N, \mathcal{F}') = \{P'x \mid P' \in \Phi_0^{\beta\eta}(P, \mathcal{F}'')\}$. Let $M' \in \Phi^{\beta\eta}(M, \mathcal{F})$, so $M' = c^n(\lambda x.P'x)$ where $n \geq 0$ and $P' \in \Phi_0^{\beta\eta}(P, \mathcal{F}'')$. By (BC), $x \neq c$. Since $x \notin FV(P)$, by lemmas 6.3.1b and 6.3.1a, $x \notin P'$. By IH and lemma 6.3.1a, $P', P'x \in \Lambda\eta_c$. By lemma 2.4, $P' \neq c$. Hence, by $(R1)$.4, $\lambda x.P'x \in \Lambda\eta_c$. We conclude using lemma 6.2.
    * Else $\Phi_0^{\beta\eta}(N, \mathcal{F}') = \{cP'x \mid P' \in \Phi^{\beta\eta}(P, \mathcal{F}'')\}$. Let $M' \in \Phi^{\beta\eta}(M, \mathcal{F})$, so $M' = c^n(\lambda x.cP'x)$ where $n \geq 0$ and $P' \in \Phi^{\beta\eta}(P, \mathcal{F}'')$. By (BC), $x \neq c$. Since $x \notin FV(P)$, by lemmas 6.3.1b, $x \notin FV(P')$, so $x \notin FV(cP')$.
      By IH and lemma 6.3.1a, $cP'x \in \Lambda\eta_c$. Since $cP' \neq c$, by $(R1)$.4, $\lambda x.cP'x \in \Lambda\eta_c$. We conclude using lemma 6.2.
  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\}$. Let $N' \in \Phi^{\beta\eta}(N, \mathcal{F}')$ and $n \geq 0$. Since by IH $N' \in \Lambda\eta_c$, by lemma 6.2 and $(R1)$.3, $c^n(\lambda x.N'[x := c(cx)]) \in \Lambda\eta_c$.
- Let $M = NP$, $\mathcal{F}_1 = \{C \mid CP \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta\eta}$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}(M, \mathcal{F}) =$ $\{c^n(N'P') \mid n \geq 0 \wedge N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(P, \mathcal{F}_2)\}$. Let $P = c^n(N'P') \in \Phi^{\beta\eta}(M, \mathcal{F})$ such that $n \geq 0$, $N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}_1)$ and $P' \in \Phi^{\beta\eta}(P, \mathcal{F}_2)$. By IH and lemma 6.3.1a, $N', P' \in \Lambda\eta_c$. Since N is an λ-abstraction then N' too. Hence, by $(R3)$, $N'P' \in \Lambda\eta_c$. By lemma 6.2, $c^n(N'P') \in \Lambda\eta_c$.
  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(N, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(P, \mathcal{F}_2)\}$. Let $c^n(cN'P') \in \Phi^{\beta\eta}(M, \mathcal{F})$ such that $n \geq 0$, $N' \in \Phi^{\beta\eta}(N, \mathcal{F}_1)$ and $P' \in \Phi^{\beta\eta}(P, \mathcal{F}_2)$. By IH, $N', P' \in \Lambda\eta_c$. Hence by $(R2)$, $cN'P' \in \Lambda\eta_c$ and by lemma 6.2, $c^n(cN'P') \in \Lambda\eta_c$.

(d) We prove this lemma by case on the belonging of $\Box$ in $\mathcal{F}$. Let $\mathcal{F}' = \{C \mid Cx \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$.

- If $\Box \in \mathcal{F}$ then $\Phi_0^{\beta\eta}(Nx, \mathcal{F}) = \{N'x \mid N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}')\}$. Hence, $P = N'x$ such that $N' \in \Phi_0^{\beta\eta}(N, \mathcal{F}')$. By (BC), $x \neq c$. Since $x \notin FV(N)$, by lemmas 6.3.1b and 6.3.1a, $x \notin FV(N')$. So $\lambda x.P =$

$\lambda x.N'x \in \mathcal{R}^{\beta\eta}$. Since $\lambda x.N'x \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_P\}$.

- Else $\Phi^{\beta\eta}_0(Nx, \mathcal{F}) = \{cN'x \mid N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\}$ and $P = cN'x$ such that $N' \in \Phi^{\beta\eta}(N, \mathcal{F}')$. By (BC), $x \neq c$. Since $x \notin FV(N)$, by lemmas 6.3.1b, $x \notin FV(N')$ and so $x \notin FV(cN')$. Since $\lambda x.cN'x \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_{\lambda x.P} = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_P\}$.

(e) Let $\mathcal{F}_1 = \{C \mid Cx \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$ and $\mathcal{F}_2 = \{C \mid NC \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_x =^{2.5} \varnothing$. We prove this lemma by case on the belonging of $\Box$ in $\mathcal{F}$.

- If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}(Nx, \mathcal{F}) = \{c^n(N'Q) \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}_1) \wedge Q \in \Phi^{\beta\eta}(x, \mathcal{F}_2)\}$. So $Px = c^n(N'Q)$ such that $n \geq 0$, $N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}_1)$ and $Q \in \Phi^{\beta\eta}(x, \mathcal{F}_2)$. So $n = 0$, $N' = P$ and $Q = x$. Since $x \in \Phi^{\beta\eta}_0(x, \varnothing)$, $Px \in \Phi^{\beta\eta}_0(Nx, \mathcal{F})$.
- Else $\Phi^{\beta\eta}(Nx, \mathcal{F}) = \{c^n(cN'Q) \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}_1) \wedge Q \in \Phi^{\beta\eta}(x, \mathcal{F}_2)\}$. So $Px = c^n(cN'Q)$ such that $n \geq 0$, $N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}_1)$ and $Q \in \Phi^{\beta\eta}(x, \mathcal{F}_2)$. So $n = 0$, $cN' = P$ and $Q = x$. Since $x \in \Phi^{\beta\eta}_0(x, \varnothing)$, $Px \in \Phi^{\beta\eta}_0(Nx, \mathcal{F})$.

(f) Easy by case on the structure of M and induction on n.

(g) By induction on the structure of M.

- Let $M \in \mathcal{V} \setminus \{c\}$. Then $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(M) \mid n \geq 0\}$ and $\mathcal{F} = \varnothing$. Now, use lemma 2.15.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}(M, \mathcal{F}) =$ $\{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')\}$. Let $c^n(\lambda x.N') \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')$.
    Then, $|c^n(\lambda x.N')|^c =^{2.15} |\lambda x.N'|^c = \lambda x.|N'|^c =^{IH,1a} \lambda x.N$.
  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(N, \mathcal{F}')\}$. Let $c^n(\lambda x.N'[x := c(cx)]) \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Phi^{\beta\eta}(N, \mathcal{F}')$. Then, $|c^n(\lambda x.N'[x := c(cx)])|^c =^{2.15} |\lambda x.N'[x := c(cx)]|^c = \lambda x.|N'[x := c(cx)]|^c =^{2.18} \lambda x.|N'|^c =^{IH} \lambda x.N$.
- Let $M = M_1M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{C \mid M_1C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If $\Box$ then $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(M_1, \mathcal{F}_1) \wedge P' \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)\}$. Let $c^n(N'P') \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$, $N' \in \Phi^{\beta\eta}_0(M_1, \mathcal{F}_1)$ and $P' \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction, N' too. Then, $|c^n(N'P')|^c =^{2.15} |N'P'|^c = |N'|^c|P'|^c =^{IH,1a} M_1M_2$.
  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \mathcal{F}_1) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)\}$. Let $c^n(cP_1P_2) \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \mathcal{F}_1)$ and $P_2 \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)$. Then $|c^n(cP_1P_2)|^c =^{2.15} |cP_1P_2|^c = |cP_1|^c|P_2|^c = |P_1|^c|P_2|^c =^{IH} M_1M_2$.

(h) We prove the statement by induction on M.

- Let $M \in \mathcal{V} \setminus \{c\}$. Then $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(x) \mid n \geq 0\}$ and $\mathcal{F} = \varnothing$. If $P \in \Phi^{\beta\eta}(M, \mathcal{F})$ then $\mathcal{R}^{\beta\eta}_P =^{2.9.5} \varnothing$. Hence, $\mathcal{F} = |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}}$.
- Let $M = \lambda x.N$ and $\mathcal{F}' = \{C \mid \lambda x.C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_N$.
  - If $\Box \in \mathcal{F}$ then $M = \lambda x.Px$ where $x \notin FV(P)$ and $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')\}$. Let $c^n(\lambda x.N') \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$ and $N' \in \Phi^{\beta\eta}_0(N, \mathcal{F}')$. $|\mathcal{R}^{\beta\eta}_{c^n(\lambda x.N')}|^c_{\mathcal{C}} =$

$\{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{c^n(\lambda x.N')}\} =^{2.9.5} \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{\lambda x.N'}\} =^{1d} \{\Box\} \cup \{|\lambda x.C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{N'}\} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{N'}|^c_{\mathcal{C}}\} =^{IH} \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{F}'\} =^{2.6} \mathcal{F}$.

  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Phi^{\beta\eta}(N, \mathcal{F}')\}$.
    Let $c^n(\lambda x.P[x := c(cx)]) \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$ and $P \in \Phi^{\beta\eta}(N, \mathcal{F}')$.
    $|\mathcal{R}^{\beta\eta}_{c^n(\lambda x.P[x:=c(cx)])}|^c_{\mathcal{C}} = \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{c^n(\lambda x.P[x:=c(cx)])}\} =^{2.9.5} \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{\lambda x.P[x:=c(cx)]}\} =^{2.9.3} \{|\lambda x.C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{P[x:=c(cx)]}\} =^{2.9.4} \{|\lambda x.C[x := c(cx)]|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{P}\} =^{2.19} \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{P}|^c_{\mathcal{C}}\}$
    $=^{IH} \{\lambda x.C \mid C \in \mathcal{F}'\} =^{2.6} \mathcal{F}$.

- Let $M = M_1M_2$, $\mathcal{F}_1 = \{C \mid CM_2 \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\mathcal{F}_2 = \{C \mid M_1C \in \mathcal{F}\} \subseteq \mathcal{R}^{\beta\eta}_{M_2}$.
  - If $\Box \in \mathcal{F}$ then $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(NP) \mid n \geq 0 \wedge N \in \Phi^{\beta\eta}_0(M_1, \mathcal{F}_1) \wedge P \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)\}$. Let $c^n(NP) \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$, $N \in \Phi^{\beta\eta}_0(M_1, \mathcal{F}_1)$ and $P \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)$. Since M_1 is a λ-abstraction, N too. By lemma 2.5, $|\mathcal{R}^{\beta\eta}_{c^n(NP)}|^c_{\mathcal{C}} = \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{c^n(NP)}\} =^{2.9.5} \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{NP}\} = \{\Box\} \cup \{|CP|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{N}\} \cup \{|NC|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{P}\} = \{\Box\} \cup \{C|P|^c \mid C \in |\mathcal{R}^{\beta\eta}_{N}|^c_{\mathcal{C}}\} \cup \{|N|^cC \mid C \in |\mathcal{R}^{\beta\eta}_{P}|^c_{\mathcal{C}}\} =^{IH} \{\Box\} \cup \{C|P|^c \mid C \in \mathcal{F}_1\} \cup \{|N|^cC \mid C \in \mathcal{F}_2\} =^{1g} \{\Box\} \cup \{CM_2 \mid C \in \mathcal{F}_1\} \cup \{M_1C \mid C \in \mathcal{F}_2\} =^{2.6} \mathcal{F}$.
  - Else $\Phi^{\beta\eta}(M, \mathcal{F}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \mathcal{F}_1) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)\}$. Let $c^n(cP_1P_2) \in \Phi^{\beta\eta}(M, \mathcal{F})$ where $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \mathcal{F}_1)$ and $P_2 \in \Phi^{\beta\eta}(M_2, \mathcal{F}_2)$. By lemma 2.5, $|\mathcal{R}^{\beta\eta}_{c^n(cP_1P_2)}|^c_{\mathcal{C}} = \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{c^n(cP_1P_2)}\} =^{2.9.5} \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{cP_1P_2}\} = \{|cCP_2|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{P_1}\} \cup \{|cP_1C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{P_2}\} = \{C|P_2|^c \mid C \in |\mathcal{R}^{\beta\eta}_{P_1}|^c_{\mathcal{C}}\} \cup \{|P_1|^cC \mid C \in |\mathcal{R}^{\beta\eta}_{P_2}|^c_{\mathcal{C}}\} =^{IH} \{C|P|^c \mid C \in \mathcal{F}_1\} \cup \{|N|^cC \mid C \in \mathcal{F}_2\} =^{1g} \{CM_2 \mid C \in \mathcal{F}_1\} \cup \{M_1C \mid C \in \mathcal{F}_2\} =^{2.6} \mathcal{F}$.

2. (a) By induction on the construction of M.
   - Let $M \in \mathcal{V} \setminus \{c\}$. So $|M|^c = M$, $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = \varnothing = \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}) = \Phi^{\beta\eta}(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$.
   - Let $M = \lambda x.N[x := c(cx)]$ where $N \in \Lambda\eta_c$. $|M|^c = \lambda x.|N|^c$. $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = \{|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_M\} =^{2.9.3} \{\lambda x.|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_{N[x:=c(cx)]}\} =^{2.9.4} \{\lambda x.|C[x := c(cx)]|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_N\} =^{2.19} \{\lambda x.|C|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_N\} = \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}\} \subseteq^{IH} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{|N|^c}\} =^{2.18} \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{|N[x:=c(cx)]|^c}\} \subseteq^{2.5} \mathcal{R}^{\beta\eta}_{\lambda x.|N[x:=c(cx)]|^c} = \mathcal{R}^{\beta\eta}_{|\lambda x.N[x:=c(cx)]|^c}$.
     Since $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = \{\lambda x.|C[x := c(cx)]|^c_{\mathcal{C}} \mid C \in \mathcal{R}^{\beta\eta}_N\}$, $\Box \notin |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}$ and $|\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}} = \{C \mid \lambda x.C \in |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}\}$. By definition, $\Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}) = \{c^n(\lambda x.N'[x := c(cx)]) \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(|N|^c, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}})\}$. By IH, $N \in \Phi^{\beta\eta}(|N|^c, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}})$, so $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$.
   - Let $M = \lambda x.Nx$ where $Nx \in \Lambda\eta_c$, $N \neq c$ and $x \notin FV(N)$. By lemma 2.4, $N \in \Lambda\eta_c$ and by lemma 2.21, $x \notin FV(|N|^c)$. $|M|^c = \lambda x.|Nx|^c = \lambda x.|N|^cx$. Since $M, |M|^c \in \mathcal{R}^{\beta\eta}$, by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{Nx}\}$, so $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = \{\Box\} \cup \{\lambda x.C \mid C \in |\mathcal{R}^{\beta\eta}_{Nx}|^c_{\mathcal{C}}\}$

$\subseteq^{IH} \{\Box\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_{|Nx|^c}\} = \mathcal{R}^{\beta\eta}_{|M|^c}$. So $|\mathcal{R}^{\beta\eta}_{Nx}|^c_{\mathcal{C}} = \{C \mid \lambda x.C \in |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}\}$. By definition, $\Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}) = \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}_0(|Nx|^c, |\mathcal{R}^{\beta\eta}_{Nx}|^c_{\mathcal{C}})\}$. By IH, $Nx \in \Phi^{\beta\eta}(|Nx|^{\beta\eta}, |\mathcal{R}^{\beta\eta}_{Nx}|^c_{\mathcal{C}})$, so by lemma 6.3.1e, $Nx \in \Phi^{\beta\eta}_0(|Nx|^{\beta\eta}, |\mathcal{R}^{\beta\eta}_{Nx}|^c_{\mathcal{C}})$.
Hence $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$.

- Let $M = cNP$ where $N, P \in \Lambda\eta_c$, so $cN \in \Lambda\eta_c$. $|M|^c = |cN|^c|P|^c = |N|^c|P|^c$. Since $M \notin \mathcal{R}^{\beta\eta}$, By lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{cCP \mid C \in \mathcal{R}^{\beta\eta}_N\} \cup \{cNC \mid C \in \mathcal{R}^{\beta\eta}_P\}$. So $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = \{C|P|^c \mid C \in |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}\} \cup \{|N|^cC \mid C \in |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}}\} \subseteq^{IH} \{C|P|^c \mid C \in \mathcal{R}^{\beta\eta}_{|N|^c}\} \cup \{|N|^cC \mid C \in \mathcal{R}^{\beta\eta}_{|P|^c}\} \subseteq^{2.5} \mathcal{R}^{\beta\eta}_{|M|^c}$. Since $\mathcal{R}^{\beta\eta}_M = \{C|P|^c \mid C \in |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}\} \cup \{|N|^cC \mid C \in |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}}\}$, $\Box \notin |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}$ and $|\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}} = \{C \mid C|P|^c \in |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}} = \{C \mid |N|^cC \in |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}\}$.
  By definition, $\Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}) = \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(|N|^c, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}) \wedge P' \in \Phi^{\beta\eta}(|P|^c, |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}})\}$.
  By IH, $N \in \Phi^{\beta\eta}(|N|^{\beta\eta}, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}})$ and $P \in \Phi^{\beta\eta}(|P|^{\beta\eta}, |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}})$, so $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$.
- Let $M = NP$ where $N, P \in \Lambda\eta_c$ and N is a λ-abstraction. So $|N|^c$ is a λ-abstraction too. $|M|^c = |N|^c|P|^c$. Since $M \in \mathcal{R}^{\beta\eta}$, By lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{\Box\} \cup \{CP \mid C \in \mathcal{R}^{\beta\eta}_N\} \cup \{NC \mid C \in \mathcal{R}^{\beta\eta}_P\}$. So $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = \{\Box\} \cup \{C|P|^c \mid C \in |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}\} \cup \{|N|^cC \mid C \in |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}}\} \subseteq^{IH} \{\Box\} \cup \{C|P|^c \mid C \in \mathcal{R}^{\beta\eta}_{|N|^c}\} \cup \{|N|^cC \mid C \in \mathcal{R}^{\beta\eta}_{|P|^c}\} =^{2.5} \mathcal{R}^{\beta\eta}_{|M|^c}$. Since $\mathcal{R}^{\beta\eta}_M = \{\Box\} \cup \{C|P|^c \mid C \in |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}\} \cup \{|N|^cC \mid C \in |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}}\}$, $|\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}} = \{C \mid C|P|^c \in |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}\}$ and $|\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}} = \{C \mid |N|^cC \in |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}\}$. By definition, $\Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}) = \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Phi^{\beta\eta}(|N|^c, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}) \wedge P' \in \Phi^{\beta\eta}(|P|^c, |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}})\}$.
  By IH, $N \in \Phi^{\beta\eta}(|N|^c, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}})$ and $P \in \Phi^{\beta\eta}(|P|^c, |\mathcal{R}^{\beta\eta}_P|^c_{\mathcal{C}})$, so $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$.
- Let $M = cN$ where $N \in \Lambda\eta_c$. $|M|^c = |N|^c$. By lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{cC \mid C \in \mathcal{R}^{\beta\eta}_N\}$ so $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}} \subseteq^{IH} \mathcal{R}^{\beta\eta}_{|N|^c} = \mathcal{R}^{\beta\eta}_{|M|^c}$. By IH, $N \in \Phi^{\beta\eta}(|N|^c, |\mathcal{R}^{\beta\eta}_N|^c_{\mathcal{C}}) = \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$, so by lemma 6.3.1f, $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$.

(b) By lemma 2.21, $c \notin FV(|M|^c)$. By lemma 6.3.2a, $|\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}} \subseteq \mathcal{R}^{\beta\eta}_{|M|^c}$ and $M \in \Phi^{\beta\eta}(|M|^c, |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}})$. To prove unicity, assume that $(N', \mathcal{F}')$ is another such pair. So $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{N'}$ and $M \in \Phi^{\beta\eta}(N', \mathcal{F}')$. By lemma 6.3.1g, $|M|^c = N'$ and by lemma 6.3.1h, $\mathcal{F}' = |\mathcal{R}^{\beta\eta}_M|^c_{\mathcal{C}}$. $\Box$

Lemma 6.4. Let $N_1 \in \Phi^{\beta\eta}(M, \mathcal{F})$. By lemma 6.3.1c, $N_1 \in \Lambda\eta_c$. By lemma 6.3.1h and lemma 2.17, there exists a unique $C_1 \in \mathcal{R}^{\beta\eta}_{N_1}$, such that $|C_1|^c_{\mathcal{C}} = C$. By definition $\exists R_1 \in \mathcal{R}^{\beta\eta}$ such that $N_1 = C_1[R_1]$. By lemma 6.3.1g, $|C_1[R_1]|^c = M$. By lemma 2.25, $|C_1[R_1]|^c \stackrel{|C_1|^c_{\mathcal{C}}}{\rightarrow}_{\beta\eta} |C_1[R'_1]|^c$ such that R'_1 is the contractum of R_1. So $M \stackrel{C}{\rightarrow}_{\beta\eta} |C_1[R'_1]|^c$, then $M' = |C_1[R'_1]|^c$. Let $\mathcal{F}' = |\mathcal{R}^{\beta\eta}_{C_1[R'_1]}|^c_{\mathcal{C}}$. Since, $N_1 = C_1[R_1] \stackrel{C_1}{\rightarrow}_{\beta\eta} C_1[R'_1]$, by lemma 2.12 and lemma 6.3.1c, $C_1[R'_1] \in \Lambda\eta_c$. By lemma 6.3.2a, $C_1[R'_1] \in \Phi^{\beta\eta}(M', \mathcal{F}')$ and $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$. By lemma 6.3.2b, if there exists a such $\mathcal{F}'$, it is unique.

Let $N_2 \in \Phi^{\beta\eta}(M, \mathcal{F})$. By lemma 6.3.1c, $N_1 \in \Lambda\eta_c$. By lemma 6.3.1h and lemma 2.17, there exists a unique $C_2 \in \mathcal{R}^{\beta\eta}_{N_2}$, such that $|C_2|^c_{\mathcal{C}} = C$. By definition $\exists R_2 \in \mathcal{R}^{\beta\eta}$ such that $N_2 = C_2[R_2]$. By lemma 6.3.1g, $|C_2[R_2]|^c = M$.

By lemma 2.25, $|C_2[R_2]|^c \overset{|C_2|^c_{\mathcal{C}}}{\rightarrow}_{\beta\eta} |C_2[R_2']|^c$ such that R_2' is the contractum of R_2. So $M \overset{C}{\rightarrow}_{\beta\eta} |C_2[R_2']|^c$, then $M' = |C_2[R_2']|^c$. Let $\mathcal{F}'' = |\mathcal{R}^{\beta\eta}_{C_2[R_2']}|^c_{\mathcal{C}}$. Since, $N_2 = C_2[R_2] \overset{C_2}{\rightarrow}_{\beta\eta} C_2[R_2']$, by lemma 2.12 and lemma 6.3.1c, $C_2[R_2'] \in \Lambda\eta_c$. By lemma 6.3.2a, $C_2[R_2'] \in \Phi^{\beta\eta}(M', \mathcal{F}'')$ and $\mathcal{F}'' \subseteq \mathcal{R}^{\beta\eta}_{M'}$.

As $N_1, N_2 \in \Phi^{\beta\eta}(M, \mathcal{F})$, by lemma 6.3.1h, $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$ and by lemma 6.3.1g, $|N_1|^c = |N_2|^c$. Finally, by lemma 2.28, $\mathcal{F}' = |\mathcal{R}^{\beta\eta}_{C_1[R_1']}|^c_{\mathcal{C}} = |\mathcal{R}^{\beta\eta}_{C_2[R_2']}|^c_{\mathcal{C}} = \mathcal{F}''$. □

Lemma 6.7. Note that $\Phi^{\beta\eta}(M, \mathcal{F}) \neq \varnothing$. Then, it is sufficient to prove:

- $(M, \mathcal{F}) \rightarrow^*_{\beta\eta d} (M', \mathcal{F}') \Rightarrow \forall N \in \Phi^{\beta\eta}(M, \mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M', \mathcal{F}'), N \rightarrow^*_{\beta\eta} N'$ by induction on the reduction $(M, \mathcal{F}) \rightarrow^*_{\beta\eta d} (M', \mathcal{F}')$.
  - If $(M, \mathcal{F}) = (M', \mathcal{F}')$ then it is done.
  - Let $(M, \mathcal{F}) \rightarrow_{\beta\eta d} (M'', \mathcal{F}'') \rightarrow^*_{\beta\eta d} (M', \mathcal{F}')$.
    By IH, $\forall N'' \in \Phi^{\beta\eta}(M'', \mathcal{F}'')$, $\exists N' \in \Phi^{\beta\eta}(M', \mathcal{F}')$ such that $N \rightarrow^*_{\beta\eta} N''$.
    By definition 6.6, $\exists C \in \mathcal{F}$ such that $M \overset{C}{\rightarrow}_{\beta\eta} M''$ and $\mathcal{F}''$ is the set of $\beta\eta$-residuals in M'' relative to C. By definition 6.5 we obtain $\forall N \in \Phi^{\beta\eta}(M, \mathcal{F}), \exists N'' \in \Phi^{\beta\eta}(M'', \mathcal{F}''), N \rightarrow_{\beta\eta} N''$.
- $\exists N \in \Phi^{\beta\eta}(M, \mathcal{F}), \exists N' \in \Phi^{\beta\eta}(M', \mathcal{F}'), N \rightarrow^*_{\beta\eta} N' \Rightarrow (M, \mathcal{F}) \rightarrow^*_{\beta\eta d} (M', \mathcal{F}')$ by induction on the reduction $N \rightarrow^*_{\beta\eta} N'$ such that $N \in \Phi^{\beta\eta}(M, \mathcal{F})$ and $N' \in \Phi^{\beta\eta}(M', \mathcal{F}')$.
  - If $N = N'$ then by lemma 6.3.2b, $M = M'$ and $\mathcal{F} = \mathcal{F}'$.
  - Let $N \rightarrow_{\beta\eta} N'' \rightarrow^*_{\beta\eta} N'$. By lemma 6.3.1c, $N \in \Lambda\eta_c$, so by lemma 2.12, $N'' \in \Lambda\eta_c$. By lemma 6.3.2b, $(|N''|^c, |\mathcal{R}^{\beta\eta}_{N''}|^c_{\mathcal{C}})$ is the one and only pair such that $c \notin FV(|N''|^c)$, $|\mathcal{R}^{\beta\eta}_{N''}|^c_{\mathcal{C}} \subseteq \mathcal{R}^{\beta\eta}_{|N''|^c}$ and $N'' \in \Phi^{\beta\eta}(|N''|^c, |\mathcal{R}^{\beta\eta}_{N''}|^c_{\mathcal{C}})$.
    So by IH, $(|N''|^c, |\mathcal{R}^{\beta\eta}_{N''}|^c_{\mathcal{C}}) \rightarrow^*_{\beta\eta d} (M', \mathcal{F}')$. Let $N \overset{C}{\rightarrow}_{\beta\eta} N''$, such that $C \in \mathcal{R}^{\beta\eta}_{N}$. By lemmas 2.26 and lemma 6.3.1g, $|N|^c = M \overset{|C|^c_{\mathcal{C}}}{\rightarrow}_{\beta\eta} |N''|^c$. So $|C|^c_{\mathcal{C}} \in \mathcal{R}^{\beta\eta}_{M}$. By definition 6.5, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{|N''|^c}$, such that $\forall P \in \Phi^{\beta\eta}(M, \mathcal{F})$, $\exists P' \in \Phi^{\beta\eta}(|N''|^c, \mathcal{F}')$ and $\exists C' \in \mathcal{R}^{\beta\eta}_{P}$ such that $P \overset{C'}{\rightarrow}_{\beta\eta} P'$ and $|C'|^c_{\mathcal{C}} = |C|^c_{\mathcal{C}}$. $\mathcal{F}'$ is called the set of $\beta\eta$-residuals of $\mathcal{F}$ in $|N''|^c$ relative to $|C|^c_{\mathcal{C}}$. Since $N \in \Phi^{\beta\eta}(M, \mathcal{F})$, $\exists P' \in \Phi^{\beta\eta}(|N''|^c, \mathcal{F}')$ and $\exists C' \in \mathcal{R}^{\beta\eta}_{N}$ such that $N \overset{C'}{\rightarrow}_{\beta\eta} P'$ and $|C'|^c_{\mathcal{C}} = |C|^c_{\mathcal{C}}$. By lemma 2.17, $C = C'$, so $P' = N''$. Since $N'' \in \Phi^{\beta\eta}(|N''|^c, \mathcal{F}')$, by lemma 6.3.2b, $\mathcal{F}' = |\mathcal{R}^{\beta\eta}_{N''}|^c_{\mathcal{C}}$. Finally, by definition 6.6, $(M, \mathcal{F}) \rightarrow_{\beta\eta d} (|N''|^c, |\mathcal{R}^{\beta\eta}_{N''}|^c_{\mathcal{C}})$.

□

Lemma 6.8. By lemma 6.3.1c, $\Phi^{\beta\eta}(M, \mathcal{F}_1), \Phi^{\beta\eta}(M, \mathcal{F}_2) \in \Lambda\eta_c$. $\forall N_1 \in \Phi^{\beta\eta}(M, \mathcal{F}_1)$ and $\forall N_2 \in \Phi^{\beta\eta}(M, \mathcal{F}_2)$, by lemma 6.3.1g, $|N_1|^c = |N_2|^c$ and by lemma 6.3.1h, $|\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} = \mathcal{F}_1 \subseteq \mathcal{F}_2 = |\mathcal{R}^{\beta\eta}_{N_2}|^c_{\mathcal{C}}$.

If $(M, \mathcal{F}_1) \rightarrow_{\beta\eta d} (M', \mathcal{F}_1')$ then by lemma 6.7, $\exists N_1 \in \Phi^{\beta\eta}(M, \mathcal{F}_1)$ and $\exists N_1' \in \Phi^{\beta\eta}(M', \mathcal{F}_1')$ such that $N_1 \rightarrow_{\beta\eta} N_1'$. Let $N_1 \overset{C_1}{\rightarrow}_{\beta\eta} N_1'$ such that $C_1 \in \mathcal{R}^{\beta\eta}_{N_1}$. Let $C_0 = |C_1|^c_{\mathcal{C}}$, so by lemma 6.3.1h, $C_0 \in \mathcal{F}_1$. By lemma 2.26 and lemma 6.3.1g, $M \overset{C_0}{\rightarrow}_{\beta\eta} M'$.

By lemma 6.4 there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that $\forall P_1 \in \Phi^{\beta\eta}(M, \mathcal{F}_1)$, $\exists P_1' \in \Phi^{\beta\eta}(M', \mathcal{F}')$ and $\exists C' \in \mathcal{R}^{\beta\eta}_{P_1}$ such that $P_1 \overset{C'}{\rightarrow}_{\beta\eta} P_1'$ and $|C'|^c_{\mathcal{C}} = C_0$.

Since, $N_1 \in \Phi^{\beta\eta}(M, \mathcal{F}_1)$, $\exists P'_1 \in \Phi^{\beta\eta}(M', \mathcal{F}')$ and $\exists C' \in \mathcal{R}^{\beta\eta}_N$ such that $N_1 \stackrel{C'}{\to}_{\beta\eta} P'_1$ and $|C'|^c_{\mathcal{C}} = C_0$. Since $C', C_1 \in \mathcal{R}^{\beta\eta}_{N_1}$, by lemma 2.17, $C' = C_1$. So, $P'_1 = N_1$. By lemma 6.3.1h, $\mathcal{F}' = |\mathcal{R}^{\beta\eta}_{N_1}|^c_{\mathcal{C}} = \mathcal{F}'_1$.

By lemma 6.4 there exists a unique set $\mathcal{F}'_2 \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that $\forall P_2 \in \Phi^{\beta\eta}(M, \mathcal{F}_2)$, $\exists P'_2 \in \Phi^{\beta\eta}(M', \mathcal{F}'_2)$ and $\exists C_2 \in \mathcal{R}^{\beta\eta}_{P_2}$ such that $P_2 \stackrel{C_2}{\to}_{\beta\eta} P'_2$ and $|C_2|^c_{\mathcal{C}} = C_0$.

Since $\Phi^{\beta\eta}(M, \mathcal{F}_2) \neq \varnothing$, let $N_1 \in \Phi^{\beta\eta}(M, \mathcal{F}_2)$. So, $\exists N'_2 \in \Phi^{\beta\eta}(M', \mathcal{F}'_2)$ and $\exists C_2 \in \mathcal{R}^{\beta\eta}_{N_2}$ such that $N_2 \stackrel{C_2}{\to}_{\beta\eta} N'_2$ and $|C_2|^c_{\mathcal{C}} = C_0$. By lemma 6.3.1h, $\mathcal{F}'_2 = |\mathcal{R}^{\beta\eta}_{N'_2}|^c_{\mathcal{C}}$.

Hence, by lemma 2.28, $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and by lemma 6.7, $(M, \mathcal{F}_2) \to_{\beta\eta d} (M', \mathcal{F}'_2)$. □

Lemma 6.9. If $M \stackrel{\mathcal{F}_1}{\to}_{\beta\eta d} M_1$ and $M \stackrel{\mathcal{F}_2}{\to}_{\beta\eta d} M_2$, then $\exists \mathcal{F}''_1, \mathcal{F}''_2$ such that $(M, \mathcal{F}_1) \to^*_{\beta\eta d} (M_1, \mathcal{F}''_1)$ and $(M, \mathcal{F}_2) \to^*_{\beta\eta d} (M_2, \mathcal{F}''_2)$. By lemma 6.8, $\exists \mathcal{F}'''_1 \subseteq \mathcal{R}^{\beta\eta}_{M_1}$ and $\exists \mathcal{F}'''_2 \subseteq \mathcal{R}^{\beta\eta}_{M_2}$ such that $(M, \mathcal{F}_1 \cup \mathcal{F}_2) \to^*_{\beta\eta d} (M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1)$ and $(M, \mathcal{F}_1 \cup \mathcal{F}_2) \to^*_{\beta\eta d} (M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$. By lemma 6.7 there exist $T, T_1, T_2 \in \Lambda\eta_c$ such that

$$T \in \Phi^{\beta\eta}(M, \mathcal{F}_1), T_1 \in \Phi^{\beta\eta}(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1), T_2 \in \Phi^{\beta\eta}(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2)$$

and $T \to^*_{\beta\eta} T_1$ and $T \to^*_{\beta\eta} T_2$. Since by lemma 6.3.1c, $T \in \Lambda\eta_c$ and by lemma 5.13.2, T is typable in the type system D, so $T \in \mathsf{CR}^{\beta\eta}$ by corollary 5.12. So, by lemma 2.12.1, there exists $T_3 \in \Lambda\eta_c$, such that $T_1 \to^*_{\beta\eta} T_3$ and $T_2 \to^*_{\beta\eta} T_3$. Let $\mathcal{F}_3 = |\mathcal{R}^{\beta\eta}_{T_3}|^c_{\mathcal{C}}$ and $M_3 = |T_3|^{\beta\eta}$, then by lemma 6.3.2a, $\mathcal{F}_3 \subseteq \mathcal{R}^{\beta\eta}_{M_3}$ and $T_3 \in \Phi^{\beta\eta}(M_3, \mathcal{F}_3)$. Hence, by lemma 6.7, $(M_1, \mathcal{F}''_1 \cup \mathcal{F}'''_1) \to^*_{\beta\eta d} (M_3, \mathcal{F}_3)$ and $(M_2, \mathcal{F}''_2 \cup \mathcal{F}'''_2) \to^*_{\beta\eta d} (M_3, \mathcal{F}_3)$, i.e., $M_1 \stackrel{\mathcal{F}''_1 \cup \mathcal{F}'''_1}{\to}_{\beta\eta d} M_3$ and $M_2 \stackrel{\mathcal{F}''_2 \cup \mathcal{F}'''_2}{\to}_{\beta\eta d} M_3$. □

Lemma 6.11. Note that $\varnothing \subseteq \mathcal{R}^{\beta\eta}_M$. We prove this statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then $\Phi^{\beta\eta}(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$ and $\mathcal{R}^{\beta\eta}_{c^n(M)} = \varnothing$, where $n \geq 0$, by lemma 2.5 and lemma 2.9.5.
- Let $M = \lambda x.N$ then $\Phi^{\beta\eta}(M, \varnothing) = \{c^n(\lambda x.Q[x := c(cx)]) \mid n \geq 0 \wedge Q \in \Phi^{\beta\eta}(N, \varnothing)\}$. Let $P \in \Phi^{\beta\eta}(M, \varnothing)$, then $P = c^n(\lambda x.Q[x := c(cx)])$ such that $n \geq 0$ and $Q \in \Phi^{\beta\eta}(N, \varnothing)$ By IH, $\mathcal{R}^{\beta\eta}_Q = \varnothing$ and by lemma 2.9.4, lemma 2.9.3 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_P = \varnothing$.
- Let $M = M_1 M_2$ then $\Phi^{\beta\eta}(M, \varnothing) = \{c^n(cQ_1Q_2) \mid n \geq 0 \wedge Q_1 \in \Phi^{\beta\eta}(M_1, \varnothing) \wedge Q_2 \in \Phi^{\beta\eta}(M_2, \varnothing)\}$. Let $P \in \Phi^{\beta\eta}(M, \varnothing)$, then $P = c^n(cQ_1Q_2)$ such that $n \geq 0$, $Q_1 \in \Phi^{\beta\eta}(M_1, \varnothing)$ and $Q_2 \in \Phi^{\beta\eta}(M_2, \varnothing)$. By IH, $\mathcal{R}^{\beta\eta}_{Q_1} = \mathcal{R}^{\beta\eta}_{Q_2} = \varnothing$ and by lemma 2.5 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_P = \varnothing$. □

Lemma 6.12. We prove the statement by induction on the structure of M.

- let $M \in \mathcal{V}$, then $\Phi^{\beta\eta}(M, \varnothing) = \{c^n(M) \mid n \geq 0\}$. Let $P \in \Phi^{\beta\eta}(M, \varnothing)$ and $Q \in \Phi^{\beta\eta}(N, \varnothing)$, then $P = c^n(M)$ where $n \geq 0$.
  - Either $M = x$, then $P[x := Q] = c^n(Q)$ and by lemma 6.3.1f and lemma 6.11, $\mathcal{R}^{\beta\eta}_{c^n(Q)} = \varnothing$.
  - Or $M \neq x$, then $P[x := Q] = P$ and by lemma 6.11, $\mathcal{R}^{\beta\eta}_P = \varnothing$.
- Let $M = \lambda y.M'$ then $\Phi^{\beta\eta}(M, \varnothing) = \{c^n(\lambda y.P'[y := c(cy)]) \mid n \geq 0 \wedge P' \in \Phi^{\beta\eta}(M', \varnothing)\}$. Let $P \in \Phi^{\beta\eta}(M, \varnothing)$ and $Q \in \Phi^{\beta\eta}(N, \varnothing)$, then $P = c^n(\lambda y.P'[y := c(cy)])$ where $n \geq 0$ and $P' \in \Phi^{\beta\eta}(M', \varnothing)$.
  So, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \mathcal{R}^{\beta\eta}_{c^n(\lambda y.P'[x:=Q][y:=c(cy)])}$. By IH, $\mathcal{R}^{\beta\eta}_{P'[x:=Q]} = \varnothing$ and by lemma 2.9.4, lemma 2.9.3 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$.

- Let $M = M_1M_2$ then $\Phi^{\beta\eta}(M, \varnothing) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \varnothing) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)\}$. Let $P \in \Phi^{\beta\eta}(M, \varnothing)$ and $Q \in \Phi^{\beta\eta}(N, \varnothing)$ then $P = c^n(cP_1P_2)$ where $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \varnothing)$ and $P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)$.
  So, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \mathcal{R}^{\beta\eta}_{c^n(cP_1[x:=Q]P_2[x:=Q])}$. By IH, $\mathcal{R}^{\beta\eta}_{P_1[x:=Q]} = \mathcal{R}^{\beta\eta}_{P_2[x:=Q]} = \varnothing$ and by lemma 2.5 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_{P[x:=Q]} = \varnothing$. $\square$

Lemma 6.13. We prove the statement by induction on the structure of M.

- Let $M \in \mathcal{V}$ then nothing to prove since by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \varnothing$.
- Let $M = \lambda x.N$.
  - If $M \in \mathcal{R}^{\beta\eta}$ then $N = N_0x$ such that $x \notin FV(N_0)$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{\square\} \cup \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_N\}$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then:
    * Either $C = \square$, then $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(\lambda x.P') \mid n \geq 0 \wedge P' \in \Phi^{\beta\eta}_0(N, \varnothing)\}$. Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(\lambda x.P')$ such that $n \geq 0$ and $P' \in \Phi^{\beta\eta}_0(N, \varnothing)$. So $P' = cP'_0x$ such that $P'_0 \in \Phi^{\beta\eta}(N_0, \varnothing)$. By lemma 6.11, $\mathcal{R}^{\beta\eta}_{P'} = \varnothing$, so if $P \to_{\beta\eta} Q$ then $Q = c^{n+1}P'_0$. By lemma 6.11, $\mathcal{R}^{\beta\eta}_{P'_0} = \varnothing$ and by lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.
    * Or $C = \lambda x.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_N$, so $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(\lambda x.P'[x := c(cx)]) \mid n \geq 0 \wedge P' \in \Phi^{\beta\eta}(N, \{C'\})\}$. Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(\lambda x.P'[x := c(cx)])$ such that $n \geq 0$ and $P' \in \Phi^{\beta\eta}(N, \{C'\})$. By lemma 2.9.4, lemma 2.9.3 and lemma 2.9.5, if $P \to_{\beta\eta} Q$ then $Q = c^n(\lambda x.Q'[x := c(cx)])$ such that $P' \to_{\beta\eta} Q'$. By IH, $\mathcal{R}^{\beta\eta}_{Q'} = \varnothing$, so by lemma 2.9.4, lemma 2.9.3 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.
  - Else, by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{\lambda x.C \mid C \in \mathcal{R}^{\beta\eta}_N\}$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then $C = \lambda x.C'$ such that $C' \in \mathcal{R}^{\beta\eta}_N$. $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(\lambda x.P'[x := c(cx)]) \mid n \geq 0 \wedge P' \in \Phi^{\beta\eta}(N, \{C'\})\}$. Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(\lambda x.P'[x := c(cx)])$ such that $n \geq 0$ and $P' \in \Phi^{\beta\eta}(N, \{C'\})$. By lemma 2.9.4, lemma 2.9.3 and lemma 2.9.5, if $P \to_{\beta\eta} Q$ then $Q = c^n(\lambda x.Q'[x := c(cx)])$ such that $P' \to_{\beta\eta} Q'$. By IH, $\mathcal{R}^{\beta\eta}_{Q'} = \varnothing$, so by lemma 2.9.4, lemma 2.9.3 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.
- Let $M = M_1M_2$.
  - Let $M \in \mathcal{R}^{\beta\eta}$, then $M_1 = \lambda x.M_0$ and by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{\square\} \cup \{CM_2 \mid C \in \mathcal{R}^{\beta\eta}_{M_1}\} \cup \{M_1C \mid C \in \mathcal{R}^{\beta\eta}_{M_2}\}$. Let $C \in \mathcal{R}^{\beta\eta}_M$ then:
    * Either $C = \square$ then $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(P_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}_0(M_1, \varnothing) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)\}$. Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(P_1P_2)$ such that $n \geq 0$, $P_1 \in \Phi^{\beta\eta}_0(M_1, \varnothing)$ and $P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)$. By lemma 6.11 and lemma 6.3.1a, $\mathcal{R}^{\beta\eta}_{P_1} = \mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. Since $P_1 \in \Phi^{\beta\eta}_0(M_1, \varnothing)$, $P_1 = \lambda x.P_0[x := c(cx)]$ such that $P_0 \in \Phi^{\beta\eta}(M_0, \varnothing)$. So, if $P \to_{\beta\eta} Q$, then $Q = c^n(P_0[x := c(cP_2)])$. By lemma 6.12 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.
    * Or $C = C'M_2$ such that $C' \in \mathcal{R}^{\beta\eta}_{M_1}$. So, $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \{C'\}) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)\}$.
      Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \{C'\})$ and $P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)$. By lemma 6.11, $\mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. So, if $P \to_{\beta\eta} Q$ then $Q = c^n(cP'_1P_2)$ and $P_1 \to_{\beta\eta} P'_1$. By IH, $\mathcal{R}^{\beta\eta}_{P'_1} = \varnothing$ and by lemma 2.5 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

* Or $C = M_1C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{M_2}$. So, $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \varnothing) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \{C'\})\}$.
  Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \varnothing)$ and $P_2 \in \Phi^{\beta\eta}(M_2, \{C'\})$. By lemma 6.11, $\mathcal{R}^{\beta\eta}_{P_1} = \varnothing$. So, if $P \to_{\beta\eta} Q$ then $Q = c^n(cP_1P'_2)$ and $P_2 \to_{\beta\eta} P'_2$. By IH, $\mathcal{R}^{\beta\eta}_{P'_2} = \varnothing$ and by lemma 2.5 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

- Let $M \notin \mathcal{R}^{\beta\eta}$, then by lemma 2.5, $\mathcal{R}^{\beta\eta}_M = \{CM_2 \mid C \in \mathcal{R}^{\beta\eta}_{M_1}\} \cup \{M_1C \mid C \in \mathcal{R}^{\beta\eta}_{M_2}\}$.
  * Or $C = C'M_2$ such that $C' \in \mathcal{R}^{\beta\eta}_{M_1}$. So, $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \{C'\}) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)\}$.
    Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \{C'\})$ and $P_2 \in \Phi^{\beta\eta}(M_2, \varnothing)$. By lemma 6.11, $\mathcal{R}^{\beta\eta}_{P_2} = \varnothing$. So, if $P \to_{\beta\eta} Q$ then $Q = c^n(cP'_1P_2)$ and $P_1 \to_{\beta\eta} P'_1$. By IH, $\mathcal{R}^{\beta\eta}_{P'_1} = \varnothing$ and by lemma 2.5 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.
  * Or $C = M_1C'$ such that $C' \in \mathcal{R}^{\beta\eta}_{M_2}$. So, $\Phi^{\beta\eta}(M, \{C\}) = \{c^n(cP_1P_2) \mid n \geq 0 \wedge P_1 \in \Phi^{\beta\eta}(M_1, \varnothing) \wedge P_2 \in \Phi^{\beta\eta}(M_2, \{C'\})\}$.
    Let $P \in \Phi^{\beta\eta}(M, \{C\})$ then $P = c^n(cP_1P_2)$ such that $n \geq 0$, $P_1 \in \Phi^{\beta\eta}(M_1, \varnothing)$ and $P_2 \in \Phi^{\beta\eta}(M_2, \{C'\})$. By lemma 6.11, $\mathcal{R}^{\beta\eta}_{P_1} = \varnothing$. So, if $P \to_{\beta\eta} Q$ then $Q = c^n(cP_1P'_2)$ and $P_2 \to_{\beta\eta} P'_2$. By IH, $\mathcal{R}^{\beta\eta}_{P'_2} = \varnothing$ and by lemma 2.5 and lemma 2.9.5, $\mathcal{R}^{\beta\eta}_Q = \varnothing$.

□

LemmA 6.14. By lemma 6.4, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}^{\beta\eta}_{M'}$, such that $\forall N \in \Phi^{\beta\eta}(M, \{C\}), \exists N' \in \Phi^{\beta\eta}(M', \mathcal{F}'), N \to_{\beta\eta} N'$. Let $N \in \Phi^{\beta\eta}(M, \{C\})$ and $N' \in \Phi^{\beta\eta}(M', \mathcal{F}')$ such that $N \to_{\beta\eta} N'$. By lemma 6.13, $\mathcal{R}^{\beta\eta}_{N'} = \varnothing$, So $|\mathcal{R}^{\beta\eta}_{N'}|^c_{\mathcal{C}} = \varnothing$ and by lemma 6.3.1h, $\mathcal{F}' = \varnothing$. Finally, by lemma 6.7, $(M, \{C\}) \to_{\beta\eta d} (M', \varnothing)$. □

Lemma 6.15. It is obvious that $\to^*_1 \subseteq \to^*_{\beta\eta}$. We only prove that $\to^*_{\beta\eta} \subseteq \to^*_1$. Let $M, M' \in \Lambda$ such that $M \to^*_{\beta\eta} M'$. We prove this claim by induction on $M \to^*_{\beta\eta} M'$.

- Let $M = M'$ then it is done since $(M, \mathcal{F}) \to^*_{\beta\eta d} (M, \mathcal{F})$.
- Let $M \to^*_{\beta\eta} M'' \to_{\beta\eta} M'$. By IH, $M \to^*_1 M''$. If $M'' = C[R] \to_{\beta\eta} C[R'] = M'$ such that R' is the contractum of R then by lemma 6.14, $(M'', \{r\}) \to_{\beta\eta d} (M', \varnothing)$, so $M'' \to_1 M'$. Hence $M \to^*_1 M'' \to_1 M'$. □

Lemma 6.16. Let $M_1, M_2 \in \Lambda$ such that $M \to^*_{\beta\eta} M_1$ and $M \to^*_{\beta\eta} M_2$. Then by lemma 6.15, $M \to^*_1 M_1$ and $M \to^*_1 M_2$. We prove the statement by induction on $M \to^*_1 M_1$.

- Let $M = M_1$. Hence $M_1 \to^*_1 M_2$ and $M_2 \to^*_1 M_2$.
- Let $M \to^*_1 M'_1 \to_1 M_1$. By IH, $\exists M'_3, M'_1 \to^*_1 M'_3$ and $M_2 \to^*_1 M'_3$. We prove that $\exists M_3, M_1 \to^*_1 M_3$ and $M'_3 \to_1 M_3$, by induction on $M'_1 \to^*_1 M'_3$.
  - let $M'_1 = M'_3$, hence $M'_3 \to_1 M_1$ and $M_1 \to^*_1 M_1$.
  - Let $M'_1 \to^*_1 M''_3 \to_1 M'_3$. By IH, $\exists M'''_3, M_1 \to^*_1 M'''_3$ and $M''_3 \to_1 M'''_3$. By lemma 2.2.1, $c \notin FV M''_3$. Since $M''_3 \to_1 M'_3$ and $M''_3 \to_1 M'''_3$, By lemma 6.9, $\exists M_3, M'_3 \to_1 M_3$ and $M'''_3 \to_1 M_3$.

□